REVISTA PROYECCIONES Nº 15: 1-20
Diciembre 1988 - ISSN 0716-0917 DOI: 10.22199/S07160917.1988.0015.00001

ON THE MULTIPLICATIVE INVERSE EIGENVALUE PROBLEM

RICARDO SOTO M.*

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1. INTRODUCCION.

An important inverse eigenvalue problem is the problem of finding a density $q(x)$ such that the operator $\ell(u) = \frac{u''}{q(x)}$, with the appropriate boundary conditions, possesses a prescribed spectrum, that is, the inverse vibrating string problem:

(1.1)
$$-\frac{u''(x)}{q(x)} = \lambda u(x).$$
$$u(0) = u(\Pi) = 0.$$

Let $h = \frac{\Pi}{n+1}$ and $u_i = u(ih)$, $q_i = q(ih)$, $i = 1,2,\ldots,n$. Then, by using finite differences to approximate $u''(x)$, (1.1) becomes

* Académico Departamento de Matemáticas, Facultad de Ciencias y Humanidades, Universidad del Norte - Chile.

$$\frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2 q_i} = \lambda u_i$$

$$u_0 = u_{n+1} = 0$$

or in matrix notation

$$V^{-1} Au = \lambda u,$$

where

$$(1.2) \qquad A = \frac{1}{h^2} \begin{bmatrix} 2 & -1 & & & & 0 \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & -1 \\ 0 & & & & -1 & 2 \end{bmatrix},$$

$V = \mathrm{diag}\{q_1, q_2, \ldots, q_n\}$, $q_i > 0$, and $u = (u_1, u_2, \ldots u_n)^T$.

The problem es therefore: Given an n by n real symmetric matrix A, find a real diagonal matrix D, with positive diagonal entries, such that DA has a given spectrum $\{\lambda_i\}_1^n$. This problem is called the **Multiplicative Inverse Eigenvalue Problem (MIEP).**

We note that the matrix DA is not symmetric and therefore its eigenvalues need not to be real. However, since D is positive definite, $DAx = \lambda x$ implies

$$(D^{1/2} A D^{1/2}) D^{-1/2} x = \lambda D^{-1/2} x,$$

where $D^{1/2}AD^{1/2}$ is now a symmetric matrix. Then, the problem of finding D such that DA has eigenvalues $\{\lambda_i\}$ is equivalent to the problem of finding $D^{1/2}$ such that $D^{1/2}AD^{1/2}$ has eigenvalues $\{\lambda_i\}$.

A more general problem is the following:

Given $n+1$ real symmetric matrices $A_0, A_1, \ldots, A_n$ and n real number $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$, find the entries of the vector $d = (d_1, d_2, \ldots, d_n)^T \in \mathbb{R}^n$ such that the matrix

$$(1.3) \qquad A(d) = A_0 + \sum_{i=1}^{n} d_i A_i$$

has the given numbers λ_i's as its eigenvalues. The MIEP is obtained when $A_0 = 0$, $A_i = e_i r_i$, $i = 1, 2, \ldots, n$, where e_i is the i^{th} unit vector and r_i the i^{th} row of A.

The MIEP was posed by Downing and Householder [4] in 1956. To compute a solution to this problem, they proposed a numerical algorithm based on the Newton's method in which they assume the existence of a solution, although they did not discuss whether a solution is posible. We quote them ([4], p. 203, 1956): "Criteria for the existence of solutions do not appear to be known. In practical applications, however, circunstantial evidence may be sufficient to justify the assumption that at least one solution exists".

There have been important advances in the question of the existence of a solution to this problem in the last 20 years. Necessary and sufficient conditions are known, which we shall discuss in the following sections.

Consider the Multiplicative Inverse Eigenvalue Problem as defined above. If the diagonal entries of A are nonzero, we may assume, without loss of generality, that they are all one, since we can divide each row of A by its diagonal element and the solution to this modified problem will be of the form $\bar{d}_i = d_i a_{ii}$, whence we may compute the d_i's.

