ON THE ARITHMETIC SUM OF MIDDLE-CANTOR SETS

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Abstract

In this article we study the arithmetic sum (difference) set $K_\alpha + K_\beta$ in $I = [0, 1]$ in terms of the parameters $(\alpha, \beta) \in I \times I$, where K_α and K_β are middle-Cantor sets contained in I . We describe two regions, A and B , in the parameter space (α, β) where the characterization of the arithmetic sum set $K_\alpha + K_\beta$ is given.

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1. Introduction

The problem of the study the arithmetic sum (difference) set of two regular Cantor sets arises in a natural way in the study of homoclinic bifurcation of one-parameter families of surface diffeomorphisms, see [7].

In [6] J. Palis propose to study the arithmetic sum (difference) set of two regular Cantor sets and he conjectures that these sets either contain intervals or have zero Lebesgue measure for most dynamically defined Cantor sets.

In [8] A. Sannami and, in [1], R. Bamón, S. Plaza and J. Vera have constructed examples of, respectively, C^∞ and C^r , $r \geq 1$, regular Cantor sets, Λ , such that $\Lambda + \Lambda$ is a Cantor set of positive Lebesgue measure. As these examples are of a very special kind, the original problem proposed by J. Palis still remains open.

In this article (section 3) we study this problem for a special kind of regular Cantor sets: the class of middle Cantor sets and describe two "white" regions, A and B , where we give an answer to the above problem. In the other region, which we will call "dark region", we get a partial answer. Besides, we want to mention that the results given in this paper were also obtained with different techniques in [4] by P. Mendes and F. Oliveira.

2. Basic Concepts

Let $\alpha \in]0, 1[$. The α -middle Cantor set $K_\alpha \subset I = [0, 1]$ is constructed as follows:

$$K_\alpha = \bigcap_{n \geq 0} I^n,$$

where $I^0 = [0, 1]$ and I^{n+1} is the union of 2^{n+1} closed disjoint intervals obtained from I^n removing the central open interval of length $\alpha \cdot \ell$ in each connected component of I^n , I_j^n , where ℓ is the length of I_j^n . Note that I^n is the union of 2^n closed disjoint intervals each of length $\left(\frac{1-\alpha}{2}\right)^n$.

It is clear that $K_\alpha = \bigcap_{n \geq 0} \varphi_\alpha^{-n}(I_0 \cup I_1)$, where $I_0 = [0, \frac{1-\alpha}{2}]$, $I_1 = [\frac{1+\alpha}{2}, 1]$ and $\varphi_\alpha : I_0 \cup I_1 \rightarrow I$ is defined by

$$\varphi_\alpha(x) = \begin{cases} \frac{2}{1-\alpha}x & x \in I_0 \\ \frac{2}{1-\alpha}x - \frac{1+\alpha}{1-\alpha} & x \in I_1. \end{cases}$$

Since φ_α/I_i , $i = 0, 1$, is an affine map, K_α is a C^∞ dynamically defined Cantor set. We call it affine Cantor set.

Furthermore, it is well known that the Hausdorff dimension, $HD(\cdot)$, and the limit capacity, $d(\cdot)$, of these Cantor sets coincide and they are given by

$$HD(K_\alpha) = d(K_\alpha) = \frac{\log(2)}{\log(2) - \log(1 - \alpha)} ,$$

it is also well known that if $HD(K_\alpha) + HD(K_\beta) < 1$, then the arithmetic sum set $K_\alpha + K_\beta = \{x + y : x \in K_\alpha, y \in K_\beta\}$ has zero Lebesgue measure. On the other hand, the thickness τ_α of K_α is given by

$$\tau_\alpha = \tau(K_\alpha) = \frac{1 - \alpha}{2\alpha} ,$$

for details see [7]. Now we have the following

Lemma 1. (*Gap Lemma, [5]*) *Let K_1 and K_2 be Cantor sets imbedded in the real line, with thicknesses τ_1 and τ_2 , respectively. Assume $\tau_1 \cdot \tau_2 > 1$. Then one, and only one, of the following alternatives occurs*

- (1) K_1 is contained in a gap of K_2 , or
- (2) K_2 is contained in a gap of K_1 , or
- (3) $K_1 \cap K_2 \neq \emptyset$; in this case $K_1 + K_2$ has nonempty interior.

