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AN ORLICZ-PETTIS THEOREM FOR CONDITIONALLY CONVERGENT SERIES

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Abstract

In this note we define the idea of signed subseries convergent series in a normed linear space, and establish a result analogous to the Orlicz-Pettis theorem; namely, that weakly signed subseries convergent series are norm signed subseries convergent.

1. Introduction.

The classical Orlicz-Pettis theorem states that a weakly subseries convergent series in a normed vector space X in subseries convergent with respect to the norm topology. In this note we show that weakly signed subseries convergent series (Definition 2.1) in X are actually norm signed subseries convergent.

2. Main Results.

Definition 2.1. Let X be a topological vector space, and $(x_n) \subseteq X$. A formal series $\sum x_n$ is ***signed subseries convergent*** if it satisfies:

a) for every subsequence (x_{n_k}) of (x_n) there is a choice signs $(s_k) \in \{-1, 1\}^N$ such that $\sum_{k=1}^{\infty} s_k x_{n_k}$ converges, and

b) if $\sum_{k=1}^{\infty} s_k x_{n_k}$ converges, and $y_l = \sum_{k \in I_l} s_k x_{n_k}$, for (I_l) and increasing sequence in I_0 (the set of all finite subintervals of $\mathbf{N}$), there is a choice of signs (t_l) such that $\sum_{l=1}^{\infty} t_l y_l$ converges. (Note that this implies $\sum y_l$ is also signed subseries convergent.)

By the following theorem of Dvoretsky and Hanani, every null sequence in a finite-dimensional normed space (i.e., $\mathbf{R}^m$) is signed subseries convergent:

Theorem 2.2. ([DH]) If (x_n) is a null sequence in $\mathbf{R}^m$, then there is a choice of signs (s_n) such that $\sum_{n=1}^{\infty} s_n x_n$ converges.

The conditions of Definition 2.1 are easily seen to be satisfied in this case, since any subsequence of a null sequence is null, so a) holds. If $\sum_{k=1}^{\infty} s_k x_{n_k}$ converges and $y_l = \sum_{k \in I_l} s_k x_{n_k}$ for (I_l) increasing, then (y_l) is a null sequence in $\mathbf{R}^m$. So by the Dvoretsky-Hanani theorem, there is a choice of signs (t_l) such that $\sum_{l=1}^{\infty} t_l y_l$ converges, which is b).

Thus, the property of signed subseries convergence is in marked contrast to subseries convergence in finite-dimensional spaces, since subseries convergence implies absolute convergence in this case.

We now establish a result analogous to the Orlicz-Pettis theorem for signed subseries convergent series in a normed linear space X. The following theorem is a generalization of the Basic Matrix Theorem of Antosik and Mikusinski, and is used in the proofs of some preliminary results.

Theorem 2.3. (The signed Basic Matrix Theorem) Let X be a Hausdorff topological group, and $x_{ij} \in X$ for $i, j \in \mathbf{N}$. Suppose

i) $\lim_i x_{ij} = x_j$ exists for each j and

ii) For each increasing sequence of positive integers (m_j) there is a subsequence (n_j) of (m_j) and a choice of signs (s_j) such that $(\sum_{j=1}^{\infty} s_j x_{in_j})_{i=1}^{\infty}$ is Cauchy.

Then $\lim_i x_{ij} = x_j$ uniformly in j. In particular, $\lim_i x_{ii} = 0$.

Proof. See ([S], Theorem 2, p. 96) for a proof of the Basic Matrix Theorem, or [St] for a proof the signed Basic Matrix Theorem, which is a straightforward generalization.

Definition 2.4. A sequence (x_n) in a topological vector space X is said to signed $\mathcal{K}$-convergent if for every subsequence of (x_n) there is a further subsequence (x_{n_k}) and a choice of signs (s_k) such that $\sum_{k=1}^{\infty} s_k x_{n_k}$ converges.

Proposition 2.5. If $(x_j) \subseteq X$ is $\sigma(X, X')$- signed-$\mathcal{K}$-convergent, then $||x_j|| \longrightarrow 0$.

Proof. The proof is essentially that of [S], Proposition 16, p. 218, using Theorem 2.3 in place of the Basic Matrix Theorem.

Theorem 2.6. If $(x_n) \subset X$ is $\sigma(X, X')$-signed- $\mathcal{K}$-convergent, then (x_k) is $||\cdot||$-signed-$\mathcal{K}$-convergent.

Proof. (The proof emulates that of [S], Theorem 19, p. 220).

