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THE STRONG STABLE FOLIATION THEOREM : A GEOMETRICAL PROOF*

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Abstract

We give a geometrical proof of the existence of the strong stable foliation for hyperbolic fixed or hyperbolic periodic points of diffeomorphisms.

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1. Introduction

In this article we give a geometrical proof of the theorem of existence of the strong stable foliation for C^r-diffeomorphisms ($r \geq 1$) in a neighborhood of an hyperbolic periodic or hyperbolic fixed point. An analytic proof of this theorem is given in [H-P-S], and an outline of the proof of the analogous theorem for vector fields appears in [Ta].

2. Statement of the theorem

Let M be a C^∞ compact boundaryless manifold. Let $f : M \to M$ be a C^r-diffeomorphism ($r \geq 1$) and $p \in M$ be a fixed point of f, i.e, $f(p) = p$. We say that p is hyperbolic if $Df(p)$ has no eigenvalues of norm one. If p is a periodic point with period ℓ, that is, $f^\ell(p) = p$ and $f^k(p) \neq p$ for $0 < k < \ell$, we say that p is hyperbolic if p is an hyperbolic fixed point of the diffeomorphism f^ℓ. It is well known that if p is an hyperbolic fixed point, then there is a splitting of T_pM (the tangent space of M at p) into two $Df(p)$-invariant subspaces, $E_p^s \oplus E_p^u$, i.e., $Df(p)E_p^s = E_p^s$ and $Df(p)E_p^u = E_p^u$, and that there exist constants $c > 0$ and $0 < \lambda < 1$ such that $||Df^n(p)/E_p^s|| \leq c\lambda^n$ and $||Df^{-n}(p)/E_p^u|| \leq c\lambda^n$, for each $n \in \mathbb{N}$ (see [dM-P]). Now, let $W^s(p) = \{x \in M : f^n(x) \to p$ as $n \to +\infty\}$ and $W^u(p) = \{x \in M : f^{-n}(x) \to p$ as $n \to +\infty\}$ be the stable and unstable sets of p, respectively. It is also well known that if p is an hyperbolic fixed point of f, then $W^s(p)$ and $W^u(p)$ are C^r-submanifolds injectively immersed in M and $T_pW^s(p) = E_p^s$, $T_pW^u(p) = E_p^u$. Now, if d denotes a distance in M and if $\varepsilon > 0$, we define the locally stable and unstable manifolds of size ε by $W_\varepsilon^s(p) = \{x \in M : d(f^n(x), p) < \varepsilon, n \geq 0\}$ and $W_\varepsilon^u(p) = \{x \in M : d(f^{-n}(x), p) < \varepsilon, n \geq 0\}$, respectively, which are C^r-submanifolds of M, and furthermore $W^s(p) = \bigcup_{n\geq 0} f^{-n}(W_\varepsilon^s(p))$, $W^u(p) = \bigcup_{n\geq 0} f^n(W_\varepsilon^u(p))$.

Let p be an hyperbolic fixed point of f. We say that f has the weakest contraction defined at p if there exists an eigenvalue, A, of $Df(p)$ such that:

i) A has multiplicity one;

ii) $|A| < 1$;

iii) if B is an eigenvalue of $Df(p)$, $B \neq A, \overline{A}$, and $|B| < 1$, then $|B| < |A|$.

Now, let p be an hyperbolic fixed point of f and $T_pM = E_p^s \oplus E_p^u$ be the splitting of T_pM. In what follows we will assume $E_p^s \neq \{0\}$ and that f has the weakest contraction defined at p; in addition, we will suppose that we have the following order for the norm of the contractive eigenvalues of $Df(p)$:

$$1 > |\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_k| > |\lambda_{k+1}| \geq \cdots \geq |\lambda_n| .$$

Theorem (of the strong stable foliation) Let M, f and p be as above and V be a neighborhood of p in $W^s(p)$. Then, in V, there exists a unique C^{r-1}-foliation, $\mathcal{F}^{ss}$, such that:

a) $W^s_\epsilon(p) = \bigcup_{x \in V} F^{ss}(x)$, where $F^{ss}(x)$ denotes the leaf of $\mathcal{F}^{ss}$ which contains the point x. Moreover, if $x, y \in V$, $x \neq y$, then $F^{ss}(x) = F^{ss}(y)$ or $F^{ss}(x) \cap F^{ss}(y) = \phi$.

