SHIL'NIKOV'S TYPE SOLUTIONS UNDER NON LINEAR. NON AUTONOMOUS PERTURBATION *

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Abstract

We study the behaviour of solutions in a neighborhood of the origin for a certain type of non autonomous systems of partial differential equations.

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To study either the bifurcations which can arise from a homoclinic orbit Γ , when a system of differential equations is perturbed or the behaviour of solutions close to Γ , it is necessary to know with some precision the return map (i.e. Poincaré map) defined in a transversal section of Γ .

The return map is defined as a composition of two applications, one of them in a neighborhood of the origin and the other in a tubular neighborhood of the homoclinic orbit Γ .

Far away from the origin we may apply continuity of the solutions with respect to initial conditions. So we require to know with some detail the solution in a neighborhood of the origin.

Here we obtain the exponential expansion (Deng [3]) of the solutions of a autonomous system of the type which can be shown in Shil'nikov Theorem, with a non linear, non autonomous perturbation.

Let us consider the equation

$$(0.1) \quad \dot{u} + Au = f(t, u)$$

where A is a sectorial operator in a Banach space X , f is locally Hölder continuous in t and smooth enough in u , $f \in C^k, k \geq 3$. Thus, the equation (0.1) has a local solution.

We assume that the origin is a hyperbolic equilibrium point i. e. $f(t, 0) = 0 \quad \forall t$ and $f(t, u) = Bu + g(t, u)$.

So equation (0.1) takes the form:

$$(0.2) \quad \dot{u} + Lu = g(t, u)$$

where

$$L = A - B.$$

We assume that

$$\sigma(L) = \sigma_1 \cup \sigma_2$$

so that

$$\text{H1 } \sigma_1 = \sigma(L) \cap \{\operatorname{Re} \lambda > 0\}$$

and there exists a simple real eigenvalue μ such that

$$0 < \mu < \operatorname{Re} \lambda \quad \forall \lambda \in \sigma_1 \setminus \{\mu\}$$

$$\text{H2 } \sigma_2 = \sigma(L) \cap \{\operatorname{Re} \lambda < 0\}$$

So that there exist two complex conjugate eigenvalues $\rho \pm i\omega$ which satisfy.

$$0 > \rho > \operatorname{Re} \lambda \quad \forall \lambda \in \sigma_2, \quad \operatorname{Re} \lambda \neq \rho$$

The space X can be written locally as $X = X_1 \oplus X_2$ where X_1 and X_2 are invariant submanifolds associated to the spectral sets σ_1 and σ_2 with projections E_1 and E_2 .

If $L_i = L/X_i$ we have that $\sigma(L_i) = \sigma_i$, $i = 1, 2$.

Let be $x = E_1 u$, $y = E_2 u$.

Thus, the equation (0.2) can be written as:

$$(0.3) \quad \begin{aligned} \dot{x} + L_1 x &= E_1 g(t, x, y) \\ \dot{y} + L_2 y &= E_2 g(t, x, y) \end{aligned}$$

we have the bounds

$$\|e^{-L_1 t} x\| \leq M e^{-\mu t} \|x\|_\alpha, M e^{-\mu t} t^{-\alpha} \|x\|, \text{ for } t > 0$$

$$\|e^{-L_2 t} x\| \leq M e^{-\rho t} \|y\|_\alpha, M e^{-\rho t} \|y\|, \text{ for } t < 0$$

H3 $\|g(t, z_1) - g(t, z_2)\| \leq k(\rho) \|z_1 - z_2\|_\alpha$ for $\|z_i\|_\alpha < \rho$ and $k(\rho) \rightarrow 0$ as $\rho \rightarrow 0$, for all $t \in \mathbf{R}$.

So, (Henry Thm. 5.2.1) [2] there exists local stable and unstable manifolds of (0.3), given by

$$\begin{aligned} W^s &: y = h(x, t) \\ W^u &: x = k(y, t) \end{aligned}$$

If we make the change of variables

$$\begin{aligned} u &= x - k(y, t) \\ v &= y - h(x, t) \end{aligned}$$

then, the equation for W_{loc}^u is $u = 0$ and for W_{loc}^s is $v = 0$.

