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ON A NEW ITERATION FOR SOLVING CHANDRASEKHAR'S H-EQUATION

IOANNIS K. ARGYROS*

Abstract.

A new iteration for solving Chandrasekhar's H-Equation is given. Under certain assumptions the iteration converges without the usual positivity assumptions on the parameters involved.

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* Department of Mathematics, New Mexico State University.
Las Cruces, New Mexico 88003, U.S.A.

Introduction.

In the theories of radiative transfer [2] and neutron transport [3] an important role is played by nonlinear integral equations of the form

$$H(s) = 1 + \lambda\, H(s) \int_0^1 k(s,t)H(t)\psi(t)dt. \tag{1}$$

In Eq. (1), λ is a real (or complex) parameter, $\psi \varepsilon X = L'[0,1]$ es given and real valued, the kernel $K(s,t)$ on $X\mathrm{x}X$ satisfies

(i) $0 < K(s,t) < 1,\ s,\ t\ \varepsilon\ X$

(ii) $K(s,t) + K(t,s) = 1,\ s,\ t\ \varepsilon\ X.$

and H is the function to be found.

The question has been answered under certain assumptions on λ (usually $|\lambda| \leq 1$) and positivity assumptions on ψ as in [1], [2], [4], [5] and the reference there.

We introduce a new iteration that converges for any λ for which a solution to Eq. (1) exists and do not require positivity assumptions on ψ and restrictions on λ as in [1], [4], [5].

I. BASIC THEOREMS.

We consider the Banach space C[0,1] of complex valued continuous functions on [0,1] with the supremum norm. We define a bounded linear operator and a bounded bilinear operator $\overline{B}$ [6] on C[0,1] by

$$(Lw)(s) = \int_0^1 K(s,t)\psi(t)w(t)dt$$

$$\overline{B}(w,z)(s) = w(s)(Lz)(s) + z(s)(Lw(s)).$$

Denote by $\overline{B}'(w)$ the Frechet derivative of $\overline{B}$ given by

$$\overline{B}(w)z = wLz + xLw \quad \text{for} \quad w,\ z \in C[0,1].$$

Define the linear operator E on a subset D of $C[0,1]$ by

$$E(w) = \int_0^1 w(t)H(t)\psi(t)dt$$

where H is a solution to Eq. 1.

Fact. Let $U = \int_0^1 H(t)\psi(t)dt$ and $\Psi = \int_0^1 \psi(t)dt$ then $U^2 - 2U + \Psi = 0$

The proof can be found in [2].

Theorem 1. Let $H = H(\lambda_0)$ be any solution to Eq. (1) with $\lambda = \lambda_0$. If $\int_0^1 \psi(t)dt \neq 0$ and $\mathrm{Kern}(L) = \{0\}$ then $\overline{B}'(H(\lambda_0))$ is an invertible linear operator on $C[0,1]$ with bounded inverse for any λ.

Proof. Let $z \in C[0,1]$ be such that

$$\overline{B}'(H(\lambda_0))z = 0$$

$$0 = \int_0^1 \psi(t)B(H(\lambda_0),z)(t)dt$$

$$= [\int_0^1 \psi(t)H(t)dt] \int_0^1 z(t)\psi(t)dt$$

Hence either $\int_0^1 z(t)\psi(t)dt = 0$ or $\int_0^1 \psi(t)H(t)dt = 0$.

By the fact before if $U = 0$ then $\Psi = 0$ which is a contradiction.

If $\int_0^1 z(t)\psi(t)dt = 0$ then

$$E(w) = 0 \quad \text{for} \quad w(t) = H^{-1}(t)z(t)$$

and since $\mathrm{Kern}(E) = \{0\} \rightarrow w = 0 \rightarrow z = 0$ and the theorem is proved.

We now introduce the iteration

$$H_n = H_{n-1} + (\overline{B}(H_{n-1}))^{-1}(H_{n-1} - 1 - \overline{B}(H_{n-1}, H_{n-1})). \tag{2}$$

Convergence of the above iteration is a consequence of Theorem 1 and Theorem 3 given in part II of this paper. Here to simplify computations as it is suggested in [4] we assume existence of a solution and seek only to show that the solution may be computed by the above iteration.