b) $\mathcal{F}^{ss}$ is f-invariant, i.e., for each $x \in V$,

$$f(F^{ss}(x)) \subseteq F^{ss}(f(x)).$$

c) We let $Diff^r(M)$ denote the space of C^r-diffeomorphisms of M. Let $g \in Diff^r(M)$ be C^r-close to f and p_g be the (unique) fixed point (hyperbolic) of g in V; we let $V(p_g)$ denote a neighborhood of p_g in $W^s(p_g, g)$. Then, in $V(p_g)$, there exists a unique C^{r-1}-foliation, $\mathcal{F}^{ss}_{p_g}$, satisfying a) and b); in addition, the leaves of $\mathcal{F}^{ss}_{p_g}$ are C^{r-1}-close to the corresponding leaves of $\mathcal{F}^{ss}$.

Remarks

1- The codimension of $\mathcal{F}^{ss}$ in $W^s(p)$ is equal to k.

2- From item (b) of the theorem we have that, for each $x \in V$, $Df(x)T_xF^{ss}(x) = T_{f(x)}F^{ss}(f(x))$.

3- If for $Df(p)/E^u_p$ we assume similar hypotheses to the ones above, we obtain that there exists a strong unstable foliation, $\mathcal{F}^{uu}$, in a neighborhood U of p in $W^u(p)$, which is f^{-1}-invariant.

4- If p is an hyperbolic periodic point of f, say with period m, then the above result applied to the diffeomorphism f^m gives us the existence of a strong stable foliation, $\mathcal{F}^{ss}_p$, in a neighborhood of p in $W^s(p)$, which satisfies properties a), b) and c) of the Theorem. Now, if we let $\theta(p) = \{p, f(p), \cdots, f^{m-1}(p)\}$ denote the orbit of p, then if the foliation $\mathcal{F}^{ss}$ is iterated by $f^i, i = 1, \cdots, m-1$, we obtain a strong stable foliation $\mathcal{F}^{ss}(\theta(p)) = \bigcup_{i=1}^{m-1} f^i(\mathcal{F}^{ss}(p))$ of a neighborhood of $\theta(p)$ in $W^s(\theta(p)) = \bigcup_{i=1}^{m-1} f^i(W^s(p))$ which satisfies properties a), b) and c) of above.

A remark about the method of the proof of the Theorem:

Let A be an $n \times n$ matrix with real entries; we say that A is hyperbolic if it is non-singular and has no eingenvalues of norm one. We let $A : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto Ax$, denote the hyperbolic isomorphism induced by A. We consider the Grassmann manifold, $G(n, k)$, $0 \leq k \leq n-1$; the elements of $G(n, k)$ are the k-planes passing through the origin in $\mathbb{R}^n$ (vectorial subspaces of $\mathbb{R}^n$ having dimension k). It is clear that if $P \subseteq \mathbb{R}^n$ is a k-plane, then $A(P)$ is a k-plane and hence A induces a diffeomorphism $\hat{A}: G(n,k) \to G(n,k)$. Now, if $f: M \to M$ is a C^r-diffeomorphism ($r \geq 1$) and $p \in M$ is an hyperbolic fixed point of f, by taking a local chart at p, f can be written as $f(x) = Df(0)x + h(x)$, where $h(0) = 0$, $Dh(0) = 0$. In addition we have that $A = Df(p)$ is an hyperbolic $n \times n$ matrix. Now, if $N \subseteq M$ is a C^r, k-dimensional, f-invariant (i.e., $f(N) \subseteq N$) submanifold and $p \in N$, then the tangent space of N at p, T_pN, is $Df(p)$-invariant and therefore T_pN is an hyperbolic fixed point of the diffeomorphism $\hat{A}$.