We can easily find some trivial examples, which show that there are cases in which for a given matrix A we can reach any desired spectrum, that is, if $\lambda(DA)$ denotes the vector of eigenvalues of DA, then $\{x \in \mathbb{R}^n : \lambda(DA) = x\} = \mathbb{R}^n$. We can also show that there are matrices for which certain spectra are inaccessible. In fact, if A is a diagonal matrix with $a_{ii} \neq 0$ for all i, then $\{\lambda(DA) : D$ is a real diagonal matrix$\} = \mathbb{R}^n$. This is also true if A is a triangular matrix with $a_{ii} \neq 0$ for all i. On the other hand, let A be any real singular matrix and $X = \{x \in \mathbb{R}^n : x_i \neq 0 \ \forall i\}$. Then, we cannot find any real diagonal matrix D such that

$$\lambda(DA) \notin X, \quad \text{for} \quad \prod_{i=1}^{n} \lambda_i(DA) = \det(DA) = \det(D)\det(A) = 0.$$

In the MIEP, the 2 by 2 case is easy to analyze completely, which we do in section 2. In section 3 we discuss the 3 by 3 case and we show that it can be completely determined when A is nonsingular or the desired spectrum is balanced. In section 4 we discuss necessary and sufficient conditions for matrices of order n. In most applications it is supposed that A is symmetric and positive definite and the desired spectrum is nonnegative.

2. THE 2 BY 2 CASE.

Let A be the real matrix $A = \begin{bmatrix} 1 & b \\ c & 1 \end{bmatrix}$, and $DA = \begin{bmatrix} d_1 & d_1 b \\ d_2 c & d_2 \end{bmatrix}$.

The characteristic polynomial of DA is

$$p(\lambda) = \lambda^2 - (d_1 + d_2)\lambda + d_1 d_2 (1 - bc).$$

If $bc = 0$, we are dealing with the trivial situation of a triangular matrix. If $bc = 1$, A is singular with spectrum $\{0, 2\}$, and DA is singular with spectrum $\{0, d_1 + d_2\}$. Excluding these cases, we note that if we want $p(\lambda)$ to have the real zeros λ_1 and λ_2, d_1 and d_2 must satisfy $d_1 + d_2 = \lambda_1 + \lambda_2$ and $d_1 d_2 = \frac{\lambda_1 \lambda_2}{1-bc}$, that is, d_1 and d_2 must be roots of

$$q(\lambda) = \lambda^2 - (\lambda_1 + \lambda_2) + \frac{\lambda_1 \lambda_2}{1-bc} = 0,$$

and the question boils down to whether or not the zeros of $q(\lambda)$ are real or imaginary. They are real if and only if $(\lambda_1 + \lambda_2)^2 \geq \frac{4\lambda_1 \lambda_2}{1 - bc}$, or equivalently

$$(\lambda_1 - \lambda_2)^2 \geq bc(\lambda_1 + \lambda_2)^2. \tag{2.1}$$

The inequality (2.1) always holds if $bc < 0$, so in that case λ_1 and λ_2 can be chosen arbitrarily. There is only a restriction on λ_1 and λ_2 if $bc > 0$. Let $\alpha = \sqrt{bc}$ and $x = \frac{\lambda_1}{\lambda_2}$. Then (2.1) becomes $|f(x)| \geq \alpha$, where $f(x) = \frac{x - 1}{x + 1}$. Now, $f(x) = \alpha$ when $x = x_o = \frac{1 + \alpha}{1 - \alpha}$ and $f(x) = -\alpha$ when $x = \frac{1}{x_o}$. As $f(x)$ is an increasing function, (2.1) holds as long as x does not lie in the interval whose end points are x_o and $\frac{1}{x_o}$. If, for example, $bc = \frac{1}{4}$ then $\alpha = \frac{1}{2}$, $x_o = 3$, and D can be formed such that DA has spectrum $\{\lambda_1, \lambda_2\}$ unless $\frac{1}{3} < \frac{\lambda_1}{\lambda_2} < 3$. Similary, if $bc = 4$,

$\alpha = 2$ and $x_o = -3$ and in this case D can be obtained unless $-3 < \frac{\lambda_1}{\lambda_2} < \frac{-1}{3}$.