Rewriting the equations (0.3) in the variables x and y , we obtain

$$(0.4) \quad \begin{aligned} \dot{x} + L_1 x &= f_1(t, x, y) x \\ \dot{y} + L_2 y &= f_2(t, x, y) y \end{aligned}$$

with

$$\begin{aligned} W_{\text{loc}}^u(0) &: x = 0 \\ W_{\text{loc}}^s(0) &: y = 0 \end{aligned}$$

and $f_i(t, 0, 0) = 0 \quad i = 1, 2$.

Let consider the equivalent integral form of (0.4)

$$(0.5) \quad \begin{aligned} x(t) &= e^{-L_1(t-t_0)} x_0 + \int_{t_0}^t e^{-L_1(t-s)} f_1(s, x(s), y(s)) x(s) ds \\ y(t) &= e^{-L_2(t-t_1)} y_1 + \int_{t_1}^t e^{-L_2(t-s)} f_2(s, x(s), y(s)) y(s) ds \end{aligned}$$

Lemma 1. For $\|x_0\|_\alpha, \|y_0\|_\alpha$ small enough the system (0.5) has a unique solution for $0 < t_0 < t_1$.

Proof. Let be $H = \{(x, y) \mid \|x\|_\alpha, \|y\|_\alpha < \rho\}$ with

$$d((x_1, y_1), (x_2, y_2)) = \|x_1 - x_2\|_\alpha + \|y_1 - y_2\|_\alpha.$$

So H is a complete metric space.

Let be $T(x, y) = (T_1(x, y), T_2(x, y))$, where T_1 and T_2 are the right hand sides of (0.5).

Then $T : H \rightarrow H$ and it is a contraction map in H , because

$$\begin{aligned} \|T_1(x, y)\|_\alpha &\leq M e^{-\mu(t-t_s)} \|x_0\|_\alpha \\ &\quad + \int_{t_0}^t M e^{-\mu(t-s)} (t-s)^{-\alpha} \|f_1(s, x(s), y(s))\| \|x(s)\|_\alpha ds \\ &\leq M e^{-\mu(t-t_s)} \|x_0\|_\alpha + \int_{t_0}^t M e^{-\mu(t-s)} (t-s)^{-\alpha} k(\epsilon) \rho ds \\ &\leq M \|x_0\|_\alpha + M k(\rho) \rho \int_{t_0}^t e^{-\mu x} x^{-\alpha} dx < \rho \end{aligned}$$

$$\begin{aligned}
 \|T_2(x, y)\|_\alpha &\leq M e^{-\rho(t-t_1)} \|y_1\|_\alpha \\
 &\quad + \int_t^{t_1} M e^{-\rho(t-s)} \|f_2\| \|y\|_\alpha ds \\
 &\leq M \|y_1\| + \alpha + \int_t^{t_1} M e^{-\rho(t-s)} k(\rho) \rho ds < \rho \\
 \|T_1(x, y) - T_1(x_1, y_1)\|_\alpha &\leq \int_{t_0}^t M e^{-\mu(t-s)} (t-s)^{-\alpha} \|f_1\| \|x(s) - x_1(s)\|_\alpha ds \\
 &\leq k(\rho) \int_{t_s}^t M e^{-\mu(t-s)} (t-s)^{-\alpha} ds \|x - x_1\|_\alpha \\
 &\leq q \|x - x_1\|_\alpha \quad \text{with } q < 1
 \end{aligned}$$

and

$$\begin{aligned}
 \|T_2(x, y) - T_2(x_1, y_1)\|_\alpha &\leq \int_t^{t_1} M e^{-\rho(t-s)} k(\rho) ds \|y - y_1\| \\
 &\leq q \|y - y_1\|_\alpha
 \end{aligned}$$

The solution of this system $(\bar{x}, \bar{y})$ is a solution of (0.4) such that

$$\begin{aligned}
 \bar{x}(t_0) &= x_0 \\
 \bar{x}(t_1, t_0, x_0, y_1) &= x(t_1) = x_1 \\
 \bar{y}(t_0, t_1, x_0, y_1) &= y(t_0) = y_0 \\
 \bar{y}(t_1) &= y_1
 \end{aligned}$$

The solution on the stable manifold has the following exponential form (see Henry Thm. 5.1.2, [2]) called the exponential expansion of the solution

$$x(t, t_0, x_0) = K(x_0, t_0) e^{-\mu(t-t_0)} + \varepsilon(t, t_0)$$

where

$$\|\varepsilon(t, t_0)\|_\alpha \leq C \|x_0\|_\alpha e^{-\delta(t-t_0)}$$

with $K(0, t_0) = 0$ and $\delta > \mu$, $\forall t$.

