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PERFECT MEASURES AND THE DUNFORD-PETTIS PROPERTY*

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Abstract

Let X be a completely regular Hausdorff space. We denote by $C_b(X)$ the Banach space of all real-valued bounded continuous function's on X endowed with the supremum-norm. $M_p(X)$ denotes the subspace of the $(C_b(X), \|\ \|)'$ of all perfect measures on X and β_p denotes a topology on $C_b(X)$ whose dual is $M_p(X)$.

In this paper we give a characterization of E-valued weakly compact operators which are β-continuous on $C_b(X)$, where E denotes a Banach space. We also prove that $(C_b(X), \beta_p)$ has strict Dunford-Pettis property and, if X contains a σ-compact dense subset, $(C_b(X), \beta_p)$ has Dunford-Pettis property.

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1. Perpect Measures

Let X be a completely regular Hausdorff space, and let $C_b(X)$ denote the space of all bounded continuous real-valued functions on X ; $M(X)$ denotes the dual of $(C_b(X), \|\|)$ which is, by The Alexandroff Representation's Theorem, the space of all zero set regular, finitely additive set functions on the Baire Algebra $Ba^*(X)$. Basic references for the theory of measure on topological spaces are on [6]. A countable additive measure μ on a measurable space $(S, \mathcal{A})$ is said to be perfect, if for every $\mathcal{A}$-measurable function $g : S \longrightarrow \mathbb{R}$, there is a Borel set B in $\mathbb{R}$ such that $B \subset g(S)$ and $|\mu|(g^{-1}(B)) = |\mu|(S)$. Our principal references about perfect measures are on [4]. If X is a completely regular space, $M_p(X)$ denotes the space of all perfect measures defined on the σ-algebra $Ba(X)$ of Baire sets. It is known $M_\sigma(X) \supset M_p(X) \supset M_t(X)$ (for definitions of $M_\sigma(X)$, and $M_t(X)$ see [6] and, if X is a separable metric space, $M_p(X) = M_t(X)$ [4].

Perfect measures were first defined by Gnedenko and Kolmogorov in 1949. In 1975 Fremlim proved a very interesting result which reveals the usefulness of perfect measures on arbitrary measurable spaces Fremlim's theorem says that every sequence of measurable scalar functions on a finite perfect measurable space has either a subsequence that converges almost everywhere or it has no measurable pointwise cluster point.

Sentilles [5] defined three topologies on $C_b(X)$, socalled strict topologies, denoted by β_0, β and β_1 whose duals are $M_t(X), M_\tau(X)$ and $M_\sigma(X)$ respectively. Following this work, Koumoullis [4] defined another topology on $C_b(X)$, denoted by β_p, and showed that $(C_b(X), \beta_p)' = M_p(X)$. The definition of this strict topology is based on some special class of sets, the distinguishable sets of Frolik: a subset G of a completely regular space Y is distinguishable if there is a continuous function φ from Y onto a separable metric space such that $G = \varphi^{-1}(\varphi(G))$. We denote by $\mathcal{D}(Y)$ the family of distinguisable set in Y. $\mathcal{D}(Y)$ is a φ-algebra containing the σ-algebra of Baire sets. For every $Q \subset \beta X - X$, the spaces $C_b(X)$ and $C_b(\beta X - Q)$ are isomorphics since $\beta(\beta(X - Q) = \beta X$. So the topology β_0 on $C_b(\beta X - Q)$, which is defined as the finest locally convex topology agreeing with the compact-open topology on all norm bounded subsets, can be regarded as a topology on $C_b(X)$ which is denoted by β_Q. It is defined the strict topology β_p by $\beta_p = Lin\{\beta_G : G \in \mathcal{D}(X), G \subset \beta X - X\}$. Since $\beta_1 = Lin\{\beta_Z : Z$ in a zero set, $Z \subset \beta X - X\}$ and every zero set is distinguishable, we have β_1 is finer than β_p.

The following results can be found in [4].