Thus, the determination of the fixed points of $\hat{A}$ in $G(n,k)$ constitutes a natural proceedure for finding k-dimensional submanifolds of M which are f-invariant and contain the point p. Moreover, the analysis of the behavior of k-dimensional f-invariant submanifolds in a neighborhood of p is reduced to the analysis of a fixed point of $\hat{A}$ in $G(n,k)$. Consequently, if $P \in G(n,k)$ is an hyperbolic fixed point of $\hat{A}$ then, by the theorem of the local stability of hyperbolic fixed points (see [dM-P]), we obtain that there exist neighborhoods $\hat{\mathcal{V}}$ of $\hat{A}$ in $Diff^r(G(n,k))$ and $\hat{\mathcal{U}}$ of P in $G(n,k)$ such that each diffeomorphism $\hat{B} \in \hat{\mathcal{V}}$ has a unique fixed point $P_{\hat{B}} \in \hat{\mathcal{U}}$ which is hyperbolic. From this we conclude that there exist neighborhoods $\mathcal{V}$ of f in $Diff^r(M)$ and U of p in M such that, for each diffeomorphism $g \in \mathcal{V}$, there exists a k-dimensional g-invariant submanifold $N_g \subseteq M$ which is C^{r-1}-close to $N = N_f$.

We use in this article the central idea of the method described above in order to give a geometrical proof of the theorem of existence of the strong stable foliation for diffeomorphisms.

3. Proof of the theorem

Let $m = dimM$. Since our problem has a local nature, by taking a local chart $\theta: W \to \mathbb{R}^m$ at p, $\theta(p) = 0$, we may suppose that $f: U \subseteq \mathbb{R}^m \to \mathbb{R}^m$ is a C^r-diffeomorphism ($r \geq 1$) and that $0 \in U$ is an hyperbolic fixed point of f. We let $W^s(0)$ and $W^u(0)$ denote the stable and unstable manifolds of f at 0, respectively. Since $A = Df(0)$ is an hyperbolic isomorphism, we have an A-invariant splitting of $\mathbb{R}^m = E^s \oplus E^u$; in addition, we may suppose that $W^s(0)$ and $W^u(0)$ are neighborhoods of 0 in E^s and E^u, respectively (see [dM-P]). Now, since we are interested in a foliation of a neighborhood V of 0 in $W^s(0)$, we restrict ourselves to this case, that is, we suppose that $\dim E^s = m$ and $f: V \subseteq W^s(0) \to \mathbb{R}^m$ is a C^r-diffeomorphism ($r \geq 1$) and $0 \in V$ is an attractive hyperbolic fixed point of f (i.e., $||Df(0)|| \leq a < 1$).

First we give the proof of the theorem under the following hypotheses:

i) there exists the weakest contraction for f which is real,

ii) every eigenvalue of $Df(0)$ has multiplicity one.

From the above case we develop a procedure which allows us to prove the theorem for the remaining cases. In what follows we will work under hypotheses i) and ii).

Let $\lambda_1, \lambda_2, \cdots, \lambda_m$ be the eigenvalues of $Df(0)$ being ordered according to the value of their norm as follows:

$$1 > |\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_m|t, \qquad \lambda_1 \in I\!R.$$

In a neighborhood of the origin $0 \in I\!R^n$, f can be written as $f(x) = Df(0)x + h(x)$, with $h(0) = 0$ and $Dh(0) = 0$. Without loss of generality (making a linear change of coordinates), we suppose that $Df(0)$ is in its Jordan canonical form. Making a change of basis (if necessary) in $I\!R^m$ we may assume that $Df(0)$ is in Jordan canonical form and moreover has the form

$$Df(0) = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_j & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \alpha_i & \beta_i & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & -\beta_i & \alpha_i & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \alpha_s & \beta_s \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -\beta_s & \alpha_s \end{bmatrix}_{m \times m}$$

Employing the diagonal notation, we write

$$Df(0) = \operatorname{diag}\left(\lambda_1, \lambda_2, \cdots, \lambda_j, \begin{pmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{pmatrix} \cdots \begin{pmatrix} \alpha_s & \beta_s \\ -\beta_s & \alpha_s \end{pmatrix}\right)_{m \times m}.$$