Although the solution on the unstable manifold do not satisfy in general an analogous condition.

Deng [3] gave some counterexamples.

Definition 1. We say that the solution y has an exponential expansion iff

$$y(t) = e^{L^*(t_1-t)} (\varphi(t, x_0, y_1) + R(t, t_0, x_0, y_1))$$

where

$$L^* = \begin{bmatrix} \rho & \omega & 0 \\ -\omega & \rho & 0 \\ 0 & 0 & \rho i_d \end{bmatrix}$$

$$\frac{\partial \varphi}{\partial y_1}(t, 0, 0) = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} \quad \frac{\partial \varphi}{\partial x_0}(t, 0, 0) = 0, \forall t.$$

R and its derivatives tend exponentially to zero as $t_1 \rightarrow \infty$.

Here we prove:

Theorem 1. Under the assumptions H1, H2, H3 there exists a change of variable such that the solution $y(t)$ has an exponential expansion.

First, we consider $\sigma_2 = \sigma_3 \cup \sigma_4$, where $\sigma_3 = \{\rho \pm i\omega\}$ and $\sigma_4 = \sigma_2 \setminus \sigma_3$.

So, $X_2 = X_3 \oplus X_4$ with projections E_3 and E_4 . $L_i = L/X_i$ $i = 3, 4$.

We redefine $y = E_3 y$, $z = E_4 y$.

The system (0.4) can be written as:

$$(0.6) \quad \begin{aligned} \dot{x} + L_1 x &= f_1(t, x, y, z)x \\ \dot{y} + L_3 y &= F_{11}(t, x, y, z)y + F_{12}(t, x, y, z)z \\ \dot{z} + L_4 z &= F_{21}(t, x, y, z)y + F_{22}(t, x, y, z)z \end{aligned}$$

where y is a two-dimensional variable and z is a $(n-2)$ dimensional variable

$$\begin{aligned} \sigma(L_3) &= \sigma_3 \\ \sigma(L_4) &= \{\lambda_i | i = 3, 4, \dots, \operatorname{Re} \lambda_i \leq \lambda < \rho < 0\} \end{aligned}$$

Lemma 2. There exist a change of variable such in (0.6)

$$\begin{aligned} F_{1j}(t, 0, y, z) &= 0 \\ F_{i1}(t, x, 0, 0) &= 0 \quad i, j = 1, 2 \end{aligned}$$

$$F_{ij} \in C^{k-1}.$$

Proof. Let be $\theta = x$, $\eta = y$, $\varepsilon = y + R_1(y, z)y + R_2(y, t)$ then

$$\begin{aligned}\dot{\varepsilon} &= \dot{y} + \dot{R}_1 y + R_1 \dot{y} + \dot{R}_2 z + R_2 \dot{z} \\ &\quad (I + R_1)(-L_3 y + F_{11} y + F_{12} z) + \dot{R}_1 y + \dot{R}_2 z \\ &\quad + R_2(-L_3 z + F_{21} y + F_{22} z)\end{aligned}$$

So

$$\dot{\varepsilon} + L_3 y + R_1 L_3 y + R_2 L_4 z = (I + R_1)(F_{11} y + F_{12} z) + \dot{R}_2 z + R_2(F_{21} y + F_{22} z)$$

$$\begin{aligned}\dot{\varepsilon} + L_3 \varepsilon &= L_3 R_1 y + L_3 R_2 z - R_1 L_3 y - R_2 L_4 z \\ &\quad + (I + R_1)(F_{11} y + F_{12} z) + \dot{R}_1 y + \dot{R}_2 z + R_2(F_{21} y + F_{22} z) \\ &= (L_3 R_1 - R_1 L_3 + \dot{R}_1 + (I - R_1)F_{11} + R_2 F_{21})y \\ &\quad + (L_3 R_2 - R_2 L_4 + \dot{R}_2 + (I + R_1)F_{12} + R_2 F_{22})z\end{aligned}$$