Note in our case that when $\tau_\alpha \cdot \tau_\beta > 1$ the last possibility of the gap Lemma always occurs.

Let us now consider the equations

$$(2.1) \quad HD(K_\alpha) + HD(K_\beta) = 1$$

$$(2.2) \quad \tau(K_\alpha) \cdot \tau(K_\beta) = 1$$

We first study equation (1), that is

$$\frac{\log 2}{\log 2 - \log(1 - \alpha)} + \frac{\log 2}{\log 2 - \log(1 - \beta)} = 1 .$$

From this we obtain $\beta = \beta(\alpha)$ as a function of α

$$\beta = \beta(\alpha) = 1 - 2 \exp \left(\frac{\log 2 \log(\frac{2}{1-\alpha})}{\log(1 - \alpha)} \right) .$$

Note that $\lim_{\alpha \rightarrow 0+} \beta(\alpha) = 1$, that $\lim_{\alpha \rightarrow 1-} \beta(\alpha) = 0$, and that

$$\frac{d\beta(\alpha)}{d\alpha} = -2 \exp\left(\frac{\log 2 \log(\frac{2}{1-\alpha})}{\log(1-\alpha)}\right) \frac{(\log 2)^2}{(1-\alpha)(\log(1-\alpha))^2} < 0.$$

Hence $\beta = \beta(\alpha)$ is strictly decreasing. This curve crosses the diagonal of the square $Q = I \times I$ at the point $\alpha = 1/2$ and, at this point, $\frac{d\beta}{d\alpha}(\frac{1}{2}) = -1$. Also note that

$$\lim_{\alpha \rightarrow 0+} \frac{d\beta(\alpha)}{d\alpha} = 0 \quad \text{and} \quad \lim_{\alpha \rightarrow 1-} \frac{d\beta(\alpha)}{d\alpha} = -\infty.$$

Figure 1

From the above, it follows that $\beta > \beta(\alpha)$ if and only if $HD(K_\alpha) + HD(K_\beta) < 1$. Therefore if $\beta > \beta(\alpha)$, $m(K_\alpha + K_\beta) = 0$, where m denotes the Lebesgue measure in $\mathbb{R}$, and this occurs in the B region of figure 1.

On the other hand, by considering equation (2) we obtain a curve, $\tilde{\beta} = \tilde{\beta}(\alpha) = \frac{1-\alpha}{1+3\alpha}$, that crosses the diagonal of Q at the point $\alpha = 1/3$, which is strictly decreasing, and $\frac{d\tilde{\beta}(0)}{d\alpha} = -4$, $\frac{d\tilde{\beta}(1)}{d\alpha} = -1/4$.

Note that, $\tau(K_\alpha) \cdot \tau(K_\beta) > 1$ if and only if $\beta < \tilde{\beta}(\alpha)$. Thus if $\beta < \tilde{\beta}(\alpha)$, then $K_\alpha + K_\beta$ has nonempty interior, and this happens in the A region of figure 1.

In figure 1 the dotted region will be called "dark region". In this region, $HD(K_\alpha) + HD(K_\beta) > 1$ and $\tau(K_\alpha) \cdot \tau(K_\beta) < 1$ simultaneously hold.

On the other hand, by considering the parameters α and β on the diagonal Δ of Q and study the sum set $K_\alpha + K_\alpha$, a straightforward calculation shows that $K_\alpha + K_\alpha$ is a Cantor set of zero Lebesgue measure if $\alpha > \frac{1}{3}$, and that $K_\alpha + K_\alpha = [0, 2]$ if $\alpha \leq \frac{1}{3}$.

3. Existence of Gaps in $K_\alpha + K_\beta$

In the A region of figure 1 we have $\tau_\alpha \cdot \tau_\beta > 1$. Therefore, the gap lemma states that the sum set $K_\alpha + K_\beta$ contains intervals. We will now prove that if $(\alpha, \beta) \in A$, then, in fact $K_\alpha + K_\beta$ does not contains gaps and therefore is equal to the interval $[0, 2]$.