Lemma 1. If $\mu \in M_\sigma^+(X)$, then μ is perfect if and only if for every continuous function f from X onto a separable metric space M , $f_*(\mu) \in M_t(M)$, where $f_*(\mu)(g) = \mu(g \circ f)$.

Theorem 2. A subset H of $M_p(X)$ is β_p-equicontinuous if and only if (a) H is bounded, and (b) for every continuous function f from X onto a separable metric space M and every $\epsilon > 0$, there is a compact K, $K \subset M$, such that $|\mu|(X - f^{-1}(K)) < \epsilon$ for all $\mu \in H$.

2. Perfects Measures and the Dunford-Pettis Property

We shall say that a Hausdorff locally convex space E has the Dunford-Pettis Property (resp. strict Dunford-Pettis Property) if for any Banach space F and every linear continuous weakly compact operator $T : E \longrightarrow F$, $T(C)$ is relatively compact (resp. $\{Tx_n\}$ is Cauchy) in F for any absolutely convex weakly compact subset C (resp. weak Cauchy sequence $\{x_n\}) \in E$.

Let T be a linear operator defined from $C_b(X)$ into a Banach E which is norm-continuous' let m denote its associated measure. The following relations will be used: $||Tf|| = \sup\{|x' \circ T(f)| : ||x'|| \leq 1\}$.

The first theorem of this work gives a characterization of the weakly compact operators defined on $C_b(X)$ into a Banach space E which are β_p-continuous.

Theorem 3: Let T be a bounded linear weakly compact operator defined on $C_b(X)$ into a Banach space E, and let m be its associated vector measure. The following statements are equivalent:

a) T is β_p-continuous.

b) For every continuous mapping g from X onto a separable metric space M, and for every $\epsilon > 0$, there exists a compact subset K_ϵ of M such that $||m||(X \backslash g^{-1}(K_\epsilon)) < \epsilon$.

Proof:

a) $\Rightarrow$ b) Suppose T is β_p-continuous, this implies that the subset $\{x' \circ m : ||x'|| \leq 1\}$ is β_p- equicontinuous; therefore, for every $\epsilon > 0$, and for every continuos map g from X onto a separable metric space M, there exists a compact subset K of M such that $|x' \circ m|(X \backslash g^{-1}(K)) < \epsilon/2$, for all $x', ||x'|| \leq 1$ [4]. The statement folows from the fact that

$$||m||(X \backslash g^{-1}(K)) = \sup\{|x' \circ m|(X \backslash g^{-1}(K)) : ||x'|| \leq 1\}.$$

b) $\Rightarrow$ a) It is enough to prove $H = \{x' \circ m : ||x'|| \leq 1\}$ is bounded, since $||m||(X) < \infty$. Also, by the fact $|x' \circ m|(A) \geq ||m||(A)$, for all $||x'|| \leq 1$ and for all Baire subset A of X, and from X onto a separable metric space M, there exists a compact subset K of M such that $|x' \circ m|(X \backslash g^{-1}(K)) < \epsilon$, for all x', $||x'|| \leq 1$. These two facts and Theorem 2 show H is β_p-equicontinuous. The statement follows from this since $||Tf|| = \sup\{|x' \circ T(f)| : ||x'|| \leq 1\}$ and $\{|x' \circ T(f)| : ||x'|| \leq 1\}$ is β_p-equicontinuous.

Theorem 4. $(C_b(X), \beta_p)$ has the strict Dunford-Pettis property.

Prooof: Since $\beta_p \geq \beta_1$, a linear operator β_p-continuous T is also β_1-continuous.

Let T be a linear β_1-continuous operator of $C_b(X)$ to E which is weakly compact, and let $\{f_n\}_{n\in \mathbb{N}}$ be a weakly Cauchy sequence in $C_b(X)$. Then $\{f_n(x)\}_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$ for each $x \in X$.

On the other hand, the associated vector measure m of T is σ-additive and then it admits a real-valued Baire control measure μ, that is, given $\epsilon > 0$, there exists $\delta > 0$ such that for every Baire subset F of $X, \mu(F) < \delta$ implies $||m||(F) < \epsilon$.

