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LOCAL STABILITY RESULTS ON A MODEL FOR TYPHOID FEVER WITH A CORE GROUP*

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Abstract

A SIRS epidemiological model with two subpopulation and vital dynamics is analized. Both subpopulations are considered constant by assuming that the birth and the death rate are equal. We analize the case where one subpopulation is a core, that is, a very infectious small group, responsible for a big fraction of the incidence. For this case a threshold is determined and the corresponding equilibrium points for the four dimensional system are shown to be locally stable by means of the classical Liapunov theorem. This system models the dynamics of typhoid fever, where the core is the group of food manipulators. Computer simulation are used to estimate the effect of vaccination on the population.

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1. Introduction.

The spread of an infectious disease in a population depends not only on the character of the disease, but also on the structure and behavior of the population. Therefore the study of *heterogeneous* epidemic models is very important. Many previous results have been obtained in this direction. Hethcote [3] analyzed the behavior of solutions of a SIR model with n subpopulations,but without vital dynamics (births and deaths). E.Beretta and V.Capasso [8] give conditions for global stability in a SIR model with n subpopulations. For a SEIRS model with n subpopulations, Thieme [4] showed global asymptotic stability under certain conditions. Hethcote and Thieme [5] analyze two different SIRS models with n subpopulations and show local stability for the endemic equilibrium. The global stability of the endemic equilibrium for the two models is still an open problem.

In the present paper we identify a threshold quantity for a SIRS model with two subpopulations and show the local stability for both the disease free and the endemic equilibrium. We consider the case of a *core-noncore* model [1], where the infectious contacts of the core people with the noncore ones are responsible for most of the total incidence. Our aim is to model the spread of typhoid fever in a modern big city, where many people eat in restaurants. In this case the *food manipulators* constitute the *core* group.

2. The Model.

Let N be the total size of the population, N_1 the size of the core and N_2 the size of the noncore. Then $N = N_1 + N_2; N_i = S_i + I_i + R_i, i = 1, 2$, where S, I, R denote the number of susceptibles, infectious and removed of each subpopulation. We assume that both subpopulations N_1 and N_2 are constant by setting the birth rate equal to the death rate. Note that, in the case of the core group, the birth and the death rate have the meaning of a *rate of change* of the food manipulators, not necesarly equal to the birth and death rate of the population. Let $x_i = I_i/N_i$, $y_i = R_i/N_i$,$i = 1, 2$; (λ_{ij}) is the *transmission matrix*, that is , λ_{ij} is the average number of infectious contacts (also called *adequate* contacts, see [1]) of an infective in group j with persons in group i per unit time. Let $p = N_1/N_2$, δ_i the birth (or death) rate of subpopulation i, γ is the recovery removal rate ($1/\gamma$ is the average infectious period), w_i is the rate of loss of temporary immunity for group i and v_i is the immunization rate by vaccination in group i. We can accept *different* values of duration of immunity because the vaccination strategy on both groups can be different (for example, food manipulators can get double dosis vaccine).

The model is then given by the following four dimensional system of autonomous differential equations :

$$
\begin{aligned}
x_1' &= (1 - x_1 - y_1)(\lambda_{11}x_1 + \frac{\lambda_{12}}{p}x_2) - x_1(\delta_1 + \gamma) \\
y_1' &= v_1 + (\gamma - v_1)x_1 - (v_1 + \delta_1 + w_1)y_1 \\
x_2' &= (1 - x_2 - y_2)(\lambda_{22}x_2 + \lambda_{21}px_1) - x_2(\delta_2 + \gamma) \\
y_2' &= v_2 + (\gamma - v_2)x_2 - (v_2 + \delta_2 + w_2)y_2
\end{aligned} \tag{1}
$$

with the conditions : $x_i, y_i \geq 0; x_i + y_i \leq 1,\ i = 1, 2.$

Let D be the domain $x_i, y_i \geq 0; x_i + y_i \leq 1, i = 1, 2$ in the four dimensional space R^4 . It is easy to see ,by applying Nagumos's Lemma, that domain D is positively invariant for system (1). This system satisfies the existence, uniquenes and prolongability conditions so that solutions of (1) starting in D exists for all time and remain in D. Thus the model is well-posed.