Set $\lambda_1 = \frac{1-\alpha}{2}$, $\lambda_2 = \frac{1-\beta}{2}$, $0 < \alpha, \beta < 1$.

Since the A region is symmetric with respect to the diagonal Δ we may, without loss of generality, assume that $\lambda_1 \leq \lambda_2$, i.e., $\beta \leq \alpha$. In fact we will suppose $\beta < \alpha$, see Figure 2, since the sum set $K_\alpha + K_\beta$ is already known when $\alpha = \beta$.

Figure 2

Lemma 2. Let m and n be positive integers, $m < n$, such that

- (1) $\lambda_1^{n-m} < \lambda_2^n < \lambda_1^{n-(m+1)}$, and
- (2) (i) $\lambda_1^{n-m} + \lambda_2^n < (1 - \lambda_1)\lambda_1^{n-(m+1)}$,
(ii) $\lambda_1^{n-m} + \lambda_2^n < (1 - \lambda_2)\lambda_2^{n-1}$.

Set $e_{n,m} = \lambda_1^{n-m} + \lambda_2^n$ and $f_{n,m} = \min\{(1-\lambda_1)\lambda_1^{n-(m+1)}, (1-\lambda_2)\lambda_2^{n-1}\}$. Then the interval $I_{n,m} =]e_{n,m}, f_{n,m}[$ is a nontrivial gap of $K_\alpha + K_\beta$.

Figure 3

Proof. Let $W \subset Q = I \times I$ the region limited by the straight lines $y = -x + e_{n,m}$ and $y = -x + f_{n,m}$. Note that a sufficient condition for the interval $]e_{n,m}, f_{n,m}[$ be a gap of $K_\alpha + K_\beta$ is that $(K_\alpha + K_\beta) \cap W \neq \emptyset$, (see figure 3). This is precisely what the hypothesis says, so the proof is finished.

Lemma 3. *There are infinitely many $n, m \in \mathbb{N}$, $m < n$, that satisfy (1) of Lemma 2.*

Proof. It is clear that (1) of Lemma 2 is equivalent to $(n-m)\log \lambda_1 < n\log \lambda_2$ and $n\log \lambda_2 < (n-(m+1))\log \lambda_1$, or equivalent to $n \left(\frac{\log \lambda_1 - \log \lambda_2}{\log \lambda_1} \right) > m$ and $n \left(\frac{\log \lambda_1 - \log \lambda_2}{\log \lambda_1} \right) - 1 < m$.

Thus,

$$n \frac{\log \lambda_1 - \log \lambda_2}{\log \lambda_1} - 1 < m < n \frac{\log \lambda_1 - \log \lambda_2}{\log \lambda_1}.$$

Furthermore, since $0 < \lambda_1 < \lambda_2 < 1$, $0 < \frac{\log \lambda_1 - \log \lambda_2}{\log \lambda_1}$ is not an integer, there are infinitely many pairs $(m, n) \in \mathbb{N} \times \mathbb{N}$ that satisfy (1) of Lemma

2. Finally, note that

$$m = \left[n \frac{\log \lambda_1 - \log \lambda_2}{\log \lambda_1} \right],$$

where $[\cdot]$ denotes the integer part function.

Lemma 4. *Let $0 < \lambda_1 < \lambda_2$ be and $m, n \in \mathbb{N}$, $m < n$. Then the following assertions are equivalent*

- (1) (i) $\lambda_1^{n-m} + \lambda_2^n < (1 - \lambda_1)\lambda_1^{n-(m+1)}$
(ii) $\lambda_1^{n-m} + \lambda_2^n < (1 - \lambda_2)\lambda_2^{n-1}$
- (2) $\frac{1}{\log \lambda_1} \left(n(\log \lambda_1 - \log \lambda_2) + \log\left(\frac{1-2\lambda_1}{\lambda_1}\right) \right) < m$
 $< \frac{1}{\log \lambda_1} \left(n(\log \lambda_1 - \log \lambda_2) + \log\left(\frac{\lambda_2}{1-2\lambda_2}\right) \right).$

Proof. It is clear that $\lambda_1^{n-m} + \lambda_2^n < (1 - \lambda_1)\lambda_1^{n-(m+1)}$ is equivalent to $\lambda_2^n < (1 - 2\lambda_1)\lambda_1^{n-(m+1)}$. Therefore, $n \log \lambda_2 < \log(1 - 2\lambda_1) + (n - m - 1) \log \lambda_1$, i.e., $\frac{1}{\log \lambda_1} \left(n(\log \lambda_1 - \log \lambda_2) + \log\left(\frac{1-2\lambda_1}{\lambda_1}\right) \right) < m$.