By the proposition above, (x_k) is norm convergent to 0. Let (y_k) be a subsequence of (x_k) such that $\sum_{k=1}^{\infty} ||y_k|| < \infty$. Next, let (z_k) be a subsequence of (y_k) such that $\sum_{k=1}^{\infty} s_k z_k$ is weakly convergent to some $z \in X$ for some choice of signs (s_k). The partial sums $S_n = \sum_{k=1}^{n} s_k z_k$ form a norm Cauchy sequence since the series $\sum ||z_k||$ is absolutely convergent. Since the norm topology is linked to the weak topology, $S_n \longrightarrow z$ in norm (see [S], Definition 16.17). The argument can be applied to any subsequence of (x_k), so (x_k) is norm signed-$\mathcal{K}$-convergent.

We next prove a lemma that is a straightforward generalization of [S], Lemma 20, p. 221.

Lemma 2.7. Let σ and τ be two vector topologies on the vector spaces E with σ linked to τ. If every series $\sum_k x_k$ in E wich is signed subseries convergent with respect to τ satisfies $\sigma - \lim x_k = 0$, then every series in E which is τ-signed subseries convergent is also σ-signed subseries convergent.

Proof. If $\sum x_n$ is τ-signed subseries convergent but fails condition a) in the definition for the σ-topology, then there exists a subseries $\sum_k x_{n_k}$ and a choice of signs (s_k) such that $\sum_k s_k x_{n_k}$ is τ-convergent, but fails to be σ-Cauchy (using [S], Lemma 18,p. 219). Therefore, there exists an increasing sequence (I_m) in I_0 and a σ-neighborhood of, 0, U, such that

$$\sum_{k \in I_m} s_k x_{n_k} = z_m \notin U.$$

The series $\sum z_m$ is τ-signed subseries convergent (using Definition 2.1 b)), so $\sigma - \lim z_m = 0$, by hypothesis. This contradicts $z_m \notin U$.

A similar argument shows that condition b) must hold for the σ-topology. Indeed, if not, there exists a subsequence (x_{n_k}) and an increasing sequence $(I_k) \subset I_0$ such that the series $\sum s_k x_{n_k}$ is τ-convergent for signs (s_k) and $\sum t_l y_l$ converges with respect to τ. where $y_l = \sum_{k \in I_l} s_k x_{n_k}$, but $\sum t_l y_l$ fails to be σ-Cauchy. Since $\sum y_l$ is τ-signed subseries convergent, the argument in the preceding paragraph can be repeated to obtain a contradiction.

We can now prove the signed Orlicz-Pettis theorem.

Theorem 2.8. If $\sum x_k$ is $\sigma(X, X')$ signed subseries convergent in X, then $\sum x_k$ is norm signed subseries convergent.

Proof. It follows from Proposition 2.5 that $\|x_k\| \longrightarrow 0$ because (x_k) is $\sigma(X, X')$ signed $\mathcal{K}$-convergent. Since the norm topology is linked to the weak topology, the theorem follows from the previous lemma.

We now state some open problems concerning signed subseries convergent series:

1) Recall that the summing basis of c_0 is the basis of vectors $x_n = \sum_{i=1}^{n} e_i$, where (e_i) is the unit vector basis of c_0. It is well known that the summing basis is a conditional basis; indeed $\sum_{n=1}^{\infty} a_n x_n$ converges in c_0 if and only if $(a_n) \in cs$, the space of all convergent series (see [Si], Example 14.1). It follows easily, using the Dvoretsky-Hanani theorem, that the summing basis is a "signed subseries" basis. That is, $\sum_{n=1}^{\infty} a_n x_n$ is signed subseries convergent whenever the series converges. It would be interesting to have an example of a Banach space with no unconditional basis that has a signed subseries basis. In particular, does $C[0,1]$ have a signed subseries basis?

2) The equivalence of unconditional convergence, bounded multiplier convergence, and subseries convergence is well known in Banach spaces (see [S]). We say that a series $\sum_n x_n$ is signed unconditionally convergent if for every permutation $\pi : \mathbf{N} \longrightarrow \mathbf{N}$. there is a choice of signs (s_n) such that $\sum_n s_n x_{\pi(n)}$ converges. Also, a series is signed bounded multiplier convergent if for every sequence $(a_n) \in l^\infty$ there is a choice of signs (s_n) such that $\sum_n s_n a_n x_n$ converges. Are these properties equivalent in Banach spaces? It is easy to see that the summing basis is signed unconditional and signed bounded multiplier convergent.

3) Finally, is part b) in Definition 2.1 implied by part a), or are they independent?

REFERENCES

[DH] Dvoretsky, A.; Hanani, C.: Sur les changements des signes d'une serie a termes complexes. *C.R. Academie Des Sciences Paris*, 225 (1947), 516-518.

[S] Swartz, C.: An Introduction to Functional Analysis. *Marcel Dekker*, 1992.

[Si] Singer, I.: Bases in Banach Spaces I. *Springer-Verlag New York*, 1970.

[St] Stuart, C.: Weak sequential completeness of β-duals. Preprint.

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