The situation can be summarized as follows:

Given $A = \begin{bmatrix} 1 & b \\ c & 1 \end{bmatrix}$ and $\{\lambda_1, \lambda_2\}$, the problem is to find $D = \mathrm{diag}\{d_1, d_2\}$ such that DA has eigenvalues $\{\lambda_1, \lambda_2\}$. Then:

If λ_1 or λ_2 is zero, the problem has no solution with d_1 and d_2 distinct from zero unless $bc = 1$, in which case there are infinitely many solutions.

If neither λ_1 nor λ_2 is zero, the problem has no solution unless $bc \neq 1$. There is always a solution if $bc \leq 0$, and if $bc > 0$ there is a solution unless $\frac{\lambda_1}{\lambda_2}$ lies inside the interval whose endpoints are $\frac{1 + \sqrt{bc}}{1 - \sqrt{bc}}$ and $\frac{1 - \sqrt{bc}}{1 + \sqrt{bc}}$. In these cases there are generally two solutions, since d_1 and d_2 can be interchanged. The solution is unique only when $d_1 = d_2$, where d_1 and d_2 are the zeros of the polynomial.

$$\lambda^2 - (\lambda_1 + \lambda_2)\lambda + \frac{\lambda_1 \lambda_2}{1 - bc} .$$

3. THE 3 BY 3 CASE.

We denote by $M(S_3)$ the following problem: Given the real symmetric matrix

$$M = \begin{bmatrix} 1 & b_3 & b_2 \\ b_3 & 1 & b_1 \\ b_2 & b_1 & 1 \end{bmatrix} . \tag{3.1}$$

with eigenvalues μ_i, and the real numbers $\lambda_1, \lambda_2, \lambda_3$, find a real diagonal matrix $D = \mathrm{diag}\{d_1, d_2, d_3\}$, such that $\sigma(DM) = \{\lambda_1, \lambda_2, \lambda_3\}$, where $\sigma(DM)$ denotes the spectrum of the matrix DM.

The characteristic polynomial of DM is

$$(3.2) \qquad \lambda^3 - (\Sigma\, d_i)\lambda^2 + (\sum_{i<j} d_i d_j - b_1^2 d_2 d_3 - b_2^2 d_1 d_3 - b_3^2 d_1 d_2)\lambda -$$

$$- \Pi\, d_i (1 + 2\Pi\, b_i - \Sigma\, b_i^2).$$

Since $\Pi(\lambda - \lambda_i) = \lambda^3 - (\Sigma\, \lambda_i)\lambda^2 + (\sum_{i<j} \lambda_i \lambda_j)\lambda - \Pi\, \lambda_i$, we have

Proposition 3.1.

A necessary and sufficient condition for the problem $M(S_3)$ to have a solution is

$$\text{i)} \quad \Sigma\, d_i = \Sigma\, \lambda_i.$$

$$(3.3) \qquad \text{ii)} \quad \sum_{i<j} d_i d_j - \sum_{\substack{j<k \\ j,k\neq i}} b_i^2 d_j d_k = \sum_{i<j} \lambda_i \lambda_j.$$

$$\text{iii)} \quad \Pi\, d_i (1 + 2\Pi\, b_i - \Sigma\, b_i^2) = \Pi\, \lambda_i.$$