The qualitative analyse of system (1) is very difficult because of the dimension, even the equilibrium points can not be found explicity. But, if we assume that $\lambda_{12} = \lambda_{22} = 0$, then system (1) become mathematically tractable. This supposition means that the model consider *only* the effect of the core on the spread of the disease in the population.

By analyzing system (1) we obtain the following threshold parameter :

$$\sigma = \frac{\lambda_{11}(\delta_1 + w_1)}{(\delta_1 + \gamma)(v_1 + \delta_1 + w_1)} \tag{2}$$

Theorem 1.

a) If $\sigma \leq 1$, then the only equilibrium point of system (1) is the disease-free equilibrium :

$$E_1 = (0, v_1/(v_1 + \delta_1 + w_1), 0, v_2/(v_2 + \delta_2 + w_2)) \tag{3}$$

which is locally stable.

b) If $\sigma > 1$, then there are two equilibrium points in D : the disease free equilibrium E_1 and the endemic equilibrium point $E_2 = (x_1^*, y_1^*, x_2^*, y_2^*)$,

where :

$$
\begin{aligned}
x_1^* &= \frac{\lambda_{11}(\delta_1 + w_1) - (\delta_1 + \gamma)(v_1 + \delta_1 + w_1)}{\lambda_{11}(\delta_1 + w_1 + \gamma)} \\
y_1^* &= \frac{\lambda_{11}\gamma + (\delta_1 + \gamma)(v_1 - \gamma)}{\lambda_{11}(\delta_1 + w_1 + \gamma)} \\
x_2^* &= \frac{px_1^*\lambda_{21}(w_2 + \delta_2)}{(\delta_2 + \gamma)(v_2 + w_2 + \delta_2) + px_1^*\lambda_{21}(w_2 + \delta_2 + \gamma)} \\
y_2^* &= \frac{v_2(\delta_2 + \gamma) + px_1^*\lambda_{21}\gamma}{(\delta_2 + \gamma)(v_2 + w_2 + \delta_2) + px_1^*\lambda_{21}(w_2 + \delta_2 + \gamma)}
\end{aligned} \tag{4}
$$

This endemic point is locally stable while the disease-free equilibrium point E_1 is unstable.

Proof. First we note that always $E_1 \in D$ but $E_2 \in D$ if and only if :

$$\lambda_{11}(\delta_1 + w_1) > (\delta_1 + \gamma)(v_1 + \gamma)(v_1 + \delta_1 + w_1)$$

This last condition is equivalent to the condition $\sigma > 1$. In order to analize the local stability of the equilibrium points we calculate the Jacobian Matrix of system (1):

$$
\begin{pmatrix}
\lambda_{11}(1 - 2x_1 - y_1) - \delta_1 - \gamma & -\lambda_{11}x_1 & 0 & 0 \\
\gamma - v_1 & -(v_1 + w_1 + \delta_1) & 0 & 0 \\
\lambda_{21}p(1 - x_2 - y_2) & 0 & -\lambda_{21}px_1 - \delta_1 - \gamma & -\lambda_{21}px_1 \\
0 & 0 & \gamma - v_2 & -(v_2 + w_2 + \delta_2)
\end{pmatrix}
$$

Evaluating in equilibrium point E_1 :

$$
\begin{pmatrix}
\frac{\lambda_{11}(1-v_1}{(v_1+w_1+\delta_1)-\delta_1-\gamma} & 0 & 0 & 0 \\
\gamma - v_1 & -(v_1 + w_1 + \delta_1) & 0 & 0 \\
\frac{\lambda_{21}p(1-v_2}{(v_2+w_2+\delta_2))} & 0 & -\delta_1 - \gamma & 0 \\
0 & 0 & \gamma - v_2 & -(v_2 + w_2 + \delta_2)
\end{pmatrix}
$$

The eigenvalues of this matrix are :

$$
\begin{aligned}
z_1 &= -(v_2 + w_2 + \delta_2) \\
z_2 &= -(\delta_1 + \gamma) \\
z_3 &= -(v_1 + w_1 + \delta_1) \\
z_4 &= \frac{\lambda_{11}(w_1 + \delta_1)}{v_1 + w_1 + \delta_1} - \delta_1 - \gamma
\end{aligned} \tag{5}
$$

Note that $z_1, z_2, and z_3$ are always negative and z_4 is negative if and only if $\sigma < 1$.