Theorem 2. Let $H(\lambda_0)$ be a solution to Eq.(1). Then there exists $\varepsilon_0 > 0$ such that if

$$||H_0 - H(\lambda_0)|| < \varepsilon_0$$

the iteration scheme given by (2) converges and

$$||H_{n+1} - H(\lambda_0)|| \leq || I - (\overline{B}(H(\lambda_0)))^{-1}||\cdot|| H_n - H(\lambda_0)||.$$

Proof. By Theorem 1, $(\overline{B}(H(\lambda_0)))^{-1}$ exist on D. As $\overline{B}(w)$ is continuous in w for fixed $\lambda = \lambda_0$, there exists $\varepsilon > 0$ such that if

$$|| H(\lambda_0) - w || < \varepsilon \quad \text{then} \quad (\overline{B}(w))^{-1} \text{ exists.}$$

Let $H_0 \varepsilon C[0,1]$ be such that

$$||H(\lambda_0) - H_0|| < \varepsilon_0 < \varepsilon$$

We show that if $||H(\lambda_0) - H_n|| < \varepsilon_0$ and ε_0 is sufficiently small then

$$||H_{n+1} - H(\lambda_0)|| < \varepsilon_0.$$

Let $d_n = H_n - H(\lambda_0)||$ and $H = H(\lambda_0)$, then

$$d_{n+1} = d_n + \overline{B}(H_n))^{-1}(H_{n-1}) - H_n$$

$$= d_n + \overline{B}(H(\lambda_0) + d_n)^{-1} [H + d_n - 1 - \overline{B}(H + d_n)(H + d_n)]$$

$$= (\overline{B}(H + d_n))^{-1}[d_n - \overline{B}(d_n, H)]$$

$$= [I - \overline{B}(H)^{-1}\overline{B}(-d_n)]^{-1}[\overline{B}(H)^{-1} - I](d_n)$$

$$= \sum_{k=0}^{\infty} [\overline{B}(H)^{-1}\overline{B}(-d_n)]^k[\overline{B}(H)^{-1} - I](d_n)$$

$$= \sum_{k=0}^{\infty} (\overline{B}(H)^{-1}B(-d_n))^k\overline{B}(H)^{-1}(d_n)$$

$$- \sum_{k=0}^{\infty} (\overline{B}(H))^{-1}\overline{B}(-d_n)^k(d_n)$$

By taking norms in the above equation we have

$$||d_n + 1|| \leq ||d_n||\ ||I - \overline{B}(H)^{-1}|| + 0\ (||d_n||^2).$$

If $||d_n|| < \varepsilon_0$ is sufficiently small, we have

$$||d_{n+1}|| \leq ||I - \overline{B}(H)^{-1}|| \cdot ||d_n|| < \varepsilon_0$$

and the theorem is proved.

II. EXISTENCE THEOREMS FOR THE SOLUTION OF THE QUADRATIC EQUATION.

We consider the quadratic equation $x = y + B(x,x)$ in a Banach space X, where $y \in X$ is fixed and B is a bounded bilinear operator on X.

Equation (1) is a special case of Eq. (3) provided that $\lambda\overline{B} = B$, $H = x$, $y = 1$ and $x = L'[0,1]$. We examine the convergence of the iterations

$$X_{n+1} = B(x_n)^{-1}(x_n - y) \quad \text{or} \quad x_{n+1} = X_n + B(x_n)^{-1}(x_n - y - B(x_n, x_n)).$$

Note that in case of convergence, a solution x of equation (3) is obtained.

We now state the following lemma. The proof can be found in [10].

Lemma 1. Let L_1 and L_2 be bounded linear operators in a Banach space X, where L_1 is invertible, and $||L_1^{-1}|| \cdot ||L_2|| < 1$. Then $(L_1 + L_2)^{-1}$ exists, and

$$|| (L_1 + L_2)^{-1} || \leq \frac{||L_1^{-1}||}{1-||L_2||\cdot||L_1^{-1}||}$$

Lemma 2. Let $z \neq 0$ be fixed in X. Assume that the linear operator $B(z)$ is invertible then $B(x)$ is also invertible for all $x \varepsilon U(z, r) = \{x \varepsilon X|\ ||x-z|| < r\}$, where $r \varepsilon (0, r_0)$ and $r_0 = [||B|| \cdot ||B(z)^{-1}||]^{-1}$.