Let us assume that M is symmetric and nonsingular. The $\mu_i \neq 0\ \forall i$ and $\det(M) = 1 + 2\Pi\, b_i - \Sigma\, b_i^2 \neq 0$. By (3.3) iii) we have

$$\Pi d_i = \frac{\Pi \lambda_i}{\det(M)} = L.$$

Let $S = \Sigma \lambda_i$ and for simplicity let $d_1 = x$, $d_2 = y$, $d_3 = S - x - y$. By (3.3) iii) and ii), we have after some simplification

$$xyS - x^2y - xy^2 = L.$$

$$(b_2^2 - 1)x^2 + (b_1^2 - 1)y^2 + (b_1^2 + b_2^2 - b_3^2 - 1)xy -$$

$$- S(b_2^2 - 1)x - S(b_1^2 - 1)y = T$$

or

$$\text{(3.4)} \quad \begin{aligned} &\text{i)} \quad x^2y + xy^2 - xyS + L = 0. \\ &\text{ii)} \quad \eta_2 x^2 + \eta_1 y^2 + (\eta_2 + \eta_3)xy - S\eta_2 x - S\eta_1 y - T = 0, \end{aligned}$$

where $\eta_1 = b_1^2 - 1$, $\eta_2 = b_2^2 - 1$, $\eta_3 = b_1^2 - b_3^2$, $T = \sum_{i<j} \lambda_i \lambda_j$. Then, we have, the following

Proposition 3.2.

Let M be a symmetric nonsingular matrix and let $d_1 = x$, $d_2 = y$, $d_3 = S - x - y$. The problem $M(S_3)$ has a solution $D = \text{diag}\{d_1, d_2, d_3\}$ if and only if the nonlinear system (3.4) has a solution $\{x, y, S - x - y\}$.

We note that if M is singular, then it is clear from (3.3) iii) that a nonzero desired spectrum $(\lambda_i \neq 0 \; \forall i)$ cannot be reached for any diagonal matrix D.

Example 3.1.

Let $M = \begin{bmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 1/2 \\ 0 & 1/2 & 1 \end{bmatrix}$ and $\lambda_i = \{-1,0,1\}$. The equations (3.4) are

$$\text{i)}\quad x^2y + xy^2 = 0.$$

$$\text{ii)}\quad -x^2 - \frac{3}{4}y^2 - xy + 1 = 0.$$

Since $D = 0$ is not a solution, we can assume that $d_i \neq 0$ for at least one i, say $d_2 = y \neq 0$. Then from i) we have $x^2 = -xy$ and from ii) $\frac{3}{4}y^2 = 1$, whence $y = \pm \frac{2\sqrt{3}}{3}$. Thus, $x = 0, \pm \frac{2\sqrt{3}}{3}$. Therefore, a solution is given by $d_i = \{0, \pm \frac{2\sqrt{3}}{3}\}$. If $y = 0$, then $x^2 = 1$, whence $d_i = \{0, \pm 1\}$ is also a solution.

The desired spectrum of the example 3.1 is balanced and has a nice behaviour in general, which allows us to obtain better conditions for the existence of a solution, even when the given matrix is not symmetric.

Proposition 3.3

Let $B = (b_{ij})$ be an arbitrary real matrix with $b_{ii} = 1$ and let the desired spectrum be $\{-\alpha, 0, \alpha\}$. A sufficient condition for the existence of a real diagonal matrix D such that DB has the spectrum $\{-\alpha, 0, \alpha\}$ is that $b_{ij}b_{ji} < 1$ for at least one off-diagonal entry of B. In this case the solution is given by $D = \operatorname{diag}\{-c, 0, c\}$, where $c = \frac{\alpha}{\sqrt{1 - b_{ij}b_{ji}}}$ and 0 is in position (k,k) in D with $k \neq i,j$.

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Proof:

Suppose $b_{ij}b_{ji} < 1$ and 0 has the position (k,k), $k = i,j$, in D. Then, the k^{th} row of DB is a row of only zeros and the characteristic equation of DB is $\lambda(\lambda^2 - c^2) + c^2 b_{ij} b_{ji} \lambda = 0$. Thus, the eigenvalues of DB are 0, and $\pm c\sqrt{1 - b_{ij}b_{ji}}$, that is, $\{0, \pm\alpha\}$.

Corollary 3.4.