Since : $\lambda_{11}(1 - 2x_* - y_1) - \delta_1 - \gamma = -\lambda_{11}x_*$ the Jacobian matrix evaluated in point E_2 is :

$$
\begin{pmatrix}
-\lambda_{11}x_1^* & -\lambda_{11}x_1^* & 0 & 0 \\
\gamma - v_1 & -(v_1 + w_1 + \delta_1) & 0 & 0 \\
\lambda_{21}p(1 - x_2^* - y_2^*) & 0 & -\lambda_{21}px_1^* - \delta_1 - \gamma & -\lambda_{21}px_1^* \\
0 & 0 & \gamma - v_2 & -(v_2 + w_2 + \delta_2)
\end{pmatrix}
$$

Hence, the characteristic polynomial decomposes in the two following quadratic equations :

$$
\begin{aligned}
&z^2 + (v_1 + w_1 + \delta_1 + \lambda_{11})z + \lambda_{11}x_1^*(w_1 + \delta_1 + \gamma) = 0 \\
&z^2 + (\lambda_{21}px_1 + \delta_1 + \delta_2 + v_2 + w_2 + \gamma)z + \\
&+\lambda_{21}px_1^*(\delta_2 + w_2 + \gamma) + (\delta_1 + \gamma)(v_2 + w_2 + \delta_2) = 0
\end{aligned} \tag{6}
$$

Since the coefficients of this polynomials are positiv, the eigenvalues have negative real parts.

3. Parameter Estimation and Vaccination Strategies by Computer Simulation.

Let us suppose that, without vaccination, the endemic values of both subpopulations are equal, and that in this case, $w_1 = w_2$ and $\delta_1 = \delta_2$. Then,

using formulae (4) :

$$x_1^* = \frac{px_1^*\lambda_{21}(w_2+\delta_2)}{(\delta_2+\gamma)(v_2+w_2+\delta_2)+px_1^*\lambda_{21}(w_2+\delta_2+\gamma)}$$

thus :

$$1 = \frac{p\lambda_{21}(w_1+\delta_1)}{(\delta_1+\gamma)(w_1+\delta_1)+px_1^*\lambda_{21}(w_1+\delta_1+\gamma)}$$

$$1 = \frac{p\lambda_{21}(w_1+\delta_1)}{(\delta_1+\gamma)(w_1+\delta_1)+p\frac{\lambda_{11}(\delta_1+w_1)-(\delta_1+\gamma)(v_1+\delta_1+w_1)}{\lambda_{11}(\delta_1+w_1+\gamma)}\lambda_{21}(w_1+\delta_1+\gamma)}$$

$$1 = \frac{p\lambda_{21}\lambda_{11}}{\lambda_{11}(\delta_1+\gamma)+p\lambda_{21}(\lambda_{11}-\delta_1-\gamma)}$$

thus:

$$\lambda_{11} = p\lambda_{21} \tag{7}$$

We use the estimation of the total prevalence given in [6]. Then:

$$x_1^* = \frac{(\delta_1+w_1)(\lambda_{11}-\delta_1-\gamma)}{\lambda_{11}(_1+w_1+\gamma)} = 0.007667$$

hence:

$$\lambda_{11} = \frac{(\delta_1+w_1)(\delta_1+\gamma)}{0.99233(\delta_1+w_1)-0.007667\gamma}$$