Proof. We have

$$\begin{aligned} ||B(x-z)|| \cdot ||B(z)^{-1}|| &\leq ||B|| \cdot ||x-z|| \cdot ||B(z)^{-1}|| \\ &\leq ||B|| \cdot ||B(z)^{-1}|| \cdot r \\ &< 1 \end{aligned}$$

for $r \varepsilon (0,r_0)$. The result now follows from Lemma 1 for $L_1 = B(z)$, $L_2 = B(x-z)$ and $x \varepsilon U(z,r)$.

Definition 1. Let $z \neq 0$ be fixed in X. Assume that the linear operator $B(z)$ is invertible.

Define the operators P,T on U(z,r) by

$$P(x) = B(x,x) + y - x, \; T(x) = (B(x))^{-1}(x-y)$$

and the real polynomials f(r), g(r) on R by

$$f(r) = a'r^2 + b'r + c', \; g(r) = ar^2 + br + c,$$

$$a' = (||B|| \cdot ||B(z)^{-1}||)$$

$$b' = -2||B|| \cdot ||B(z)^{-1}||$$

$$c' = 1 - ||B(z)^{-1}|| - ||B|| \cdot ||B(z)^{-1}||^2 \cdot ||z-y||$$

$$a = r_0^{-1}$$

$$b = ||B(z)^{-1}(I-B(z))|| - 1$$

$$c = ||B(z)^{-1}P(z)||.$$

From now on we assume that B is a bounded symmetric bilinear operator on XxX.

Theorem 3. Let $x \in X$ be such that B(z) is invertible and that the following are true:

a) $c' > 0$;

b) $b < 0$, $b^2 - 4ac > 0$, and

c) there exists $r > 0$ such that $f(r) > 0$ and $g(r) \leq 0$.

Then the iteration

$$x_{n+1} = B(x_n)^{-1}(x_n - y), \quad n = 0, 1, 2, \ldots$$

is well defined and it converges to a unique solution x in $\overline{U}(z,r)$ of (3) for any $x_0 \in \overline{U}(z,r)$.

Proof. T is well defined by lemma 2.

Claim 1. T maps $\overline{U}(z,r)$ **into** $\overline{U}(z,r)$.

If $x \in \overline{U}(z,r)$ then

$$T(x) - z = B(x)^{-1}(x-y) - z$$

$$= B(x)^{-1}[(I - B(z))\ (x-z) - P(z)]$$

so

$$||\ T(x) - z\ || \leq r \quad \text{if}$$

$$\frac{1}{1-||B|| \cdot ||B(z)^{-1}||r}[||B(z)^{-1}(I - B(z))\ ||r + ||B(z)^{-1}P(z)||] \leq r$$

(using lemma 1 for $L_1 = B(z)$ and $L_2 = B(x-z)$) or $g(r) \leq 0$ which is true by hypothesis.

Claim 2. T is a contraction operator on $\overline{U}(z,r)$.

If $w, v \in \overline{U}(z,r)$ then

$$||T(w) - T(v)|| = ||B(w)^{-1}(w-y) - B(v)^{-1}(v-y)|| = ||B(w)^{-1}[I - B(B(v)^{-1}(v-y)](w-v)||$$

$$= ||B(w)^{-1}[I - B(B(v)^{-1}(v-z)) + B(B(v)^{-1}(z-y))](w-v)||$$

$$\leq \frac{}{1-||B||\cdot||B(z)^{-1}||\cdot r}\left[||B(z)^{-1}||+\frac{||B||\cdot||B(z)^{-1}||^2 r + ||B||\cdot||B(z)^{-1}||^2||z-y||}{1-||B||\cdot||B(z)^{-1}||\cdot r}\right]\cdot||w-v||$$

$$= q \cdot ||w-v||$$

So T is a contraction on $\overline{U}(z-r)$ if $0 < q < 1$, where

$$q = \frac{1}{1-||B|| \cdot ||B(z)^{-1}|| \cdot r}\left[||B(z)^{-1}||+\frac{||B|| \cdot ||B(z)^{-1}||^2 r + ||B||\,||B(z)^{-1}||^2||z-y||}{1 - ||B|| \cdot ||B(z)^{-1}|| \cdot r}\right]$$

which is true since $f(r) > 0$.

Note that Theorems 1 and 3 can now be used to compute the solution to Eq. (1) by iteration (4), if $z = H$ and $\overline{U}(H,r) \subset D$.

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