Let M be the matrix in (3.1). A sufficient condition for the problem $M(S_3)$ to have a solution is that $b_i^2 < 1$ for least one i $(i = 1,2,3)$. A solution is $D = \mathrm{diag}\{0, \pm c\}$, where $c = \dfrac{\alpha}{(1 - b_i^2)^{1/2}}$ and 0 has the position (i,i) in D.

Corollary 3.5.

The spectrum $\{-\alpha, 0, \alpha\}$, $\alpha \in \mathbb{R}$, is always obtainable for a 3 by 3 tridiagonal matrix and a solution is $D = \mathrm{diag}\{-\alpha, 0, \alpha\}$.

Example 3.2.

1. Let M be the matrix in (3.1) with $b_2^2 < 1$. Then, if

$D = \mathrm{diag}\{-c, 0, c\}$, $DM = \begin{bmatrix} -c & -cb_3 & -cb_2 \\ 0 & 0 & 0 \\ cb_2 & cb_1 & c \end{bmatrix}$ has the characteristic equation

$\lambda(\lambda^2 - c^2) + c^2 b_2^2 \lambda = 0$, with roots 0, and $\pm\alpha$, provided that $c = \alpha(1 - b_2^2)^{-1/2}$

2. $M = \begin{bmatrix} 1 & \sqrt{2}/2 & 0 \\ \sqrt{2}/2 & 1 & \sqrt{2}/2 \\ 0 & \sqrt{2}/2 & 1 \end{bmatrix}$, $\lambda_i = \{-\sqrt{17}, 0, \sqrt{17}\}$. M is

singular. A solution is $D^{(1)} = \text{diag}\{-3, -2, 5\}$, and if we apply corollary 3.4 we find two more solutions $D^{(2)} = \text{diag}\{-\sqrt{17}, 0, \sqrt{17}\}$ and $D^{(3)} = \text{diag}\{0, -\sqrt{34}, \sqrt{34}\}$.

3. $B = \begin{bmatrix} 1 & 1/2 & 2 \\ -2 & 1 & 3 \\ 1/4 & 0 & 1 \end{bmatrix}$, $\lambda_i = \{-\alpha, 0, \alpha\}$, $\alpha \in \mathbb{R}$. B is not symmetric. According to proposition 3.3 we must have at least 3 different solutions. In fact, they are $D^{(1)} = \text{diag}\{\frac{-\sqrt{2}}{2}\alpha, \frac{\sqrt{2}}{2}\alpha, 0\}$, $D^{(2)} = \text{diag}\{0, -\alpha, \alpha\}$ and $D^{(3)} = \text{diag}\{-\sqrt{2}\,\alpha, 0, \sqrt{2}\,\alpha\}$.

Remark 3.3.

We comment that the MIEP for matrices of order 3 can be explicitly solved when the desired spectrum is of the form $\{-\alpha, 0, \alpha\}$. In this case the matrix M can be nonsymmetric and also singular. In the case that the desired spectrum is not of the form $\{-\alpha, 0, \alpha\}$, but it contains one zero eigenvalue, we also have an explicit solution from (3.4) if we assume that M is symmetric nonsingular. In fact, if $\lambda_i = 0$ for some i, then $d_j = 0$ for some j, which leads to $\eta_1 y^2 - S\eta_1 y = T$ (if $x = 0$ and $\eta_1 \neq 0$) or $\eta_2 x^2 - S\eta_2 x = T$ (if $y = 0$ and $\eta_2 \neq 0$) or $(\eta_3 - \eta_1)xy = T$ (if $x + y = S$ and $\eta_3 - \eta_1 \neq 0$).

For an arbitrary spectrum $\lambda_i \neq 0\ \forall i$, we have a necessary condition, which says that the matrix M must be nonsingular. In this case we can still find an explicit solution (if it exists) from (3.4), as it is shown in the following example:

Example 3.4.

$$M = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}, \quad \lambda_i = -1,\ 1,\ 2. \quad \text{Here,} \quad \eta_1 = \eta_2 = 0, \quad \eta_3 = 1,$$

$T = -1, L = S = 2$. By (3.4)

$$x^2y + xy^2 - 2xy = -2$$

$$xy = -1,$$

whence $x,y = 2 \pm \sqrt{5}$. Thus, $d_1 = 2 - \sqrt{5}$, $d_2 = 2 + \sqrt{5}$, and $d_3 = -2$.