Now we consider the C^∞ manifold $G(V, m, m-1) = \{(x, P) : x \in V, P \subseteq T_x I\!R^m$ subspace $(m-1)$–dimensinal $\}$, and let $\pi: G(V, m, m-1) \to V$ be the projection $\pi(x, P) = x$. We have that π is a C^∞ submersion and that $\pi^{-1}(x) = G(x, m, m-1)$, the Grassmann manifold of $(m-1)$ afine planes of $I\!R^m$ passing through the point x. Since for each $x \in V$, $G(x, m, m-1)$ is diffeomorphic to $G(m, m-1)$, we have that $G(x, m, m-1)$ is compact and therefore $\pi: G(V, m, m-1) \to V$ is a fibre bundle with fibre $G(m, m-1)$ (see [J-B]) and thus there exists a neighborhood U of 0 in $\mathbb{R}^m$ such that $\pi^{-1}(U)$ is diffeomorphic to $U \times G(m, m-1)$ and hence we have that the following diagram commutes

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\eta} & U \times G(m, m-1) \\ \pi \searrow & & \swarrow \pi_1 \\ & U & \end{array}$$

where $\pi_1(x, P) = x$ and η is a C^∞ diffeomorphism. We now define the maps $\Psi, \Phi: G(V, m, m-1) \longrightarrow G(V, m, m-1), \Psi(x, P) = (f(x), Df(0)P)$ and $\Phi(x, P) = (f(x), Df(x)P)$. We have that Ψ is C^r and Φ is C^{r-1}; in addition, Φ and Ψ are C^{r-1}- close diffeomorphisms.

Now, E^{ss} will denote the maximal $Df(0)$-invariant subspace which corresponds to the eigenvalues $\lambda_2, \lambda_3, \cdots, \lambda_m$ and E will denote the $Df(0)$-invariant subspace which corresponds to λ_1. It is clear that $\dim E^{ss} = m-1$, $\dim E = 1$ and $\mathbb{R}^m = E^{ss} \oplus E$. We have that $\Phi(0, E^{ss}) = (0, E^{ss}) = \Psi(0, E^{ss})$, that is, $(0, E^{ss}) \in G(V, m, m-1)$ is a fixed point for the diffeomorphisms Φ and Ψ.

We will now prove that $(0, E^{ss})$ is an hyperbolic fixed point for Φ and Ψ.

We have seen that there exists a neighborhood U of 0 in $\mathbb{R}^m$ and a C^∞ diffeomorphism $\eta: \pi^{-1}(U) \to U \times G(m, m-1)$ such that the following diagram commutes

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\Psi, \Phi} & \pi^{-1}(U) \\ \eta \downarrow & & \downarrow \eta \\ U \times G(m, m-1) & \underset{\tilde{\Psi}, \tilde{\Phi}}{\rightrightarrows} & U \times G(m, m-1) \end{array}$$

where $\tilde{\Psi} = \eta \circ \Psi \circ \eta^{-1}$ and $\tilde{\Phi} = \eta \circ \Phi \circ \eta^{-1}$.

Furthermore, we may choose η such that $\eta(0, E^{ss}) = (0, E^{ss})$; consequently we have

$$D\tilde{\Psi}(0, E^{ss}) = D\eta(0, E^{ss}) \circ D\Psi(0, E^{ss}) \circ (D\eta(0, E^{ss}))^{-1}, \quad \text{and}$$
$$D\tilde{\Phi}(0, E^{ss}) = D\eta(0, E^{ss}) \circ D\Phi(0, E^{ss}) \circ (D\eta(0, E^{ss}))^{-1}$$

Therefore $D\tilde{\Psi}(0, E^{ss})$ and $D\tilde{\Phi}(0, E^{ss})$ have the same eigenvalues as $D\Psi(0, E^{ss})$ and $D\Phi(0, E^{ss})$, respectively. Then it is suffices to prove that $(0, E^{ss})$ is an hyperbolic fixed point for $\tilde{\Psi}$ and $\tilde{\Phi}$. We next prove the latter. In effect, we have that $\tilde{\Psi}(x, P) = (\Psi_1(x, P), \Psi_2(x, P))$, where $\Psi_1(x, P) = x$ and $\Psi_2(x, P) =$

