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GLOBAL SMOOTHNESS PRESERVATION BY SINGULAR INTEGRALS

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Abstract

Here is established that the well-known singular integrals of Picard, Poisson–Cauchy and Gauss–Weierstrass fulfill the "global smoothness preservation" property. I.e., they "ripple" less than the function they are applied on, that is producing a nice approximation to the unit. The associated inequalities are sharp.

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1. Introduction

Let f be a function from R into itself. We call

$$\omega_1(f,h) := \sup_{\substack{x,y \\ |x-y|\le h}} |f(x)-f(y)|, \quad h>0 \tag{1.1}$$

the first modulus of continuity of f. Here we consider f such that $\omega_1(f,h) < +\infty$, for any $h>0$.

For $\zeta > 0$ with $\zeta \to 0$, we consider the well-known Picard, Poisson–Cauchy and Gauss–Weierstrass singular integrals

$$P_\zeta(f,x) := \frac{1}{2\zeta}\cdot\int_{-\infty}^{\infty} f(x+t)\cdot e^{-|t|/\zeta}\cdot dt, \tag{1.2}$$

$$Q_\zeta(f,x) := \frac{\zeta}{\pi}\cdot\int_{-\pi}^{\pi}\left(\frac{f(x+t)}{t^2+\zeta^2}\right)\cdot dt, \tag{1.3}$$

and

$$W_\zeta(f,x) := \frac{1}{\sqrt{\pi\cdot\zeta}}\cdot\int_{-\pi}^{\pi} f(x+t)\cdot e^{-t^2/\zeta}\cdot dt. \tag{1.4}$$

The above integrals are positive linear operators with respect to f. Furthermore, here we consider only f such that $P_\zeta(f,x), Q_\zeta(f,x), W_\zeta(f,x) \in$ R, for all $x \in$ R. In particular, notice that (for $id(x)=x$)

$$P_\zeta(id,x) = x \in \mathrm{R}, \tag{1.5}$$

$$Q_\zeta(id,x) = \frac{2x}{\pi}\cdot\tan^{-1}\frac{\pi}{\zeta} \in \mathrm{R}, \tag{1.6}$$

and

$$W_\zeta(id,x) = x\cdot Erf\left(\frac{\pi}{\sqrt{\zeta}}\right) \in \mathrm{R}, \quad \text{all } x\in \mathrm{R}. \tag{1.7}$$

To get (1.5) we use that

$$\frac{1}{2\zeta}\cdot\int_{-\infty}^{\infty} e^{-|t|/\zeta}dt = 1, \tag{1.8}$$

to get (1.6) we use that

$$\frac{\zeta}{\pi}\cdot\int_{-\pi}^{\pi}\frac{dt}{t^2+\zeta^2} = \frac{2}{\pi}\cdot\tan^{-1}\frac{\pi}{\zeta}, \tag{1.9}$$

finally, to get (1.7) we use that

$$Erf(x) := \frac{2}{\sqrt{\pi}} \cdot \int_0^x e^{-t^2} \cdot dt. \tag{1.10}$$

Obviously, id is a function from $\mathbf{R} \to \mathbf{R}$ such that

$$\omega_1(id, h) = h < +\infty, \quad \text{any } h > 0.$$

The above operators (1.2), (1.3), (1.4) have been studied extensively in [2]. In [2] the authors obtain the degree of convergence of these operators to the unit with rates over the class of Hölder-continuous functions.

In [1] the author obtains some more refined results, in the same direction as in [2], however, only over the set of $(C_{2\pi})$ 2π-periodic continuous functions on $\mathbf{R}$, where a monotonicity assumption from [2] regarding the modulus of continuity there is dropped.

In this note the author establishes that the operators (1.2), (1.3), (1.4) when applied to f do not "ripple" more than f. I.e., operators (1.2), (1.3), (1.4) fulfill the "global smoothness preservation" property, and the associated inequalities are sharp, and more precisely are attained. The above "smoothness preservation" property indicates that the operators (1.2), (1.3), (1.4) approximate nicely the unit operator.

2. Main Results

Theorem 1. *Let the function $f\colon R \to R$ with $\omega_1(f,h) < +\infty$, for any $h > 0$, such that $P_\zeta(f,x)$, $Q_\zeta(f,x)$, $W_\zeta(f,x) \in R$, for all $x \in R$. Here take $\zeta > 0$ and $\zeta \to 0$. Then*

$$\omega_1(P_\zeta(f), h) \le \omega_1(f,h), \tag{2.1}$$

$$\omega_1(Q_\zeta(f), h) \le \frac{2}{\pi} \cdot \tan^{-1} \frac{\pi}{\zeta} \cdot \omega_1(f,h), \tag{2.2}$$

and

$$\omega_1(W_\zeta(f), h) \le Erf\left(\frac{\pi}{\sqrt{\zeta}}\right) \cdot \omega_1(f,h), \tag{2.3}$$

for any $h > 0$.

Proof.

Inequality (2.1): Note that

$$P_\zeta(f,x) - P_\zeta(f,y) = \frac{1}{2\zeta} \cdot \int_{-\infty}^{\infty} (f(x+t) - f(y+t)) \cdot e^{-|t|/\zeta} \cdot dt.$$

Thus

$$\begin{aligned} |P_\zeta(f,x) - P_\zeta(f,y)| &\le \frac{1}{2\zeta} \cdot \int_{-\infty}^{\infty} |f(x+t) - f(y+t)| \cdot e^{-|t|/\zeta} \cdot dt \\ &\le \frac{1}{2\zeta} \cdot \int_{-\infty}^{\infty} \omega_1(f,|x-y|) \cdot e^{-|t|/\zeta} \cdot dt \\ &= \omega_1(f,|x-y|), \quad \text{by (1.8)}. \end{aligned}$$

Now (2.1) is obvious.