4. NECESSARY AND SUFFICIENT CONDITIONS FOR MATRICES OF ORDER n.

In this section we shall discuss necessary and sufficient conditions for the solvability of the problem $M(S_n)$, that is, the problem of finding a real diagonal matrix D such that DA has a prescribed spectrum $\{\lambda_i\}_{i=1}^n$ for a given $n \times n$ real symmetric matrix $A = (a_{ij})$. We assume that $a_{ii} = 1\ \forall i$.

Necessary Conditions.

The simplest necessary condition can be derived from the fact that $\det(DA) = \det(D)\det(A)$. That is,

Proposition 4.1.

Let the desired spectrum $\{\lambda_i\}$ be such that $\lambda_i \neq 0$ for all i and assume that the problem $M(S_n)$ has a solution. Then A is a nonsingular matrix.

Proposition 4.2.

If the problem $M(S_n)$ has a solution $D = \mathrm{diag}\{d_1,\ldots,d_n\}$, then

$$2\sum_{i<j} d_i d_j \leq \left(\sum_1^n \lambda_i\right)^2. \tag{4.1}$$

Proof:

Let $x = (x_1,x_2,\ldots,x_n)$ and let $E_k(x)$ denote the elementary symmetric function of degree k of the n elements $x_1,x_2,\ldots,x_n$, that is, $E_1(x) = \sum_{i=1}^n x_i$, $E_2(x) = \sum_{i<j}^n x_i x_j$, and so on.

We denote by $A[i_1,i_2,\ldots,i_k|i_1,i_2,\ldots,i_k]$ the principal submatrix of A lying in rows and columns $i_1,i_2,\ldots,i_k$. Then we have,

$$\det(\lambda I - DA) = \lambda^n + \sum_{k=1}^n (-1)^k \lambda^{n-k}\Big(\sum_{1\leq i_1<\ldots<i_k\leq n} \det A[i_1,\ldots,i_k|i_1,\ldots,i_k]\times$$

$$\times(d_{i_1} d_{i_2} \cdots d_{i_k})\Big).$$

Since $\prod_{i=1}^n (\lambda - \lambda_i) = \lambda^n + \sum_{k=1}^n (-1)^k E_k(\lambda_1,\lambda_2,\ldots,\lambda_n)\lambda^{n-k}$, then

$$\sum_{1\leq i_1<\ldots<i_k\leq n} \det A[i_1,\ldots,i_k|i_1,\ldots i_k] \prod_{j=1}^k d_{i_j} = E_k(\lambda_1,\ldots,\lambda_n).$$

Thus, for $k = 1,2$, we have $\sum_1^n d_i = \sum_1^n \lambda_i$ and

$$\sum_{i<j} d_i d_j (1 - a_{ij}^2) = \sum_{i<j} \lambda_i \lambda_j . \tag{4.2}$$

Since $\operatorname{tr}[(DA)^2] = \sum d_i^2 + 2 \sum_{i<j} d_i d_j a_{ij}^2 = \sum \lambda_i^2$,

$$\sum_1^n \lambda_i^2 - 2 \sum_{i<j} d_i d_j a_{ij}^2 \geq 0. \tag{4.3}$$

By (4.2) and (4.3), $\sum_{i<j} d_i d_j a_{ij}^2 = \sum_{i<j} d_i d_j - \sum_{i<j} \lambda_i \lambda_j$

$$\sum_1^n \lambda_i^2 - 2 \sum_{i<j} d_i d_j + 2 \sum_{i<j} \lambda_i \lambda_j \geq 0.$$

Thus, $2 \sum_{i<j} d_i d_j \leq \sum_1^n \lambda_i^2 + 2 \sum_{i<j} \lambda_i \lambda_j = (\sum_1^n \lambda_i)^2$, or

$$2 \sum_{i<j} d_i d_j \leq (\sum_1^n \lambda_i)^2 .$$

Theorem 4.3. (Friedland, 1977, [5]).