$Df(0)P$, then

$$D\tilde{\Psi}(0,E^{ss}) = \begin{bmatrix} D_1\Psi_1(0,E^{ss})_{m\times m} & O_{m\times(m-1)} \\ & \\ O_{(m-1)\times m} & D_2\Psi_2(0,E^{ss})_{(m-1)\times(m-1)} \end{bmatrix}_{(2m-1)\times(2m-1)}$$

where D_1 and D_2 are the partial derivatives with respect to the first and second variables, respectively, and $O_{k\times\ell}$ denotes the null matrix. It is clear that $D_1\Psi_1(0,E^{ss}) = Df(0)$. Moreover, since Ψ_2 does not depend on x, we may suppose that $\Psi_2\colon G(m,m-1) \to G(m,m-1)$, $\Psi_2(P) = Df(0)P$. To compute $D\Psi_2(E^{ss})$, we choose a neighborhood Z of E^{ss} in $G(m,m-1)$ such that each $(m-1)$-plane, P, in Z is represented by a matrix having the form (see [L])

$$P = \begin{bmatrix} x_1 & x_2 & x_3 & & x_{m-2} & x_{m-1} \\ 1 & 0 & 0 & & 0 & 0 \\ 0 & 1 & 0 & & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & & 0 & 1 \end{bmatrix}_{m\times(m-1)}$$

Thus, $Df(0)P$ is represented by the matrix

$$\begin{bmatrix} \lambda_1 x_1 & \lambda_1 x_2 & \lambda_1 x_j & \lambda_1 x_{j+1} & \lambda_1 x_{j+2} & \lambda_1 x_{m-2} & \lambda_1 x_{m-1} \\ \lambda_2 & 0 & 0 & 0 & 0 & 0 & \\ 0 & \lambda_3 & 0 & 0 & 0 & 0 & \\ & & & & & & \\ 0 & 0 & \lambda_j & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_i & \beta_i & 0 & \\ 0 & 0 & 0 & -\beta_i & \alpha_i & 0 & \\ & & & & & & \\ 0 & 0 & 0 & 0 & 0 & \alpha_s & \beta_s \\ 0 & 0 & 0 & 0 & 0 & -\beta_s & \alpha_s \end{bmatrix}_{(m-1)\times(m-1)}$$

Therefore the grassmannian coordinates of $Df(0)P$ are given by $Df(0)P \cdot B^{-1}$, where

$$B = \operatorname{diag}\left(\lambda_1,\cdots,\lambda_j, \begin{pmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{pmatrix} \cdots \begin{pmatrix} \alpha_s & \beta_s \\ -\beta_s & \alpha_s \end{pmatrix}\right),$$

and hence

$$B^{-1} = \operatorname{diag}\left(\lambda_1^{-1},\cdots,\lambda_j^{-1}, \begin{pmatrix} \alpha_i' & -\beta_i' \\ \beta_i' & \alpha_i' \end{pmatrix},\cdots, \begin{pmatrix} \alpha_s' & -\beta_s' \\ \beta_s' & \alpha_s' \end{pmatrix}\right)_{(m-1)\times(m-1)}$$

where

$$\lambda'_k = \lambda_k^{-1} \text{ and } \alpha'_k = \frac{\alpha_k}{\alpha_k^2 + \beta_k^2} \text{ and } \beta'_k = \frac{\beta_k}{\alpha_k^2 + \beta_k^2}.$$

Thus the grassmannian coordinates of $Df(0)P$ are given by the vector

$$v = \Big(\frac{\lambda_1}{\lambda_2} x_1, \cdots, \frac{\lambda_1}{\lambda_j} x_{j-1}, \lambda_1(\alpha'_i x_j + \beta'_i x_{j+1}), \lambda_1(-\beta'_i x_j + \alpha'_i x_{j+1}), \cdots,$$
$$\lambda_1(\alpha'_s x_{m-2} + \beta'_s x_{m-1}), \lambda_1(-\beta'_s x_{m-2} + \alpha'_s x_{m-1})\Big)$$