Inequality (2.2): See that

$$Q_\zeta(f,x) - Q_\zeta(f,y) = \frac{\zeta}{\pi} \cdot \int_{-\pi}^{\pi} \frac{(f(x+t) - f(y+t))}{t^2+\zeta^2} \cdot dt.$$

Hence

$$\begin{aligned} |Q_\zeta(f,x) - Q_\zeta(f,y)| &\le \frac{\zeta}{\pi} \cdot \int_{-\pi}^{\pi} \frac{|f(x+t) - f(y+t)|}{t^2+\zeta^2} \cdot dt \\ &\le \frac{\zeta}{\pi} \cdot \int_{-\pi}^{\pi} \frac{\omega_1(f,|x-y|)}{t^2+\zeta^2} \cdot dt \\ &= \omega_1(f,|x-y|) \cdot \frac{2}{\pi} \cdot \tan^{-1}\frac{\pi}{\zeta}, \quad \text{by (1.9)}. \end{aligned}$$

The validity of (2.2) now is clear.

Inequality (2.3): Observe that

$$W_\zeta(f,x) - W_\zeta(f,y) = \frac{1}{\sqrt{\pi\zeta}} \cdot \int_{-\pi}^{\pi} (f(x+t) - f(y+t)) \cdot e^{-t^2/\zeta} \cdot dt.$$

Therefore

$$\begin{aligned} |W_\zeta(f,x) - W_\zeta(f,y)| &\le \frac{1}{\sqrt{\pi\zeta}} \cdot \int_{-\pi}^{\pi} |f(x+t) - f(y+t)| \cdot e^{-t^2/\zeta} \cdot dt \\ &\le \frac{1}{\sqrt{\pi\zeta}} \cdot \omega_1(f,|x-y|) \cdot \int_{-\pi}^{\pi} e^{-t^2/\zeta} \cdot dt \\ &= Erf\left(\frac{\pi}{\sqrt{\zeta}}\right) \cdot \omega_1(f,|x-y|), \quad \text{from (1.10)}. \end{aligned}$$

I.e.,

$$|W_\zeta(f,x) - W_\zeta(f,y)| \leq Erf\left(\frac{\pi}{\sqrt{\zeta}}\right) \cdot \omega_1(f, |x-y|), \quad \text{all } x, y \in \mathbf{R}.$$

The last establishes (2.3). Q.E.D.

Optimality is obtained in

Theorem 2. *Inequalities* (2.1), (2.2), (2.3) *are sharp, in fact they are attained by the identity function.*

Proof. Note that the *id*-function belongs to the class of functions under consideration, for this see the related discussion in §1 and (1.5), (1.6), (1.7).

Inequality (2.1): See that

$$\begin{aligned} P_\zeta(id,x) - P_\zeta(id,y) &= \frac{1}{2\zeta} \cdot \int_{-\infty}^{\infty} ((x+t)-(y+t)) \cdot e^{-|t|/\zeta} \cdot dt \\ &= (x-y) \cdot \frac{1}{2\zeta} \cdot \int_{-\infty}^{\infty} e^{-|t|/\zeta} \cdot dt \\ &= x - y, \quad \text{by (1.8).} \end{aligned}$$

That is

$$\omega_1(P_\zeta(id), h) = \omega_1(id, h),$$

any $h > 0$.

Inequality (2.2): Observe that

$$\begin{aligned} Q_\zeta(id,x) - Q_\zeta(id,y) &= (x-y) \cdot \frac{\zeta}{\pi} \cdot \int_{-\pi}^{\pi} \frac{dt}{t^2+\zeta^2} \\ &= (x-y) \cdot \frac{2}{\pi} \cdot \tan^{-1}\frac{\pi}{\zeta}, \quad \text{by (1.9).} \end{aligned}$$

I.e.,

$$\omega_1(Q_\zeta(id), h) = \omega_1(id, h) \cdot \frac{2}{\pi} \cdot \tan^{-1}\frac{\pi}{\zeta},$$

any $h > 0$.

Inequality (2.3): Similarly we have that

$$\begin{aligned} W_\zeta(id,x) - W_\zeta(id,y) &= (x-y)\cdot\frac{1}{\sqrt{\pi\zeta}}\cdot\int_{-\pi}^{\pi} e^{-t^2/\zeta}\cdot dt \\ &= (x-y)\cdot Erf\left(\frac{\pi}{\sqrt{\zeta}}\right), \quad \text{by (1.10).} \end{aligned}$$

That is,

$$\omega_1(W_\zeta(id),h) = \omega_1(id,h)\cdot Erf\left(\frac{\pi}{\sqrt{\zeta}}\right),$$

for any $h > 0$. Q.E.D.

Remark 1. As $\zeta \to 0$ note that both $\left(\frac{2}{\pi}\cdot\tan^{-1}\frac{\pi}{\zeta}\right)$ and $Erf\left(\frac{\pi}{\sqrt{\zeta}}\right)$ tend to 1, showing that the (R.H.S.)'s of (2.1), (2.2), (2.3) for very small ζ are about the same.

References

[1] S.G. Gal, "Remark on the Degree of Approximation of Continuous Functions by Singular Integrals," *Math. Nachr.* 164, pp. 197 – 199 (1993).

[2] R.N. Mohapatra and R.S. Rodriguez, "On the Rate of Convergence of Singular Integrals for Hölder Continuous Functions," *Math. Nachr.* 149, pp. 117 – 124 (1990).

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