Let $A = (a_{ij})$ be a real symmetric positive definite matrix and let $M = (m_{ij})$, with $m_{ii} = a_{ii}$ $\forall i$, be any real symmetric matrix, then

$$\sum_{i=1}^n \lambda_i \lambda_i (A^{-1}M) \geq \sum_{i=1}^n \lambda_i \tag{4.4}$$

is a necessary condition for the existence of a real diagonal matrix D such that DA has the prescribed spectrum $\{\lambda_i\}$.

Friedland [5] also showed that if $\lambda_i \geq 0$, the assumption $m_{ii} = a_{ii}$ can be removed. In this case (4.4) becomes

$$(4.5) \qquad \sum_{i=1}^{n} \lambda_i \lambda_i (A^{-1}M) \geq (\sum_{i=1}^{n} \lambda_i)[\min_j m_{jj}].$$

Proposition 4.4.

Let A be a nonnegative matrix with $a_{ii} = 1$ $\forall i$. Suppose that $D = \mathrm{diag}\{d_1, d_2, \ldots d_n\}$ is such that $|d_i| \leq 1$ $\forall i$. If $\sigma(DA) = \{\lambda_i\}_1^n$, then

$$(4.6) \qquad |\lambda_i| \leq \rho(A), \quad \text{for all } i.$$

Proof:

Since $|d_i| \leq 1$ $\forall i$, $|d_i a_{ij}| \leq a_{ij}$ $\forall i,j$. Then by a theorem of Ky Fan (see Marcus and Minc, [7], p. 152), each eigenvalues of DA lies in at least one of the discs C_i: $|z - d_i| \leq \rho(A) - 1$. Hence, $|\lambda_i - d_i| \leq \rho(A) - 1$, whence $d_i - \rho(A) + 1 \leq \lambda_i \leq d_i + \rho(A) - 1$ and (4.6) follows.

Example 4.1.

1. $A = \begin{bmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 1/2 \\ 0 & 1/2 & 1 \end{bmatrix}$ has eigenvalues $\mu_i = 1 - \frac{\sqrt{2}}{2}$, 1, $1 + \frac{\sqrt{2}}{2}$. Let the desired spectrum be $\{-2, 0, 2\}$, so that for $\lambda = \pm 2$ we have $|\lambda_i| > \rho(A) = 1 + \frac{\sqrt{2}}{2}$. Any solution d_1, d_2, d_3, must satisfy $\Pi d_i = \frac{\Pi\lambda_i}{\Pi\mu_i} = 0$ and $\Sigma d_i = \Sigma\lambda_i = 0$, which implies that at least one of the d_i's must be zero, say $d_2 = 0$. Then $d_1 = -d_3$. By (4.2) we have

$$\sum_{i<j} d_i d_j (1 - a_{ij}^2) = -d_1^2 = \sum_{i<j} \lambda_i \lambda_j = -4,$$

which contradicts $|d_i| \leq 1 \ \forall i$. Thus, the given spectrum is inaccessible if the condition $|d_i| \leq 1 \ \forall i$ must be satisfied.

Sufficient Conditions.

All the known sufficient conditions for the MIEP have been derived by the use of the Brouwer fixed point theorem:

Theorem 4.5. (Hadeler, 1969, [6]).

Let $A = (a_{ij})$ be an n by n symmetric positive definite matrix with $a_{ii} = 1$ for all i and let $0 < \lambda_1 < \lambda_2 < \ldots\ldots < \lambda_n$ be the prescribed spectrum. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. if

$$\min_i (\lambda_{i+1} - \lambda_i) \geq 4 \| \lambda \|_\infty \max_i \Big(\sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}^2 \Big)^{1/2}, \tag{4.7}$$

then there exists $D = (d_i \delta_{ij})$ such that $(DA) = \{\lambda_i\}_{i=1}^n$.

