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ON THE CODOMINATION NUMBER OF A GRAPH

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Abstract

Given a graph $G = (V, E)$, set $S \subset V$ is a dominating set if each node of $V - S$ is adjacent to at least one node in S. The domination number of G is the smallest size of a dominating set and the codomination number is the domination number of its complement. We determine the codomination number of a graph having diameter at least three. Further we explore the effects of this result on the open problem of characterizing graphs having equal domination and codomination numbers.

1. Introduction.

In general, we follow the notation and terminology of [8]. Each node of a graph $G = (V, E)$ dominates every node in its closed neighborhood. A set $S \subset V$ is a dominating set if each node in V is dominated by some node of S. The domination number $d = d(G)$ is the smallest size of a dominating set. The domination number $\overline{d} = d(\overline{G})$ of its complement is called the codomination number of G. Domination in both a graph and its complement has been the topic of several recent papers [2, 3, 5, 6, 7, 15, 16]. These authors were concerned with finding a smallest set $S \subset V$ which dominates both G and $\overline{G}$, called a "factor dominating set" in [5] and a "global dominating set" in [16]. Others [9, 12, 13] found Nordhaus-Gaddum type inequalities involving domination and codomination numbers. For a recent bibliography on domination see [11]. The investigation of graphs G for which an arbitrary invariant has the same value in G and $\overline{G}$ is surveyed in [1].

The diameter, denoted $diam(G)$, is the maximum distance between any two nodes of G. We establish a relationship between the diameter of a graph and its codomination number and consider the ramification of this results on characterizing graphs for which $d = \overline{d}$.

2. Codomination Number.

A bound on the diameter of $\overline{G}$ for the case when $diam(G) \geq 3$ is given in [10] as a lemma in proving that every nontrivial self-complementary graph has diameter 2 or 3.

Theorem A. [10] If $diam(G) \geq 3$, then $diam(\overline{G}) \leq 3$.

Furthermore, it is shown in [14] that if "regular" is added to the hypotheses, then the diameter of the complement is at most two. Brigham, Chinn, and Dutton [4] obtained an interesting relationship between the diameter and codomination number of a graph.

Theorem B. [4] If $\overline{d} \leq 3$, then $diam(G) \leq 2$.

If G is not connected, let $diam(G) = \infty$. We now determine the codomination number of any graph having diameter at least three.

Theorem 1. If G has no isolated nodes and $diam(G) \geq 3$, then $\overline{d} = 2$.

Proof. Let x and y be nodes of G such that $d(x, y) = diam(G) \geq 3$. Obviously, x and y form a dominating set for the complement of G. For any node z which is adjacent to neither x nor y in $\overline{G}$ is adjacent to both nodes in G, contrary to the hypothesis that $d(x, y) \geq 3$. It follows that the $\overline{d} \leq 2$. If $\overline{d} = 1$, then G would have an isolated node, again contrary to the hypothesis.

Although we have not characterized graphs which have equal domination and codomination numbers, our next result provides insight about their structure.

Corollary 1.1. Let G be a nontrivial connected graph having $d = \overline{d}$. Then either $d = \overline{d} = 2$ or $diam(G) = 2$.

Proof. Let G be a nontrivial graph with $d = \overline{d}$. If $diam(G) \geq 3$, then $d = \overline{d} = 2$ by Theorem 1. If $diam(G) = 1$, then $\overline{G}$ is disconnected and $d = 1 \neq \overline{d}$. In all other cases $diam(G) = 2$.

Corollary 1.2. Cycles C_n and paths P_n have $d = \overline{d}$ if and only if $n = 4, 5, 6$.

Obviously, self-complementary graphs have $d = \overline{d}$ and as mentioned above it follows directly from Theorem A that every nontrivial self-complementary graph has diameter 2 or 3.

Corollary 1.3. If G is a nontrivial self-complementary graph, then $diam(G) = diam(\overline{G}) = 2$ or $diam(G) = diam(\overline{G}) = 3$ and $d = \overline{d} = 2$.

Examples.

(1) The path P_4 is a self-complementary graph having $d = \overline{d} = 2$ and $diam = 3$.

(2) The composition of C_5 with itself, written $G = C_5[C_5]$, is a self-complementary graph having $d = \overline{d} = 3$ and $diam = 2$.

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