Thus, the first condition is satisfied if

$$\begin{aligned}\dot{R}_1 + L_3 R_1 - R_1 L_3 &= -(I + R_1)F_{11}(0, y, z) - R_2 F_{21}(0, y, z) \\ \dot{R}_2 + L_3 R_2 - R_2 L_4 &= -(I + R_1)F_{12}(0, y, z) - R_2 F_{22}(0, y, z).\end{aligned}$$

Plus the equations for y and z with $x = 0$.

If we write the elements of the matrices R_1 and R_2 as columns we obtain a system of ordinary differential equations, with the origin as an equilibrium point. Its linearization has eigenvalues with non negative real parts, meanwhile the elements of σ_3 and σ_4 have negative real part.

So, there exist in a neighborhood of the origin C^{k-1} invariant manifolds, given by

$$\begin{aligned}R_1 &= R_1(y, z) \\ R_2 &= R_2(y, z) \quad \text{as we wished.}\end{aligned}$$

The second proof is analogous.

According, to this lemma the equation (0.6) has the form

$$\begin{aligned}(0.7) \quad \dot{x} + L_1 x &= f_1(t, z, y, z)x \\ \dot{y} + L_3 y &= G_{11}xy^2 + G_{12}xyz + G_{12}xz \\ \dot{z} + L_4 z &= G_{21}y^2 + G_{22}yz + F_{22}z\end{aligned}$$

where

$$\begin{aligned}f_1(t, 0, 0, 0) &= 0 \\ F_{22}(t, 0, 0, 0) &= 0\end{aligned}$$

and G_{ij} are bounded in t , $i, j = 1, 2$.

If we use the integral form equivalent to (0.7), we can prove.

Lemma 3. For $\|x_0\|_\alpha$, $\|y_1\|_\alpha$, $\|z_1\|_\alpha$ small enough, $t_0, t_1 > 0$ we have

$$\begin{aligned} \|Dx(t)\|_\alpha &\leq M e^{-\mu(t-t_0)} & t \geq t_0 \\ \|Dy(t)\|_\alpha &\leq M e^{\rho(t_1-t)} & t < t_1 \\ \|Dz(t)\|_\alpha &\leq M e^{(\rho+\sigma)(t_1-t)} & t < t_1 \end{aligned}$$

For the functions and its derivatives with respect to the initial conditions, for

$$\begin{aligned} \lambda - \rho &< \sigma < 0 \\ \rho &< \sigma < 0, \quad M \text{ is a positive constant.} \end{aligned}$$

Proof. Let be $H = \{(x, y, z) | x, y, z \text{ satisfy the inequalities } \}$ and $P(x, y, z) = (P_1(x, y, z), P_2(x, y, z), P_3(x, y, z))$ where P_1 , P_2 y P_3 are the corresponding right hand sides of the integral equations.

Then, $P : H \rightarrow H$ and P is a contraction mapping, because

$$\begin{aligned} \|P_1\|_\alpha &\leq M e^{-\mu(t-t_0)} \|x_0\|_\alpha + \int_{t_0}^t M e^{-\mu(t-s)} (t-s)^{-\alpha} \|f_1\| \|x(s)\| ds \\ &\leq M e^{-\mu(t-t_0)} \left(\|x_0\|_\alpha + \int_{t_0}^t M e^{-\mu s} \|f_1\| (t-s)^{-\alpha} ds \right) \\ &\leq \overline{M} e^{-\mu(t-t_s)} \end{aligned}$$