On the other hand, $\lambda_1^{n-m} + \lambda_2^n < (1 - 2\lambda_2)\lambda_2^{n-1}$ is equivalent to

$$m < \frac{1}{\log \lambda_1} \left(\log \left(\frac{\lambda_2}{1 - 2\lambda_2} \right) + n(\log \lambda_1 - \log \lambda_2) \right),$$

which completes the proof.

Now set

$$(3.1) \quad a = \frac{1}{\log \lambda_1} \log \left(\frac{1-2\lambda_1}{\lambda_1} \right), \quad b = \frac{1}{\log \lambda_1} \log \left(\frac{\lambda_2}{1-2\lambda_2} \right),$$

and $\varphi = \frac{\log \lambda_1 - \log \lambda_2}{\log \lambda_1}.$

Thus part (2) of Lemma 4 takes the simple form

$$(2') \quad n\varphi + a < m < n\varphi + b.$$

Henceforth, $K_{\lambda_1} + K_{\lambda_2}$ contains a gap if and only if (2') holds.

Lemma 5. *Let $]c, d[$ be an interval and let $\theta \in]0, 1[$ be an irrational number. Then there exists $n \in \mathbb{N}$ such that $[n\theta] \in]c + n\theta, b + n\theta[$ if and only if $c < 0$ and $d > -1$.*

Proof. Let $n \in \mathbb{N}$ such that $c+n\theta < [n\theta] < d+n\theta$. Then $c+n\theta < [n\theta] \leq n\theta$, so $c < 0$. Now $[n\theta] < d+n\theta$ implies $-1 < [n\theta] - n < d$, and so $d > -1$.

For the converse we have two cases to consider:

I.- $d > 0$,

II.- $-1 < d \leq 0$.

We first let $T_\theta : \mathbb{R} \rightarrow \mathbb{R}$ be the translation $T_\theta(x) = x + \theta$, hence $T_\theta^n(0) = n\theta$.

Case I.- If $d > 0$, then $0 \in]c, d[$ and hence $\theta \in]c + \theta, d + \theta[$, so $n\theta \in]c + n\theta, d + n\theta[$, all $n \in \mathbb{N}$. Thus, since the rotation in the circle S^1 induced by T_θ has all of its orbits dense in S^1 , we conclude that there exist $n, k \in \mathbb{N}$ such that $c + n\theta < k < n\theta < d + n\theta$ and thus $c + n\theta < k \leq [n\theta] \leq n\theta < d + n\theta$.

Note that there are infinitely many pairs $(n, k) \in \mathbb{N} \times \mathbb{N}$ with the above property.

Case II.- If $-1 < d \leq 0$, we set $\varepsilon = d - c$. Let $s \in \mathbb{N}$ such that $\frac{\varepsilon}{s} < d + 1$. For such s there are infinitely many pairs $(n, k) \in \mathbb{N} \times \mathbb{N}$ such that, $0 < T_\theta^n(d) - k < \varepsilon/s$, i.e., $0 < d + n\theta - k < \varepsilon/s$. From this we have $k < d + n\theta$. Since $\varepsilon/s = (d - c)/s < d - c$, we conclude that $c < d - \varepsilon/s$ therefore $c + n\theta < d + n\theta - \varepsilon/s < k$, and so $k \in]c + n\theta, d + n\theta[$. Finally, since $0 \leq -d < 1$ and $0 < d + n\theta - k$ we have $k < k - d < n\theta < k - d + \varepsilon/s < k + 1$, hence taking $k = [n\theta]$ the assertion follows, and the proof has been completed.

