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COUNTING CIRCULAR CATERPILLARS

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Abstract

Our objects are (2.1) to introduce the graphical concept of a circular caterpillar together with cyclical "code" of non-negative integers, and (2.2) to provide an elementary example of the simplest kind of application of the celebrated Polya Enumeration Theorem.

Dedicated to one who counts, Edgar M. Palmer on his 60th birthday.

1. Introduction: Caterpillars

In general we follow the notation and terminology of [1,4]. The now well-known species of trees called caterpillars were first discovered [9] when I asked my then doctoral student, Allen Schwenk, which trees T have a Hamiltonian square. The characterization is that T^2 is hamiltonian if and only if T does not contain $SK_{1,3}$, the subdivision graph of the star $K_{1,3}$ as a subtree. Of course this criterion says that T is a *caterpillar*, a tree whose "pruned subtree" T' (obtained by deleting all the endnodes of T) is a path, called its spine, P_s . We then [10] derived a formula for the number of caterpillars with n nodes having given spine P_s . This derivation involved the "code" of a caterpillar T defined as follows. The *end-degree* $ed(u)$ of a node u of tree T is the number of nodes of degree 1 (endnodes) adjacent to u . Consider caterpillar T whose spine P_s is labeled $v_1, v_2, \dots, v_s$. Writing ed_i for $ed(v_i)$, we defined the code of caterpillar T as the larger of the two sequences $(ed_1, ed_2, \dots, ed_s)$ and $(ed_s, ed_{s-1}, \dots, ed_1)$. The code of a caterpillar was utilized in [3] as a structural model in anthropology.

The square G^2 of a graph G was first defined in [8] where we determined which graphs are isomorphic to the square of some tree. This result was substantially generalized in [6] by studying the topic from the viewpoint of chordal graphs.

Because the notation T' for the pruned subtree of T resembles in appearance the notation $y' = \frac{dy}{dx}$ for the derivative, it seemed natural [5] to define the integral $\int T$ of a tree T as the supertree of T obtained by appending a new endnode to each node of T . Thus $(\int T)' = T$ and $\int T$ is precisely the same as the corona $T \circ K_1$. Any additional number of endnodes can also be attached to a node of T and these may be analogously regarded as a constant of integration. Furthermore, for a given subset $S \subset V(T)$ we can define $\int T dS$ as the result of adding one endnode to each node in S only. We shall use the *integral of a cycle*, defined precisely as for trees, in discussing circular caterpillars.

Iterated integrals of K_2 rooted at one of its two nodes give the family of "broadcast trees". It is shown in [2] how these can be completed to form the hypercubes, providing another criterion for this important family of graphs.

2. Circular caterpillars: Codes

A *circular caterpillar* takes the cycle C_n and appends $p \geq 0$ pendant edges at its nodes. We label the nodes of C_n by $v_1, \dots, v_n$. Let G be a generic circular caterpillar with p_i endnodes at v_i for $i = 1$ to n . Consider the following set of $2n$ vectors:

$$(2.1) \quad \begin{aligned} & (p_1, p_2, \dots, p_n), (p_2, p_3, \dots, p_n, p_1), (p_3, p_4, \dots, p_n, p_1, p_2), \dots \\ & (p_n, p_{n-1}, \dots, p_1), (p_{n-1}, \dots, p_1, p_n), (p_{n-2}, \dots, p_1, p_n, p_{n-1}), \dots \end{aligned}$$

Given two vectors of nonnegative integers $\alpha = (a_1, \dots, a_n)$, $\beta = (b_1, \dots, b_n)$, we say that $\alpha > \beta$ if one of the following n conditions is met:

$$(2.2) \quad \begin{aligned} & a_1 > b_1 \\ & a_1 = b_1, \quad \text{and} \quad a_2 > b_2 \\ & a_1 = b_1, a_2 = b_2 \quad \text{and} \quad a_3 > b_3 \\ & a_i = b_i \quad \text{for } i = 1, 2, \dots, n-1 \quad \text{and} \quad a_n > b_n \end{aligned}$$

Then we define $ccc(G)$, the *circular caterpillar code* of G , as the maximum of the vectors in (2.1). We now list $ccc(G)$ for all G with $n = 5$, $p = 4$:

Table I. The circular caterpillar codes for $n = 5$, $p = 4$.

These 10 circular caterpillars are shown in Figure 1.

4	0	0	0	0
3	1	0	0	0
3	0	1	0	0
2	2	0	0	0
2	0	2	0	0
2	1	1	0	0
2	1	0	1	0
2	1	0	0	1
2	0	1	1	0
1	1	1	1	0

Figure 1. All the circular caterpillars with $n = 5$, $p = 4$.

3. Circular caterpillars: Enumeration

After one fully understands Pólya's Enumeration Theorem (PET), it is a routine exercise to determine the number $cc(n, p)$ of circular caterpillars for given values of n and p . We follow the expositions in [4, p. 182] and [7, p. 41].

It is well known that the automorphism group of the cycle graph C_n is the dihedral group D_n . Let Z_n be the cyclic group of order n and degree n generated by the permutation $(1\ 2\ 3\ \dots\ n)$. The cycle indexes of Z_n and D_n , see [4, p. 184] or [7, p. 37], are:

$$(3.1) \quad CI(Z_n) = \frac{1}{n} \sum_{k|n} \phi(k) z_k^{n/k}$$

$$(3.2) \quad CI(D_n) = \frac{1}{2} CI(Z_n) + \begin{cases} \frac{1}{2} z_1 z_2^{(n-1)/2} & n \text{ odd} \\ \frac{1}{4} \left(z_2^{n/2} + z_1^2 z_2^{(n-2)/2} \right) & n \text{ even} \end{cases}$$

where $\phi(h)$ is the "Euler ϕ -function" also called the "totient" of h , defined as the number of positive integers less than h and relatively prime to h , and the symbols $z_1, z_2, \dots$ are indeterminates. We use the convention:

$$(3.3) \quad \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

In (3.1) we replace each z_i by $1/(1-x^i)$, and write

$$(3.4) \quad CI\left(D_n, \frac{1}{1-x}\right)$$

for the resulting polynomial in x .

Then by the PET, the number $cc(n, p)$ is the coefficient of x^p in equation (3.4).

We illustrate with the small example $n = 5$, $p = 4$ for which

$$(3.5) \quad CI(D_5) = \frac{1}{10} (z_1^5 + 4z_5 + 5z_1z_2^2)$$

$$(3.6) CI\left(D_5, \frac{1}{1-x}\right) = \frac{1}{10} \left\{ \left(\frac{1}{1-x}\right)^5 + \frac{4}{1-x^5} + \frac{5}{1-x} \left(\frac{1}{1-x^2}\right)^2 \right\}$$

Thus the coefficient of x^4 in (3.6) is precisely that in:

$$\frac{1}{10} \left[(1+x+x^2+x^3+x^4)^5 + 5(1+x+x^2+x^3+x^4)(1+x^2+x^4)^2 \right]$$

which is seen by very elementary manipulations to be 10, as verified in Figure 1.

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