Consequently, the map Ψ_2 in local coordinates in a neighborhood of E^{ss} is given by the map $\overline{\Psi}_2 : \mathbb{R}^{m-1} \to \mathbb{R}^{m-1}$, $\overline{\Psi}_2(x_1, \cdots, x_{m-1}) = v$, and from this we obtain that

$$D\overline{\Psi}_2(0) = \lambda_1 \text{diag}\left(\frac{1}{\lambda_2}, \cdots, \frac{1}{\lambda_j}, \begin{pmatrix} \alpha'_i & \beta'_i \\ -\beta'_i & \alpha'_i \end{pmatrix}, \cdots, \begin{pmatrix} \alpha'_s & \beta'_s \\ -\beta'_s & \alpha_s \end{pmatrix}\right)$$

and also that the eigenvalues of $D\overline{\Psi}_2(0)$ are

$$\mu_1 = \frac{\lambda_1}{\lambda_2}, \cdots, \mu_{j-1} = \frac{\lambda_1}{\lambda_j}, \mu_j = \lambda_1(\alpha'_i + i\beta'_i), \cdots, \mu_{m-1} = \lambda_1(\alpha'_s + i\beta'_s).$$

Accordingly, the eingenvalues of $D\tilde{\Psi}(0, E^{ss})$ are:

$$\lambda_1, \cdots, \lambda_j \lambda_{j+1} = \alpha_i + i\beta_i, \cdots, \lambda_m = al_s + i\beta_s, \mu_1, \mu_2 \cdots \mu_{m-1}.$$

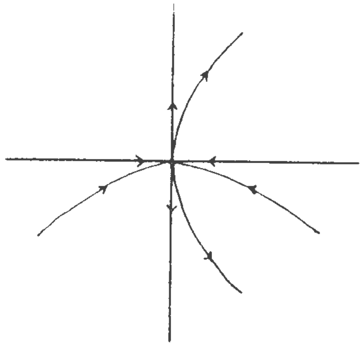
Since $1 > |\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_m|$, we have that $|\mu_1| = \frac{|\lambda_1|}{|\lambda_2|} > 1, \cdots, |\mu_{j-1}| = \frac{|\lambda_1|}{|\lambda_j|} > 1$.

Therefore $(0, E^{ss})$ is an hyperbolic fixed point of Ψ and, since the diffeomorphisms Ψ and Φ are C^{r-1}-close and the eigenvalues of an hyperbolic matrix depend continuously on the matrix (see [dM-P]), we have that $(0, E^{ss})$ is an hyperbolic fixed point of Φ.

Let $W^s_\Psi(0, E^{ss}) = \{(x, P) : \Psi^\ell(x, P) \to (0, E^{ss}), \ell \to +\infty\} = \{(x, P) : (f^\ell(x), Df^\ell(0)P) \to (0, E^{ss}), \ell \to +\infty\}$ be the stable manifold of $(0, E^{ss})$ in $G(V, m, m-1)$ with respect to Ψ. We have that $(x, P) \in W^s_\Psi(0, E^{ss})$ if and only if $x \in U$ and $Df^\ell(0)P \to E^{ss}$ as a distribution of $(m-1)$-planes, and, by the λ-lemma (in the linear case) (see [dM-P]), we obtain that $(x, P) \in W^s_\Psi(0, E^{ss})$ if and only if $P = x + E^{ss}$ (the parallel translation of E^{ss} to the point x). This characterization of P allows us to define a C^∞ distribution, D_0, of $(m-1)$-planes in $\mathbb{R}^m$ by $D_0(x) = x + E^{ss}$, $x \in U$. Thus $W^s_\Psi(0, E^{ss}) = \{(x, D_0(x)) : x \in U\}$. It is clear that the distribution D_0 defines a unique $C^\infty(m-1)$- dimensional foliation in U whose leaves are the submanifolds $D_0(x) \cap U$, $x \in U$. Thus we have obtained a C^∞ $\mathcal{F}_{\mathcal{L}}{}^{ss}$ strong stable foliation of $W^s_L(0) \cap U$, $L = Df(0)$, which is L-invariant.

We next prove the existence of a C^{r-1} f-invariant strong stable foliation, $\mathcal{F}^{ss}$, in a neighborhood of 0 in $W^s_f(0)$.