Next, by replacing $\lambda_1 = \frac{1-\alpha}{2}$ and $\lambda_2 = \frac{1-\beta}{2}$ in (3) we obtain

$$a = \frac{1}{\log(\frac{1-\alpha}{2})} \log(\frac{1}{\tau_\alpha}), \quad b = \frac{1}{\log(\frac{1-\beta}{2})} \log(\tau_\beta),$$

$$\text{and } \varphi = \frac{\log(1-\alpha) - \log(1-\beta)}{\log(1-\alpha) - \log 2},$$

and thus we have the following

Theorem 1. Suppose $\varphi = \frac{\log(1-\alpha) - \log(1-\beta)}{\log(1-\alpha) - \log 2}$ is an irrational number. Then there exists a gap in $K_\alpha + K_\beta$ if and only if $\tau_\alpha \cdot \tau_\beta < 1$.

Proof. It is a straightforward application of Lemma 5.

Now, by taking (α, β) in the A region and c, d and θ of Lemma 5 as the, respectively, a, b and φ of above, we have that

(i) $a < 0$ if and only if $\log(1/\tau_\alpha) > 0$,

(ii) $-1 < b$ if and only if $\log(2/(1-\alpha)) > \log \tau_\beta$ if and only if $\beta > \frac{1-\alpha}{5-\alpha}$, and it is easy to see that the curve $\beta = \bar{\beta}(\alpha) = \frac{1-\alpha}{5-\alpha}$ is contained in the A region.

(iii) We must always have that $a < b$, i.e.,

$$\frac{1}{\log((1-\alpha)/2)} \log(1/\tau_\alpha) < \frac{1}{\log((1-\alpha)/2)} \log(\tau_\beta).$$

Thus $\log(1/\tau_\alpha) > \log(\tau_\beta)$ and hence $\tau_\alpha \cdot \tau_\beta < 1$.

Note that K_α is not contained in a gap of K_β , and that K_β is not contained in a gap of K_α , then the gap Lemma implies that $K_\alpha + K_\beta$ contains intervals. On the other hand, by Theorem 1 we have that $K_\alpha + K_\beta$ is in fact the interval $[0, 2]$ when $\varphi = \frac{\log(1-\alpha)-\log(1-\beta)}{\log(1-\alpha)-\log 2}$ is an irrational number. In particular we have that, in this case, there exist gaps in $K_\alpha + K_\beta$ for the parameter values (α, β) in the dark region. However, this does not imply that $K_\alpha + K_\beta$ is a Cantor set for these parameter values, since this set may be an M -Cantorval (see [4]).

Lemma 6. Suppose $\varphi = \frac{\log \lambda_1 - \log \lambda_2}{\log \lambda_1} = \frac{p}{q} < 1$ is a rational number. Then there exists $n \in \mathbb{N}$ such that $m = [n\varphi] \in]a + n\varphi, b + n\varphi[$ if and only if there exists $r \in \mathbb{N}$, $0 \leq r < q$, such that $[r\varphi] \in]a + r\varphi, b + r\varphi[$.

Proof. Set $\bar{k} = [n\varphi]$ and $\varepsilon = n\varphi - \bar{k}$ ($0 \leq \varepsilon < 1$). Then $\bar{k} = mp + k$, $0 \leq k < p$ ($< q$) and $\varepsilon + k = \varepsilon + \bar{k} - mp = n\varphi - mp = n\varphi - mq p/q = (n - mq)\varphi < k + 1$, so $k = [(n - mq)\varphi] < p < q$. On the other hand, $\bar{k} = [n\varphi] \in]a + n\varphi, b + n\varphi[$ if and only if $a + n\varphi < mp + k < b + n\varphi$, i.e., $a + (n - mq)\varphi < k < b + (n - q)\varphi$. Thus $[(n - mq)\varphi] \in]a + (n - mq)\varphi, b + (n - mq)\varphi[$.

Set $r = n - mq$. Then $k < \varepsilon + k = r\varphi < k + 1 < p$, so $r < \frac{p}{\varphi} = q$. Therefore, there exists $r \in \mathbb{N}$, $0 \leq r < q$, such that $[r\varphi] \in]a + r\varphi, b + r\varphi[$.

The converse is trivial.