$$\begin{aligned} \|P_2\|_\alpha &\leq M e^{\rho(t_1-t)} \|y_1\|_\alpha + \int_t^{t_1} M e^{\rho(t-s)} \|G_{11}xy^2 + G_{12}xyz + G_{13}xz\| ds \\ &\leq M e^{\rho(t_1-t)} \left(\|y_1\| + \int_t^{t_1} M e^{-\mu s} e^{\sigma(t_1-s)} ds \right) \\ &\leq \overline{M} e^{\rho(t_1-t)} \end{aligned}$$

$$\begin{aligned} \|P_3\|_\alpha &= M e^{(\rho+\sigma)(t_1-t)} \left(\|z_1\|_\alpha + \int_t^{t_1} e^{(\sigma+\rho)(t_1-s)} \|g_{21}y^2 + G_{22}yz + \right. \\ &\quad \left. + F_{22}z\| ds \right) \\ &\leq M e^{(\rho+\sigma)(t_1-t)} \left(\|z_1\|_\alpha + \int_t^{t_1} \left(\|g_{21}\| e^{(\rho-\sigma)(t_1-s)} + \|G_{22}\| e^{\sigma(t_1-s)} + \right. \right. \\ &\quad \left. \left. + \|F_{22}\| \right) ds \right) \\ &\leq \overline{M} e^{(\rho+\sigma)(t_1-t)} \end{aligned}$$

analogously

$$\|P_i(x, y, z) - P_i(x_1, y_1, z_1)\|_\alpha \leq q d((x, y, z)(x_1, y_1, z_1))$$

such as in [1], we can prove

$$\begin{aligned}\left\| \frac{\partial Dx(t)}{\partial t_0} \right\|_{\alpha} &\leq K e^{-\mu(t-t_0)+\rho(t_1-t)} \\ \left\| \frac{\partial Dy(t)}{\partial t_0} \right\|_{\alpha} &\leq K e^{\rho(t_1-t)} \\ \left\| \frac{\partial Dz(t)}{\partial t_0} \right\|_{\alpha} &\leq K e^{(\rho+\sigma)(t_1-t)}\end{aligned}$$

Let consider

$$\begin{pmatrix} Y \\ Z \end{pmatrix} = e^{-L^*(t_1-t)} \begin{pmatrix} y \\ z \end{pmatrix}$$

i.e.

$$\begin{aligned}Y(t) &= e^{-L_3(t_1-t)}y(t) \\ Z(t) &= e^{-\rho I_d(t_1-t)}z(t)\end{aligned}$$

We have for the functions and it derivatives with respect to the initial conditions.

Lemma 4.

$$\left\| \frac{\partial DY}{\partial t_1} \right\|_{\alpha}, \left\| \frac{\partial DZ}{\partial t_1} \right\|_{\alpha} \leq K e^{\sigma(t_1-t)}$$

Proof. [Blázquez-Tuma] So, to finish the Proof of the theorem.

We have that $\bar{Y} = \begin{pmatrix} DY \\ DZ \end{pmatrix}$ is Cauchy in t_0 uniformly on compact sets, because

$$\begin{aligned}& \|\bar{Y}(t, t_1^1, x_0, y_1, z_1) - \bar{Y}(t, t_1^2, x_0, y_1, z_1)\|_{\alpha} \\ & \leq \int_{t_1^1}^{t_1^2} \left\| \frac{\partial}{\partial T} \left(\bar{Y}(t, T, x_0, y_1, z_1) \right) \right\| dT \\ & \leq \int_{t_1^1}^{t_1^2} K e^{\sigma(T-t)} dT \\ & \leq \left| \frac{K}{\sigma} \right| e^{\sigma(t_1^1-t)} \rightarrow 0 \\ & \text{as } t_1' \rightarrow \infty\end{aligned}$$

then, there exists

$$\varphi(t, x_0, y_1, z_1) = \lim_{t_1 \rightarrow \infty} \bar{Y}(t, t_1, x_0, y_1, z_1)$$

and

$$\begin{aligned} R(t, t_1, x_0, y_1, z_1) &= e^{L^*(t, -t)} \left(\bar{Y}(t, t_1, x_0, y_1, z_1) - \varphi(t, x_0, y_1, z_1) - \right. \\ &\quad \left. - \varphi(t, x_0, y_1, z_1) \right) \end{aligned}$$

■

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