Recall that $\Phi(x, P) = (f(x), Df(x), P)$ is a C^{r-1} diffeomorphism and that $(0, E^{ss})$ is an hyperbolic fixed point of Φ. Since $W^s_\Phi(0, E^{ss})$ is tangent to $W^s_\Psi(0, E^{ss})$, and $W^u_\Phi(0, E^{ss})$ is transversal to $W^s_\Psi(0, E^{ss})$ at the point $(0, E^{ss})$, by the λ-lemma (see [dM-P]) it follows that $\Phi^{-k}(W^s_\Psi(0, E^{ss})) \to W^s_\Phi(0, E^{ss})$ when $k \to +\infty$, and the convergence is uniform in the compact parts of $W^s_\Phi(0, E^{ss})$.



Using the above, for each $k \in \mathbb{N}$, we may define a C^{r-1} distribution of $(m-1)$-planes, D_k, in U by

$$D_k(x) = Df^{-k}(f^k(x))D_0(f^k(x)).$$

We have that, for each $k \in \mathbb{N}$, D_k is completely integrable, because D_0 is integrable, and the integrable submanifolds of D_k are the submanifolds $f^k(D_0(x) \cap U)$.

We let D denote the C^{r-1} distribution of $(m-1)$-planes in U given by

$$D(x) = P \text{ if and only if } (x, P) \in W^s_\Phi(0, E^{ss}).$$

From the uniform convergence in the compact parts given by the λ-lemma, we have that if $B \subseteq W^s_\Phi(0, E^{ss})$ is a compact disc with center at $(0, E^{ss})$, then D_k converges uniformly in B, when $k \to +\infty$. From the latter and from the fact that for each $k \in \mathbb{N}$, D_k is completely integrable (the Fröbenius Theorem (see [L-C])), we have that D is completely integrable. Therefore D defines a C^{r-1} f-invariant foliation, $\mathcal{F}^{ss}$, of dimension $m-1$ in a neighborhood of 0 in $W_f^s(0)$ whose leaves are the integrable manifolds of D. We have thus proven the theorem in this case.

We will next see how the proof of the theorem can be obtained in the remaining cases.

1.- If the weakest contraction, λ_1, is complex, then it follows from a similar reasoning as above that there exists a C^{r-1} f-invariant strong stable foliation, $\mathcal{F}^{ss}$, of dimension $m-2$ in a neighborhood of 0 in $W_f^s(0)$.

2.- If the weakest contraction, λ_1, is real (resp. complex) and one of the eingenvalues, say $\lambda = \lambda_2$, has multiplicity $k > 1$ and, furthermore, $\lambda \in \mathbb{R}$, then, again, by reasoning as above, we obtain a C^{r-1} f- invariant strong stable foliation, $\mathcal{F}^{ss}$, of dimension $m-1$ (resp. of dimension $m-2$) in a neigborhood of 0 in $W_f^s(0)$. We remark here that when we must procure the grassmannian coordinates of $Df(0)P$, we must find the inverse of the submatrix B of $Df(0)P$ given by

$$\begin{bmatrix}
\lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & \lambda_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_4 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \lambda_j & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \alpha_i & \beta_i & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\beta_i & \alpha_i & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_s & \beta_s \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta_s & \alpha_s
\end{bmatrix}_{(m-1)\times(m-1)}$$

for which it is suffices to find the inverse of the submatrix B_1 of B given by

$$\begin{bmatrix}
\lambda & 0 & 0 & \cdots & 0 & 0 \\
1 & \lambda & 0 & \cdots & 0 & 0 \\
0 & 1 & \lambda & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \lambda & 0 \\
0 & 0 & 0 & \cdots & 1 & \lambda
\end{bmatrix}_{k\times k}$$

which is

$$B_1^{-1} = \begin{bmatrix} \lambda^{-1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ -\lambda^{-2} & \lambda^{-1} & 0 & 0 & \cdots & 0 & 0 \\ \lambda^{-3} & -\lambda^{-2} & \lambda^{-1} & 0 & \cdots & 0 & 0 \\ -\lambda^{-4} & \lambda^{-3} & -\lambda^{-2} & \lambda^{-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ (-1)^{2k+1}\lambda^{-k} & . & . & . & \cdots & -\lambda^{-4}\lambda^{-3} & -\lambda^{-2}\lambda^{-1} \end{bmatrix}.$$

A computation shows that the eigenvalues of $D\Psi(0, E^{ss})$ are: $\lambda_1, \lambda = \lambda_2$ of multiplicity $k, \lambda_3, \cdots, \lambda_j$
$\alpha_i + i\beta_i, \cdots, \alpha_s + i\beta_s; \mu_1 = \frac{\lambda_1}{\lambda}$ of multiplicity k, $\frac{\lambda_1}{\lambda_3}; \cdots, \frac{\lambda_1}{\lambda_j}, \lambda_1(\alpha_i' + i\beta_i), \cdots, \lambda_1(\alpha_s' + i\beta_s')$.