Lemma 7. If $\varphi = \frac{\log \lambda_1 - \log \lambda_2}{\log \lambda_1} = \frac{p}{q}$ ($p < q$) is a rational number, then the following assertions are equivalent

- (1) there exists $r \in \mathbb{N}$, $0 \leq r < q$, such that $[r\varphi] \in]a + r\varphi, b + r\varphi[$
- (2) (i) $a < 0$ and $b > 0$ or
 (ii) $a < 0$, $-1 < b \leq 0$, and there exists at least one integer k , $0 \leq k < q$, such that $-\frac{k}{q} \in]a, b[$.

Proof. (1) $\implies$ (2) Suppose there exists an integer r , $0 \leq r < q$, such that $[r\varphi] \in]a + r\varphi, b + r\varphi[$. Then $a + r\varphi < [r\varphi] \leq r\varphi$, so $a < 0$. On the other hand, $[r\varphi] < b + r\varphi$ implies $-1 < [r\varphi] - r\varphi < b$ and we have two possibilities: (I) $b > 0$ or (II) $-1 < b \leq 0$.

Case I.- If $b > 0$, there is nothing to prove.

Case II.- If $-1 < b \leq 0$, set $\bar{k} = [r\varphi]$ and let $0 \leq \varepsilon < 1$ such that $r\varphi = \bar{k} + \varepsilon$. We claim that there exists $n \in \mathbb{N}$ and $0 \leq u < p$ such that $np = \bar{k}q + u$. In fact, by considering the following p consecutive integers $\bar{k}q, \bar{k}q + 1, \dots, \bar{k}q + (p - 1)$, we conclude that there exists u such that $\bar{k}q + u$ is a multiple of p , and therefore there exist $n \in \mathbb{N}$ and $0 \leq u < p$ such that $np = \bar{k}q + u$. From this we have that $0 \leq u/q = n\varphi - \bar{k} < 1$, since $u < p < q$ and $\bar{k} = [n\varphi] = [r\varphi]$. Therefore, $\varepsilon = r\varphi + n\varphi + u/q$, since $u/q = n\varphi - \bar{k}$. Thus $0 \leq \varepsilon = \frac{(r+n)p+u}{q} < 1$. Let $k = (r - n)p + u$. It is clear that $0 \leq k < q$. Furthermore, from (1) we have $a < \bar{k} - r\varphi < b$, but $\bar{k} - r\varphi = -\varepsilon$ and hence there exists $k \in \mathbb{N}$, $0 \leq k < q$, such that $a < -\frac{k}{q} < b$.

(2) $\implies$ (1) First we suppose (i). Taking $r = 0$ we have $0 = [0\varphi] \in]a + 0\varphi, b + 0\varphi[$, and assertion (1) holds.

We now suppose that $a < 0$ and $-1 < b \leq 0$ and that the interval $]a, b[$ contains at least one point of the form $-k/q$, $0 \leq k < q$. We claim that there exist $r, \bar{k} \in \mathbb{N}$ such that $-k + rp = \bar{k}q$. In fact, let us consider the q integers

$$\begin{aligned} -k &= k_0q + r_0, \quad 0 \leq r_0 < 1 \\ -k + p &= k_1q + r_1, \quad 0 \leq r_1 < 1 \\ &\vdots \\ -k + (q-1)p &= k_{q-1}q + r_{q-1}, \quad 0 \leq r_{q-1} < 1. \end{aligned}$$

If the remainders r_i are all different, then there is an index i such that $r_i = 0$ and $-k + ip = kq$. If there are indexes $0 \leq j < i < q$ such that $r_j = r_i$, then $(i-j)p = (-k + ip) - (-k + jp) = (k_iq + r_i) - (k_jq + r_j) = (k_i - k_j)q$. Since p/q is irreducible (i.e., p and q have no common prime factors), we have that q divides $i - j$ which is a contradiction, since $0 < i - j < q$. So there exist $\bar{k}, r \in \mathbb{N}$, $0 \leq r < q$ such that $-k + rp = \bar{k}q$. From this, we have $\bar{k} = -\frac{k}{q} + r\varphi$ and $\bar{k} < \bar{k} + \frac{k}{q} = r\varphi < \bar{k} + 1$, since $0 \leq \frac{k}{q} < 1$, and so $\bar{k} = [r\varphi]$. On the other hand, $a < -\frac{k}{q} < b$ implies $a + r\varphi < -\frac{k}{q} + r\varphi < b + r\varphi$, i.e., $[r\varphi] \in]a + r\varphi, b + r\varphi[$. Therefore, the proof has been completed.