Now, considering that [6]: $\delta_1 = 0.02$; $\gamma = 0.111$ and that the mean duration of the vaccine has been estimated in 5 years, that is : $w_1 = w_2 = 0.2$, and that we can estimate for Valparaiso : $p = 0.017$, we obtain finally :

$$\lambda_{11} = 0.1325 \ ; \lambda_{21} = 7.794$$

Thus, the set of parameter values we use to model the endemic situation of typhoid fever in Valparaiso is :

$$\delta_1 = 0.02; \ \delta_2 = 0.02; \ \gamma = 0.111; \ p = 0.017$$

$$\lambda_{11} = 0.1325; \ \lambda_{21} = 7.794; \ \lambda_{12} = 0 \ \lambda_{22} = 0$$

(8) $$w_1 = 0.2; \quad w_2 = 0.2$$

With this parameter values we can calculate:

$$y_1^* = 0.0040 \ ; y_2^* = 0.0049$$

. The values of the threshold parameter σ can then be calculated for each intensity of the vaccination strategy of food manipulators. For example :

$$\sigma_0 = 1.1726; \quad \sigma_{10} = 0.806; \quad \sigma_{50} = 0.3583; \quad \sigma_{100} = 0.21145.$$

Where the subindices indicate the yearly percentage of immunization by vaccination of the corresponding population. We note that, even a very low level of immunization (10 percent), yields to the disease-free state. In order to estimate *the time* necesary to reduce the endemic level we have simulated the behavior of the system for three vaccination intensities (see Fig 1) :

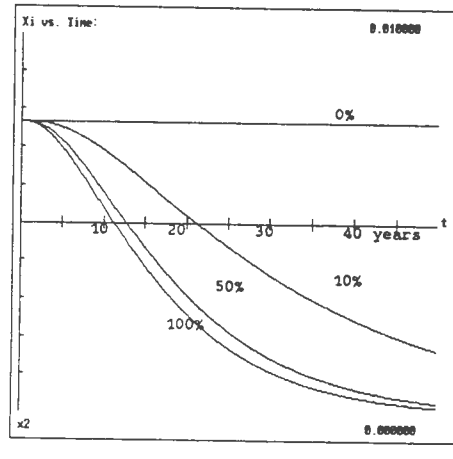

Fig 1

In order to compare the effect of a vaccination policy *in the whole population* see Fig 2 :

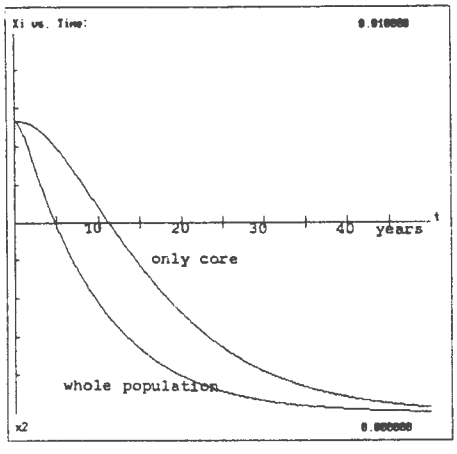

Fig 2

4. Discussion.

Typhoid fever is still a severe disease in our country. After a surprising increase of the endemic level from 50 reported cases per 100.000 habitants in 1960 to 120.2 in 1978, the number of cases has decreased during the last ten years to 5652 (45.1 per 100.000 habitants) in 1987. Hence we have only returned to the old level. This endemic level is too high if we compare with other countries (for example Argentina has a rate of about 2 per 100.000 and U.S.A. has a rate of 0.2 per 100.00). Moreover, considering an historical undernotification of 50 percent, the real endemicity is about 100 cases per 100.000 habitants.

In the last years several control policies have been proved, in particular the Ty21a oral vaccine of René Germanier [10]. This vaccine has a 69 percent efficacy for at least four years but only if it is applied with three doses, while adding a fourth dose enhance significantly the protection [10]. Therefore is very important to estimate the possibility of focalize the control to the most important contagion agents: the food manipulators. The mo[illegible] discussed in this paper consider precisely this question.

The computer simulations presented here have only an illustrative character, because the model is still too simple and parameter estimation are not enough accurate to be practical. Nevertheless they give, in my opinion, an interesting orientation about the complex problem of controlling this disease.

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