3.- If, in case 2), the eigenvalue λ of multiplicity $k > 1$ is complex, then, by reasoning analogously as before, the result follows.

From all of the above, we have proven the theorem for the case in which the weakest contraction for f exists.

Finally, if the eigenvalues of $Df(0)$ are ordered as follows: $1 > |\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_k| > |\lambda_{k+1}| \geq \cdots |\lambda_m|$, E^{ss} will denote the $Df(0)$-invariant maximal subspace which corresponds to the eigenvalues $\lambda_{k+1}, \cdots, \lambda_m$, and E will denote the $Df(0)$-invariant maximal subspace corresponding to the eigenvalues $\lambda_1, \cdots, \lambda_k$. As before, we define the C^∞ manifold $G(V, m, m-k) = \{(x, P)/x \in V, P \subseteq T_x\mathbb{R}^m$ codimension k subspace $\}$, and the diffeomorphisms $\Psi, \Phi: G(V, m, m-k) \to G(V, m, m-k)$, by $\Psi(x, P) = (f(x), Df(0)P)$ and $\Phi(x, P) = (f(x), Df(0)P)$. Now, by reasoning analogously as before, we obtain that $(0, E^{ss})$ is an hyperbolic fixed point for Ψ and Φ. For it is it suffices to note that the eingenvalues of $D\Psi_2(0)$ satisfy the following relations:

$$|\mu_j| \geq \frac{\min\{|\lambda_i|, \quad i = 1, \cdots, k\}}{\max\{|\lambda_s|, \quad s = k+1, \cdots, m\}}, \quad j = 1, \cdots, m-1.$$

Note that in this case the strong stable foliation, $\mathcal{F}^{ss}$, has codimension k in $W_f^s(0)$.

Remark We note that the main ingredient in the proof of the Theorem, is that there exists an attracting (resp. expanding) hyperbolic part in the splitting of the tangent space at M in p. Hence, when $f: M \to M$ is a C^r, $r \geq 1$, diffeomorphism which has a saddle-node fixed (or periodic) point, p, (see [N-P-T]) we may apply a similar argument to the above, obtaining a C^{r-1} strong stable (resp. strong unstable) foliation $\mathcal{F}^{ss}$ (resp. $\mathcal{F}^{uu}$) of $W^s(p)$ (resp. of $W^u(p)$) having the strong stable (resp. unstable) manifold, $W^{ss}(p)$ (resp. $W^{uu}(p)$) as a leaf; furthermore, we may extend this foliation to a C^{r-1} strong stable (resp. strong unstable) foliation of any center-stable (resp. center-unstable) manifold, $W^{cs}(p)$, (resp $W^{cu}(p)$) of p.

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REFERENCE

[dM-P] *de Melo, W.; Palis J.*: Geometric Theory of Dynamical Systems. *Springer-Verlag*, 1982.

[H-P-S] *Hirsch, M.; Pugh, C.; Shub, M.*: Invariant Manifolds, *Lectures Notes in Math., 583. Springer-Verlag*, 1977.

[Ta] *Takens F.*: Moduli of Stability for Gradient Vector Fields. *North-Holland Math. Studies 103, Singularities & Dynamical Systems. S.N. Pnevmatikos (Ed.)* Apendix 2 pp. 77-79, 1985.

[J-B] *Bröker, T.; Jänich K.*: Introduction to Differential Topology. *Springer-Verlag.*

[C-L] *Lins, A.; Camacho C.*: Geometric Theory of Foliations, *Birkhauser* 1985.

[L] *Lima, E.*: Variedades Diferenciáveis, *Monografias IMPA*,1973.

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