Now we will study the existence of gaps in the sum set $K_\alpha + K_\beta$ when

$$\varphi = \frac{\log \lambda_1 - \log \lambda_2}{\log \lambda_1} = \frac{\log(1-\alpha) - \log(1-\beta)}{\log(1-\alpha) - \log 2} = \frac{p}{q} \in \mathbb{Q} \quad p < q.$$

Case (I). We have that $a < 0$ if and only if $\tau_\alpha = \frac{1-\alpha}{2\alpha} < 1$ if and only if $\alpha > \frac{1}{3}$. On the other hand $b > 0$ if and only if $\tau_\beta < 1$ if and only if $\beta > \frac{1}{3}$. This is shown in the dotted region of figure 4.

Figure 4.

Case (II). That is, $a < 0$, $-1 < b \leq 0$ and there exists at least one integer k , $0 \leq k < q$, such that $-\frac{k}{q} \in]a, b[$. We know that $a < 0$ if and only if $\alpha > \frac{1}{3}$, and $b < 0$ if and only if $\beta > \frac{1}{3}$. Furthermore, $-1 < b$ if and only if $\log\left(\frac{1}{1-\alpha}\right) > \log \beta$, if and only if $\bar{\beta} = \frac{1-\alpha}{5-\alpha} < \beta$. The latter has no contribution to our conclusion, because the relation $-1 < b \leq 0$ is always true in the region we are considering.

Therefore, we will have to analyze the existence of an integer k , $0 \leq k < q$ such that $-\frac{k}{q} \in]a, b[$, i.e.,

$$\frac{\log\left(\frac{1}{\tau_\alpha}\right)}{\log\left(\frac{1-\alpha}{2}\right)} < -\frac{k}{q} < \frac{\log \tau_\beta}{\log\left(\frac{1-\alpha}{2}\right)}.$$

This last inequality is equivalent to

$$\frac{q}{\beta} < \log\left(\frac{2}{1-\alpha}\right)^k < \log\left(\frac{1}{\tau_\alpha}\right)^q$$

i.e., $\tau_\beta^q < \left(\frac{2}{1-\alpha}\right)^k < \left(\frac{1}{\tau_\alpha}\right)^q$.

Now, $\frac{1-\beta}{2\beta} < \frac{2}{1-\alpha}$ if and only if $\beta > \frac{1-\alpha}{5-\alpha} = \bar{\beta}(\alpha)$. But this is always true in the dark region. Thus, $\frac{1-\beta}{2\beta} < \frac{2}{1-\alpha}$ and hence $\tau_\beta^k < \left(\frac{2}{1-\alpha}\right)^k$, all integer $k \geq 1$. setting $q = k + \ell$ we have

$$\tau_\beta^q < \left(\frac{2}{1-\alpha}\right)^k \tau_\beta^\ell < \left(\frac{2}{1-\alpha}\right)^k$$

since $\tau_\beta < 1$ in the dark region.

On the other hand, $\left(\frac{2}{1-\alpha}\right)^k < \left(\frac{2\alpha}{1-\alpha}\right)^q$ if and only if $(1-\alpha)^{q-k} < 2^{q-k} \alpha^q$. Define $g_k(\alpha) = (1-\alpha)^{q-k}$ and $h_k(\alpha) = 2^{q-k} \alpha^q$. It is clear that g_k is strictly decreasing and that h_k is strictly increasing. Thus, there exists k , $0 \leq k < q$ as above if and only if $g_k(\alpha) < h_k(\alpha)$, but this is true only for $k = 0$.

Final conclusion: In the A region there no gaps, hence for arameters $(\alpha, \beta) \in A$, $K_\alpha + K_\beta = [0, 2]$. Furthermore, in the dark region there are always gaps.

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