Theorem 4.6. (de Oliveira, 1972, [2]).

Let $A = (a_{ij})$ be an n by n real matrix with $a_{ii} = 1$ for all i. If

$$\max_i S_i = \max_i \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \leq \frac{1}{2} \tag{4.8}$$

and the intervals

$$I_k = [\lambda_k(1 - 2S_k), \frac{\lambda_k(1 + S_k)}{1 - S_k}], \tag{4.9}$$

$k = 1,2,\ldots,n,$ are pairwise disjoint, then there exists a real diagonal matrix $D = (d_i \delta_{ij})$ such that $\sigma(DA) = \{\lambda_i\}_{i=1}^n$.

Next, we derive a sufficient condition for the problem $M(S_n)$ using the Brouwer fixed point theorem.

Let $K = \prod_{j=1}^{n} I_j$, where $I_j = [\lambda_j - r, \lambda_j + r]$, where $r > 0$ will be determined. Let $d = (d_1, d_2, \ldots, d_n) \in K$. Then $\| d - \lambda \|_\infty \leq r$, whence $\| d \|_\infty \leq \| \lambda \|_\infty + r$. We consider the Gerschgorin circles of DA

$$C_j : \{z \in \mathbb{C}: |z - d_j| \leq |d_j| \sum_{\substack{k=1 \\ k \neq j}} |a_{jk}|\}$$

which are disjoint if $\min_i |\lambda_{i+1} - \lambda_i| > 2r + 2m(DA)$, where

$$m(DA) = \max_i \sum_{\substack{j=1 \\ j \neq i}}^{n} |d_i||a_{ij}| = \max_i |d_i| \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \quad \text{and} \quad \lambda_i < \lambda_{i+1}, \ 1 \leq i \leq n-1$$

Since $m(DA) \leq \| d \|_\infty (\| A \|_\infty - 1) \leq (\| \lambda \|_\infty + r)(\| A \|_\infty - 1)$, the Gerschgorin circles of DA are disjoint if

$$\min_i |\lambda_{i+1} - \lambda_i| > 2r + 2(\| \lambda \|_\infty + r)(\| A \|_\infty 1\ 1). \tag{4.10}$$

With the assumption (4.10) and by the Gerschgorin theorem, DA has n real eigenvalues λ_j^*, such that

$$|d_j - \lambda_j^*| \leq |d_j| \sum_{\substack{k=1 \\ k\neq j}}^{n} |a_{jk}| \leq ||\ d\ ||_\infty(||\ A\ ||_\infty - 1)$$

$$\leq (||\ \lambda\ ||_\infty + r)(||\ A\ ||_\infty - 1).$$

Here we assume that λ_j^* lies in C_j. The mapping $T: K \longrightarrow \mathbb{R}^n$, defined by $T(d) = \lambda + d - \lambda^*$ maps K into a subset of itself. In fact,

$$|T(d)_j - \lambda_j| = |d_j - \lambda_j^*| \leq (||\ \lambda\ ||_\infty + r)(||\ A\ ||_\infty - 1).$$

Now, we define r in such a way that $(||\ \lambda\ ||_\infty + r)(||\ A\ ||_\infty - 1) \leq r$, that is,

$$r = \frac{||\ \lambda\ ||_\infty(||\ A\ ||_\infty - 1)}{2 - ||\ A\ ||_\infty}, \quad 1 \leq ||\ A\ ||_\infty < 2.$$

With r defined in that form, (4.10) becomes

$$\min|\lambda_{i+1} - \lambda_i| > 2r||\ A\ ||_\infty + 2||\ \lambda\ ||_\infty(||\ A\ ||_\infty - 1) = 4r$$

and also $T(d) \in K$, whence T has a fixed point d^* in K. Hence, the problem $M(S_n)$ has a solution $D = (d_i^* \delta_{ij})$. