COMMUTATIVITY RESULTS FOR RINGS WITH CERTAIN CONSTRAINTS ON COMMUTATORS

HAMZA A. S. ABUJABAL
King Abdul Aziz University, Saudi Arabia

VESELIN PERIC
University of Sarajevo, Bosnia-Herzegovina

Abstract

We investigate here the commutativity of a left (resp. right) s -unital ring R satisfying the polynomial identity $y^r [x^n y] x^t = \pm y^s [x, y^m]$ (resp. $y^r [x^n, y] x^t = \pm [x, y^m] y^s$) for some non-negative integers $m > 0, n > 0, r, s$ and t such that $n + t > 1$ (resp. $m + s > 1$ for $r = 0$). For such a ring R , we prove the commutativity if $n + t > 1$, and the commutators in R are n -torsion free ($Q(n)$ property) for $m > 1, n > 1$, and $(t + 1)$ -torsion free for $n = 1$ (and $t > 0$). If $r = 0$, then R is commutative provided $m + s > 1$ and R has $Q(m)$ property for $m > 1, n > 1$, and $Q(s + 1)$ property for $m = 1$ (and $s > 0$). Especially, for $r = 0, R$ is commutative, if m and n are relatively prime integers (not both equal to one).

(1991) AMS Subject Classification : 16U80.

Key words and phrases : Left or right s -unital rings, commutativity of rings and groups, torsion freeness of commutators in a ring, polynomial identity.

1. Introduction.

Throughout this paper, R will be an associative ring. Let $Z(R)$ denote the center of R , $C(R)$ the commutator ideal of R , $N(R)$ the set of all nilpotent elements of R and $N'(R)$ the set of all zero-divisors of R . For any x and y in R , we denote by $[x, y]$ the commutators $xy - yx$ of x and y .

A ring R is called left (resp. right) s -unital ring if $x \in Rx$ (resp. $x \in xR$) for every x in R . Further, R is called s -unital if R is both left and right s -unital. If R is s -unital (resp. left or right s -unital), then for any finite subset F of R , there exists an element e in R such that $ex = xe = x$ (resp. $ex = x$ or $xe = x$) for all x in F . Obviously, a ring R is s -unital (resp. left or right s -unital) if and only if the opposite ring R' of R is s -unital (resp. right or left s -unital).

For a positive integer n , we will consider, in a ring R , the following property :

$Q(n)$: For every x and y in R , $n[x, y] = 0$ implies $[x, y] = 0$.

The property $Q(n)$ is an H -property in the sense of [9]. Obviously, every n -torsion free ring R has the property $Q(n)$ and every ring R has the property $Q(1)$. If a ring R has the property $Q(n)$, then R has the property $Q(m)$ for any factor m of n . For any positive integer n , a ring R has the property $Q(n)$ if and only if the opposite ring R' of R has that property.

The objective of this paper is to study commutativity of a left (resp. right) s -unital ring R satisfying the polynomial identity.

$$(1.1) \quad y^r [x^n, y] x^t = \pm y^s [x, y^m],$$

(resp.

$$(1.2) \quad y^r [x^n, y] x^t = \pm [x, y^m] y^s$$

for some given non-negative integers m, n, r and t , which will be more specified later.

If $m = n = 0$, then any ring R satisfies both identities (1.1) and (1.2). Hence, it is naturally to assume that $m > 0$ or $n > 0$. Since the cases $m = 0, n > 0$, and $m > 0, n = 0$ were considered earlier (see [2] and [3]), so we will assume here that $m > 0$ and $n > 0$. Moreover, in any ring R with the identity

$$(1.3) \quad [x, [x, y]] = 0$$

each of the identities (1.1) and (1.2) reduces to the identity

$$(1.4) \quad ny^r [x, y] x^{n+t-1} = \pm my^{m+s-1} [x, y]$$

(see Lemma 1). Therefore, such a ring satisfies both identities (1.1) and (1.2) without the signs $\pm$ if $m = n, m + s = r + 1$ and $n + t = 1$. Thus it is naturally to assume that, for $m = n, m + s \neq r + 1$ or $n + t \neq 1$, i. e. $n + t > 1$. In the main result we will assume $n + t > 1$, and in the case of a right s -unital ring R with polynomial identity (1.2), also $m + s > 1$ for $r = 0$ (see Theorem 1). In the complementary results concerning the case $r = 0$, one or both of these assumptions will be made also (see Theorem 2 and Theorem 3). The assumption $n + 1 > 1$ (resp. $m + s > 1$) for $r = 0$ is necessary to prove that a left (resp. right) s -unital ring R with polynomial identity (1.1) (resp. (1.2)) is in fact s -unital (see Lemma 6), and thus the first of the two additional assumption is essential only in Theorem 1, and the second in Theorem 2 if we deal with an s -unital ring R .

There are left (resp. right) s -unital rings satisfying the polynomial identity (1.1) (resp. (1.2)) for some non-negative integers $m > 0, n > 0, r, s$ and t such that $n + t > 1, m + s > 1$ for $r = 0$, which are not commutative. To establish the commutativity of such a ring some additional assumptions are necessary. The assumptions we will made here concern some torsion freeness of commutators in R and are of the for $Q(n), Q(t+1), Q(m)$ and $Q(s+1)$.

For a (multiplicative) group G , the identities (1.1) and (1.2) are mean to be without the signs $\pm$. An example is given to show that a group G satisfying polynomial identity (1.1) or (1.2) is not necessarily commutative. However, it is known that a group G with polynomial identity (1.1) is commutative when $r = s = t = 0$ and $m = n + 1$.

2. Preliminaries.

We begin with the following well-known results.

Lemma 1 ([11, Lemma 3]). Let R be a ring such that $[x, [x, y]] = 0$ for some x and y in R . Then $[x^k, y] = kx^{k-1}[x, y]$ for any positive integers k .

Lemma 2 ([4, Lemma 2]). Let R be a ring with unity 1, and let x and y be elements of R . If $kx^m[x, y] = 0$ and $k(x+1)^m[x, y] = 0$ for some integers $m \geq 1$ and $k \geq 1$, then necessarily $k[x, y] = 0$.

Lemma 3 ([10, Theorem]). Let f be a polynomial in n non-commuting indeterminates $x_1, x_2, \dots, x_n$ with relatively prime integral coefficients. Then the following are equivalent :

1. Every ring satisfying the polynomial identity $f = 0$ has nil commutator ideal.
2. Every semi-prime ring satisfying $f = 0$ is commutative.
3. For every prime p , the ring $\mathbf{Z}_p^{2 \times 2}$ of all 2×2 matrices over $\mathbf{Z}_p$ fails to satisfy $f = 0$.

Lemma 4 ([14, Lemma 3]). Let R be a ring with unity 1, and let k and m be positive integers. If $(1 - y^k)x = 0$, then $(1 - y^{km})x = 0$ for all x and y in R .

The following theorem is due to Herstein.

Theorem H ([8, Theorem 18]). Let R be a ring, and let $n > 1$ be a fixed integer. If $x^n - x \in Z(R)$ for each x in R , then R is commutative.

Lemma 5 ([15, Lemma]). Let R be a left (resp. right) s -unital ring. If for each pair x and y in R there exist a positive integer $k = k(x, y)$ and an element $e = e(x, y)$ of R such that $x^k e = x^k$ and $y^k e = y^k$ (resp. $e x^k = x^k$ and $e y^k = y^k$), then R is s -unital.

3. Main results.

The main result of this paper is the following :

Theorem 1. Let R be a left (resp. right) s -unital ring satisfying the polynomial identity (1.1) (resp. (1.2)) for some given non-negative integers $m > 0, n > 0, s$ and t such that $n + 1 > 1$, (resp. $m + s > 1$ for $r = 0$). If $n + t > 1$, and R has $Q(n)$ property for $m > 1, n > 1$, and $Q(t + 1)$ property for $n = 1$ ($t > 0$), then R is commutative.

First we prove the following result.

Lemma 6. Let R be a left (resp. right) s -unital ring satisfying the polynomial identity (1.1) (resp. (1.2) for some given non-negative integers $m > 0, n > 0, r, s$ and t such that $n + t > 1$ (resp. $m + s > 1$ for $r = 0$). Then R is s -unital.

Proof. For arbitrary elements x and y in a left (resp. right) s -unital ring R , let e be an element in R such that $ex = x$ and $ey = y$ (resp. $xe = x$ and $ye = y$). Let R be a left s -unital ring satisfying the polynomial identity (1.1) for some given non-negative integers $m > 0, n > 0, r, s$ and t such that $n + t > 1$. Set e for y in (1.1) to derive $(x^n e - x^n) x^t = x e^m - x$. If $t = 0$, then $n > 1$, and hence $x = x^n - x^n e + x e^m \in xR$. Thus, R is s -unital. For $t > 0$, $x = x e^m$, and R is s -unital again.

Suppose now, R is right s -unital and R satisfies the polynomial identity (1.2) for some given non-negative integers $m > 0, n > 0, r, s$ and t such that $m + s > 1$ for $r = 0$. Setting e for x in (1.2) one gets $y^r e^n y - y^{r+1} = e y^{m+s} - y^{m+s}$. For $r > 0$, this yields $e y^{m+s} = y^{m+s}$. Similarly, for $r > 0$, $e x^{m+s} = x^{m+s} = x^{m+s}$, and since $m + s > 0$, R is s -unital in view of Lemma 5.

If $r = 0$, then $m + s > 1$, hence $y = e^n y - e y^{m+s} + y^{m+s} \in Ry$, and R is s -unital.

Lemma 7. Let R be a ring with unity 1 satisfying the polynomial identity (1.1), (resp. (1.2)) for some given non-negative integers $m > 0, n > 0, r, s$ and t such that $n + t > 1$ and R has $Q(n)$ property for $m > 1, n > 1$. Then $N(R) \subseteq Z(R)$.

Proof. Let $a \in N(R)$. Then there exists the minimal positive integer p such that $a^k \in Z(R)$ for all integer $k \geq p$. If $p = 1$, then $a \in Z(R)$. For $p > 1$, set $b = a^{p-1}$. Then

$$(1.5) \quad b^k [b, x] = [b, x] b^k = [b^k, x] = 0 \quad \text{for all } x \text{ in } R \text{ and } k > 1$$

Now, set b for x in (1.1) (resp. (1.2)). Since $n + t > 1$, then $n > 1$, or $t > 1$ or $n = t = 1$. In the first two cases, by (1.5), one gets

$$y^s [b, y^m] = 0 \quad (\text{resp. } [b, y^m] y^s = 0)$$

If $m = 1$, by Lemma 2, This yields $[b, y] = 0$ for all y in R , hence we have a contradiction $a^{p-1} \in Z(R)$.

If $m > 1$, we set $x + 1$ for x in (1.1) (resp. (1.2)). In view of (1.5) and the above two identities obtained from (1.1) (resp. (1.2)) for $x = b$, we get

$$y^r (n[b, y]) = 0 \quad \text{for all } y \text{ in } R.$$

Now, Lemma 2 and $Q(n)$ property (if $n > 1$) yield $[b, y] = 0$ for all y in R , that is $a^{p-1} \in Z(R)$, which is a contradiction.

In the remaining case $n = t = 1$, R is commutative, and thus, surely $N(R) \subseteq Z(R)$. Namely, setting in (1.1) (resp. in (1.2)), $1 + x$ for x , one gets $y^r [x, y](1 + x) = y^r [x, y]x$, hence $y^r [x, y] = 0$ for all x, y in R . By Lemma 2, the last identity yields $[x, y] = 0$ for all x, y in R .

Lemma 8. Let R be a ring satisfying the polynomial identity (1.1) (resp. (1.2)) for some given non-negative integers $m > 0, n > 0, r, s$ and t such that $n + t > 1$ (resp. $n + t > 1$ or $m + s > 1$). Then the commutator ideal $C(R)$ of R is nil, that is $C(R) \subseteq N(R)$.

Proof. Let $n + t > 1$. Then the elements $x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, y = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ in $\mathbf{Z}_p^{2 \times 2}$ show that the ring $\mathbf{Z}_p^{2 \times 2}$ fails to satisfy (1.1) and for $s = 0$, also (1.2). If $s \geq 1$, then the elements $x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, y = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ in $\mathbf{Z}_p^{2 \times 2}$ don't satisfy (1.2). Let now, $m + s > 1$. Then the elements $x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ in $\mathbf{Z}_p^{2 \times 2}$ show that the ring $\mathbf{Z}_p^{2 \times 2}$ fails to satisfy (1.2). Therefore, $C(R) \subseteq N(R)$, by Lemma 3.

Lemma 9. Let R be a ring with unity 1 satisfying the identity (1.4) for some given non-negative integers $m > 0, n > 0, r, s$ and t . Then, for any x, y in R , $n[x, y] = 0$ if and only if $m[x, y] = 0$. Especially, R has $Q(n)$ property if and only if R has $Q(m)$ property. If m, n are relatively prime integers, then R has both $Q(m)$ and $Q(n)$ properties.

Proof. Suppose that, for some x, y in R , $n[x, y] = 0$. Then

$$nv^r [u, v] u^{n+t-1} = 0 \quad \text{for } u \in \{x, 1+x\} \quad \text{and } v \in \{y, 1+y\},$$

and thus, in view of (1.4).

$$mv^{m+s-1}[u, v] = 0 \text{ for } u \in \{x, 1+x\} \text{ and } v \in \{y, 1+y\},$$

By Lemma 2, the last equation implies $m[x, y] = 0$. Similarly, $n[x, y] = 0$, follows from $m[x, y] = 0$.

Let now, R be a ring with $Q(m)$ property. If for some x, y in R , $n[x, y] = 0$, then also $m[x, y] = 0$, and thus, $[x, y] = 0$. Therefore, R has also $Q(n)$ property. Similarly, one can prove that if R has $Q(n)$ property, then R has also $Q(m)$ property.

Finally, suppose that m, n are since m, n are relatively prime integers. If $m[x, y] = 0$ for some x, y in R , then also $n[x, y] = 0$, and since m, n are relatively prime, then $[x, y] = 0$. Thus, R has $Q(m)$, and therefore, also $Q(n)$ property.

Proof of Theorem 1. By Lemma 6, R is s -unital. Therefore, in view of Proposition 1 of [9], we may assume that R is a ring with unity 1. Hence, by Lemma 3 and Lemma 7, for the ring R we have

$$(1.6) \quad C(R) \subseteq N(R) \subseteq Z(R).$$

According to (1.6), R satisfies the identity (1.3), and thus, we can apply Lemma 1, which we will do without special mention. In view of (1.3), the identities (1.1) and (1.2) are equivalent each other, for the ring R in Theorem 1, and we can assume that R satisfies the identity (1.1).

In (1.1) replace x by $2x$ to get $2^{n+t}y^r[x^n, y]x^t = \pm 2y^s[x, y^m]$. Combining this identity with (1.1) we obtain $(2^{n+t} - 2)y^s[x, y^m] = 0$, hence, by Lemma 1,

$$(2^{n+t} - 2)my^{m+s-1}[x, y] = 0 \text{ for all } x, y \text{ in } R.$$

This and Lemma 2 imply that

$$k[x, y] = 0 \text{ for all } x, y \text{ in } R, \text{ and } k = (2^{n+t} - 2)m > 1.$$

Therefore, $[x^k, y] = kx^{k-1}[x, y] = 0$, that is

$$(1.7) \quad x^k \in Z(R) \text{ for all } x \text{ in } R, \text{ and } k = (2^{n+t} - 2)m.$$

Now, suppose that $n > 1$. Replace x by x^n in (1.1) to obtain

$$y^r[(x^n)^n, y]x^{nt} = \pm y^s[x^n, y^m].$$

Then

$$\begin{aligned} y^r [(x^n)^n, y] x^{nt} &= ny^r [x^n, y] x^{nt+n(n-1)} \\ &= ny^r [x^n, y] x^t x^{(n-1)(n+1)} \\ &= \pm ny^s [x, y^m] x^{(n-1)(n+t)}, \end{aligned}$$

and

$$y^s [x^n, y^m] = ny^s [x, y^m] x^{n-1}.$$

Thus,

$$ny^s [x, y^m] x^{(n-1)(n+t)} = ny^s [x, y^m] x^{n-1},$$

and hence

$$ny^s [x, y^m] x^{n-1} (1 - x^{(n-1)(n+t-1)}) = 0.$$

By Lemma 4, we get

$$(1.8) \quad ny^s [x, y^m] x^{n-1} (1 - x^{k(n-1)(n+t-1)}) = 0.$$

It is easy to prove that the polynomial identity (1.1) implies that

$$(1.9) \quad y^{r'} [x^{n^2}, y] x^{t'} = y^{s'} [x, y^{m^2}]$$

for all x, y in R , and $r' = mr + r$, $s' = ms + s$ and $t' = nt + t$.

The ring R is isomorphic to a subdirect sum of subdirectly irreducible rings R_i ($i \in I$). As homomorphic image of R , each of the ring R_i has a unity 1 and satisfies all the identities. As homomorphic image of R , each of the rings R_i has a unity 1 and satisfies all the identities satisfying by R . But R_i is not necessarily satisfying $Q(n)$, (resp. $Q(t+1)$) for $m > 1, n > 1$ (resp. for $n = 1$ ($t > 0$)).

Now, we consider the ring R_i for some fixed index i in I . Let S be the intersection of all non-zero ideals of R_i . Then $S \neq (0)$ and $S_d = (0)$ for all central zero divisor d of R_i . Let u be any zero divisor of R_i . Then

$$ny^s [u, y^m] u^{n-1} (1 - u^{k(n-1)(n+t-1)}) = 0.$$

If $ny^s [u, y^m] u^{n-1} \neq 0$, then $1 - u^{k(n-1)(n+t-1)}$ will be a central zero divisor of R_i . Hence, we would have

$$(0) = S (1 - u^{k(n-1)(n+t-1)}) = S \neq 0,$$

a contradiction. Therefore, $ny^s[u, y^m]u^{n-1} = 0$, and consequently, $mny^{m+s-1}[u, y]u^{n-1} = 0$ for all y in R_i and any zero divisor u of R_i . Thus, Lemma 2 implies that

$$(1.10) \quad mn[u, y]u^{n-1} = 0 \text{ for all } y \text{ in } R_i \text{ and any } u \text{ in } N'(R_i).$$

Combining (1.9) and (1.10), we see that

$$\begin{aligned} y^{r'}[u^{n^2}, y]u^{t'} &= ny^{mr+r}[u^n, y]u^{nt+t+n^2-n} \\ &= ny^{mr}(y^r[u^n, y]u^t)u^{n(n+t-1)} \\ &= \pm ny^{mr}(y^s[u, y^m])u^{n(n+t-1)} \\ &= \pm mny^{mr+s+m-1}[u, y]u^{n(n+t-1)} \\ &= y^{mr+s+m-1}(mn[u, y]u^{n-1})u^{n(n+t-1)-(n-1)} \\ &= 0 \end{aligned}$$

Therefore, $y^r[u^{n^2}, y]u^{t'} = 0$ for all y in R_i and u in $N'(R_i)$. By Lemma 2, we obtain

$$[u^{n^2}, y]u^{t'} = 0 \text{ for all } y \text{ in } R_i \text{ and } u \text{ in } N'(R_i),$$

and thus in account of (1.9), Lemma 1 and Lemma 2,

$$m^2[u, y] = 0 \text{ for all } y \text{ in } R_i \text{ and } u \text{ in } N'(R_i).$$

By the above identity, we get $[x^{m^2}, u] = m^2x^{m^2-1}(x, u) = 0$ for all x in R_i and u in $N'(R_i)$. Thus,

$$(1.11) \quad [x^{m^2}, u] = 0 \text{ for all } x \text{ in } R_i \text{ and } u \text{ in } N'(R_i).$$

Now, let $c \in Z(R_i)$. In (1.1) replace x by cx to get

$$c^{n+1}y^r[x^n, y]x^t = \pm cy^s[x, y^m] = cy^r[x^n, y]x^t,$$

and hence,

$$(c^{n+t} - c)y^r[x^n, y]x^t = 0.$$

Applying Lemma 2, we obtain

$$(c^{n+t} - c)[x^n, y]x^t = 0,$$

or, by Lemma 1,

$$n \left(e^{n+t} - e \right) [x, y] x^{n+t-1} = 0.$$

Now, Lemma 2 gives $n \left(e^{n+t} - e \right) [x, y] = 0$, and hence by Lemma 1,

$$\left(e^{n+t} - e \right) [x^n, y] = n \left(e^{n+t} - e \right) [x, y] x^{n-1} = 0$$

for all x, y in R_i and any e in $Z(R_i)$. Thus,

$$(1.12) \quad \left(e^{n+t} - e \right) [x^n, y] = 0 \text{ for all } x, y \text{ in } R_i \text{ and } e \text{ in } Z(R_i).$$

In particular, by (1.1) and (1.12), we obtain

$$(1.13) \quad \left(y^{k(n+1)} - y^k \right) [x^n, y] = 0 \text{ for all } x \text{ and } y \text{ in } R_i.$$

Now, let $y \in R_i$. If $[x^{m^{2n}}, y] = 0$, then clearly $[x^{m^{2n}}, y^q - y] = 0$ for all positive integers $q > 1$.

If $[x^{m^{2n}}, y] \neq 0$, then $[x^n, y] \neq 0$. Since $[x^n, y] \neq 0$, then by (1.13), $y^{k(n+1)} - y^k$ is a zero divisor. Therefore, $y^{k(n+1)+1} - y$ is also a zero divisor. Hence (1.11) implies

$$(1.14) \quad \begin{aligned} [x^{m^{2n}}, y^p - y] &= 0 \text{ for all } x, y \text{ in } R_i \text{ and} \\ p &= k(n+1)+1 > 1. \end{aligned}$$

Since R_i ($i \in I$) satisfies (1.14), the original ring R also satisfies (1.14). But, R has $Q(n)$ property, and thus, by Lemma 9, also $Q(m)$ property. Now, Combining (1.14) along with Lemma 1 and Lemma 2, we finally obtain $[x, y^p - y] = 0$ for all x, y in R and some positive integer $p > 1$. Therefore, R is commutative by Theorem H. For $n = 1$ ($t > 0$) the identity (1.1), in view of Lemma 1 implies the identity

$$y^{r+mt} [x^{t+1}, y] = \pm y^s [x, y^{m(t+1)}],$$

and thus, R is commutative by the $Q(t+1)$ property.

For the special case $r = 0$, we have the following two complementary results.

Theorem 2. Let R be a left (resp. right) s -unital ring satisfying the polynomial identity (1.1) (resp. (1.2)) for some given non-negative integers $m > 0, n > 0, r, s$ and t such that $n + t > 1$ (resp. $m + s > 1$). If $r = 0, m + s > 1$, and R has $Q(m)$ property for $m > 1, n > 1$, and $Q(s + 1)$ property for $m = 1 (s > 0)$, then R is commutative.

Proof. If $r = 0$ and R satisfies the identity (1.1) (resp. (1.2)), then the opposite ring R' of R (resp. the ring R itself) satisfies the polynomial identity

$$[y^m, x]y^s = \pm x^t[y, x^n], \text{ (resp. } [y^m, x]y^s = \pm [y, x^n]x^t,$$

and therefore, the ring R in Theorem 2 is commutative by Theorem 1.

Theorem 3. Let R be a left (resp. right) s -unital ring satisfying the polynomial identity (1.1) (resp. (1.2)) for some given non-negative integers $m > 0, n > 0, r, s$ and t such that $n + t > 1$. (resp. $m + s > 1$). If $r = 0$, and m, n are relatively prime integers, then R is commutative.

Proof. By Lemma 6, R is s -unital, and therefore, we may assume that R is a ring with unity 1. In view of Theorem 1 and Theorem 2, it suffices to show that R has the properties $Q(m)$ and $Q(n)$.

For b as in the proof of Lemma 7, we have, for $r = 0$,

$$(1.15) \quad n[b, y] = 0 \text{ for all } y \text{ in } R,$$

Since $r = 0$, we can similarly obtain also

$$(1.16) \quad m[x, b] = 0 \text{ for all } x \text{ in } R.$$

But, m, n are relatively prime integers, and therefore, (1.15) and (1.16) yield

$$[x, b] = 0 \text{ for all } x \text{ in } R, \text{ that is } b \in Z(R).$$

Thus, $N(R) \subseteq Z(R)$, and by Lemma 8, R satisfies (1.6), hence also (1.4).

Therefore, in view of Lemma 9, $m[x, y] = 0$ is equivalent to $n[x, y] = 0$ for all x, y in R , and since m, n are relatively prime, obviously R has both $Q(m)$ and $Q(n)$ properties.

The above three theorems improve some earlier results of the authors, and also some other results due to different authors. Some of them we will state here as corollaries.

Corollary 1 ([1, Theorem 1]). Let R be a left or right s -unital ring satisfying the polynomial identity

$$(1.17) \quad x^t [x^n, y] = [x, y^n] x^s$$

or

$$(1.18) \quad x^t [x^n, y] = x^s [x, y^n],$$

for some given non-negative integers $m > 1, n \geq 1, s$ and t . If R has the property $Q(n)$ for $n > 1$, then R is commutative.

Proof. If R is left or right s -unital and satisfies the identity (1.17) or (1.18), then the opposite ring R' of R is right or left s -unital and satisfies the identity

$$(1.19) \quad x^s [y^m, x] = [y, x^n] x^t,$$

or the ring R itself satisfies the identity

$$(1.20) \quad x^s [y^m, x] = x^t [y, x^n].$$

Now, it is easy to see that a left or right s -unital ring R satisfying the identity (1.19) or (1.20) must be s -unital. Therefore, Corollary 1 follows from Theorem 1.

Corollary 2 ([3, Theorem 1]). Let R be an s -unital ring with the polynomial identity

$$(1.21) \quad x^t [x^n, y] = [x, y^m] y^s$$

or

$$(1.22) \quad x^t [x^n, y] = y^s [x, y^m],$$

where m, n, s and t are some given non-negative integers. If $m > 1, n \geq 1$, (resp. $n > 1, m \geq 1$) and R has the property $Q(m)$ (resp. $Q(n)$), then R is commutative. If m, n are relatively primer positive integers, then R is also commutative.

Proof. If the ring R satisfies the identity [1.21] or (1.22), then the opposite ring R' of R satisfies the identity

$$(1.23) \quad [x^n, y] x^t = y^s [x, y^m]$$

or

$$(1.24) \quad [x^n, y] x^t = [x, y^m] y^s.$$

Hence, Corollary 2 follows from Theorem 1 and Theorem 2.

Let us remark that Corollary 2 improves Theorem 1 and Theorem 2 of [12].

Corollary 3. Let $m \geq 1$ and $n \geq 1$ be fixed integers. Suppose a left (resp. right) s -unital ring R satisfies the identity.

$$(1.25) \quad [yx, x^n \pm y^m] = 0.$$

Then R is commutative if R satisfies one of the following conditions :

- (i) R has the property $Q(m)$.
- (ii) R has the property $Q(n)$.
- (iii) m and n are relatively prime.

Proof. Actually, R satisfies the polynomial identity

$$(1.26) \quad [x^n, y] x = \pm y [x, y^m]$$

Therefore, for a left s -unital ring R , Corollary 3 follows from Theorem 2 and Theorem 3. But a right s -unital ring R with polynomial identity (1.26) is obviously s -unital, and thus Corollary 2 is true also for a right s -unital ring R .

Especially, for $m = n = 1$, the Corollary 3 shows that a left (resp. right) s -unital ring R in which the product of any two elements commutes with their sum (or difference) is necessarily commutative.

Corollary 4. ([6, -Theorem 5]). Let R be a ring with 1 and $n > 1$ be a fixed integer. If R is n -torsion free and satisfies $x^n y - y x^n = x y^n - y^n x$ for all x and y in R , then R is commutative.

Corollary 5. ([12, Theorem 2]). Let $n \geq m \geq 1$ be fixed integers with $mn > 1$ and let R be an s -unital ring. Suppose that every commutator is $m!$ -torsion free. If, further, R satisfies $[x^n, y] = [x, y^m]$ for all $x, y \in R$, then R is commutative.

Actually, the ring R in Corollary 4 is commutative by Theorem 1 and also by Theorem 2. Since in Corollary 5, surely $n > 1$, and the $m!$ -torsion freeness of commutators obviously implies $Q(m)$ property, then the ring R in this corollary is commutative by Theorem 1 for $m = 1$, and by Theorem 2 for $m > 1$.

Remark. If a left (resp. right) s -unital ring R satisfies the identity (1.1) (resp. (1.2)), then the opposite ring R' of R is a right (resp. left) s -unital ring which satisfies the identity

$$(1.27) \quad x^l [x^n, y] y^r = \pm [x, y^m] y^s,$$

(resp.

$$(1.28) \quad x^l [x^n, y] y^r = \pm y^s [x, y^m]$$

and conversely. Moreover, R is commutative if and only if the opposite ring R' of R is commutative. Therefore, from Theorem 1, Theorem 2 and Theorem 3 we obtain analogous commutativity theorems for a right (resp. left) s -unital ring R satisfying the identity (1.27) (resp. (1.28)).

There are rings with unity satisfying the identity (1.1) or (1.2) which are not commutative. Now, we give an example to show that a group which satisfies such one identity need not be commutative.

Example. Let G be a (multiplicative) group with the center $Z(G)$. Suppose that the group $G/Z(G)$ is a periodical group of finite period p . Then, for any $x \in G$, $x^p \in Z(G)$ and thus, such a group G satisfies the identity

$$(1.29) \quad [x^n, y] = [x, y]^m$$

for $n = 1$ and $m = p + 1$. Since any finite group G satisfies the condition of the example, we see that a group G satisfying the identity (1.29) for some given relatively prime positive integers m and n need not be commutative. But, if $m = n + 1$, such a group G is necessarily commutative (see [13, Theorem 3]).

References

- [1] H. A. S. Abujabal and V. Peric, Commutativity of s -unital rings through a streb result, *Radovi Math.*, 7, 73 – 92 (1991).
- [2] H. A. S. Abujabal and V. Peric, Commutativity theorems for s -unital rings with constraints on commutators, *Results Math.*, 21, 256 – 263 (1991).
- [3] H. A. S. Abujabal and V. Peric, Some commutativity theorems for s -unital rings with constraints on commutators. *Publ. Inst. Math. (Beograd)*, 52 (66), 86 – 94 (1992).
- [4] H. Abu-Khuzam and A. Yaqub, Rings and groups with commuting powers, *Internat. J. Math. and Math. Sci.*, 4 (1), 101 – 107 (1981).
- [5] H. E. Bell, A Commutativity condition for tings, *Canad. Math. Bull.*, 28, 486 – 491 (1976).
- [6] H. E. Bell, On the power map and ring commutativity, *Canad. Math. Bull.*, 21, 399 – 404 (1978).
- [7] H. E. Bell, On rings with commuting powers, *Math. Japon.*, 24 (4), 473 – 478 (1979).
- [8] I. N. Herstein, A generalization of a theorem of Jacobson, *Amer. J. Math.* 73, 756 – 762 (1951).
- [9] Y. Hirano, Y. Kobayashi and H. Tominaga, Some polynomial identities and commutativity of s -unital rings, *Math. J. Okayama Univ.*, 24, 7 – 13 (1982).
- [10] T. P. Kezlan, A note on commutativity of semi prime PI-rings, *Math. Japon.*, 27, 267 – 268 (1982).
- [11] W. K. Nicholson and A. Yaqub, A commutativity theorem for rings, and groups with constraints on commutators, *Canad. Math. Bull.*, 22, 419 – 423 (1979).
- [12] E. Psomopoulos, H. Tominaga and A. Yaqub, Some commutativity theorems for n -torsion free rings, *Math. J. Okayama Univ.*, 23, 37 – 39 (1981).

- [13] E. Psomopoulos, Commutativity theorems for rings and groups with constraints on commutators, *Internat. J. Math. and Math. Sci.*, 7 (3), 513 – 517 (1984).
- [14] M. A. Quadri and M. A. Khan, A commutativity theorem for left s -unital rings, *Bull. Inst. Math. Acad. Sinica*, 15, 323 – 327 (1987).
- [15] H. Tominaga and A. Yaqub, A commutativity theorem for one-sided s -unital rings, *Math. J. Okayama Univ.*, 26, 125 – 128 (1984).

Received : January 03, 1995

Hamza A. S. Abujabal
Department of Mathematics
Faculty of Science
King Abdul Aziz University
P. O. Box 31464
Jeddah 21497, Saudi Arabia

Veselin Peric
Department of Mathematics
University of Sarajevo
Vojvode Putnika 43
71000 Sarajevo, Bosnia-Herzegovina