Now, since $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise in X, we have that, by Egoroff's Theorem, there exists a Baire subset F_δ of X such that $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly on $X - F_\delta$ and $\mu(F_\delta) < \delta$.

Let $n_0 \in \mathbb{N}$ such that for $n, m \leq n_0$

$$\sup\{||f_n(x) - f_m(x)|| : x \in X - F_\delta\} < \epsilon/(2M), \quad \text{where} \quad M = ||m||(X).$$

Thus

$$\begin{aligned} ||Tf_n - Tf_m|| &\leq || \int_{X-F_\delta} (f_n - f_m) dm|| + || \int_{F_\delta} (f_n - f_m) dm|| \\ &\leq \sup\{||f_n(x) - f_m(x)|| : x \in X - F_\delta\} ||m||(X) + L||m||(F_\delta) < \epsilon, \end{aligned}$$

where $||f_n|| \leq L$ for all $n \in \mathbb{N}$.

Theorem 5. If X is σ-compact, then $(C_b(X), \beta_p)$ has the Dunford-Pettis property.

Proof: Let $\{K_n\}$ be an increasing sequence of compact subset of X whose union $\cup K_n$ in dense in X. Denoting by B_n the closed unit ball of $(C(K_n), ||\ ||)'$, we get for any $\mu \in B_n$, $\tilde{\mu} \in M_p(X)$ defined by $\tilde{\mu}(f) = \mu(f_{|K_n})$. In fact, first of all observe that $\tilde{\mu} \in M_\sigma(X)$ since if $f_m \downarrow 0$, then $f_m \longrightarrow 0$ uniformly on K_n (by Dini's Theorem) which implies $\tilde{\mu}(f_m) = \mu(f_{m|K_n}) \longrightarrow 0$. Let g be a continuous function from X onto a separable metric space M, and consider the induced map $g_* : M_\sigma(X) \longrightarrow M_\sigma(M)$, defined by $g_*(\mu)(B) = \mu(g^{-1}(B))$ for all Baire subset of M. In order to prove $\tilde{\mu} \in M_p(X)$, by Lemma 1, we have to show $g_*(|\tilde{\mu}|) \in M_t(M)$. Take a $h_\alpha \longrightarrow 0$ in the compact-open topology on the unit ball of $C_b(M)$, without lost of generality we assume $h_\alpha \geq 0$, $g_*(|\tilde{\mu}|)(h_\alpha) = |\tilde{\mu}|(h_\alpha \circ g) \leq |\mu|((h_\alpha \circ g)_{|K_n}) \longrightarrow 0$ since $h_\alpha \longrightarrow 0$ uniformly on the compact subset $g(K_n)$, this says $g_*(|\mu|) \in M_t(M)$.

Putting $\tilde{B}_n = \{\tilde{\mu} : \mu \in B_n\}$ and taking a net $\{\tilde{\mu}_\beta\}$ such that $\mu_\beta \longrightarrow \mu \in B_n$ (note B_n is compact in weak*-topology), i.e. $\mu_\beta(f) \longrightarrow$

$\mu(f), \forall f \in C(K_n)$. This implies $\tilde{\mu}_\beta(f) \longrightarrow \tilde{\mu}(f) \quad \forall f \in C_b(X)$ and so $\tilde{B}_n$ is compact in $(M_p(X), \sigma(M_p(X), C_b(X))$.

The vector subspace $B = \cup_{n,m \in \mathbb{N}} m\tilde{B}_n$ is dense in $(M_p(X), \sigma(M_p(X), C_b(X))$, since if $f \in C_b(X)$ and $f \equiv 0$ on X. Therefore, $(M_p(X), \sigma(M_p(X), C_b(X))$ is σ-compact. The result now tollows from the fact that $(C_b(X), \beta_p)$ possesses the strict Dunford-Pettis property and Theorem 1 [3].

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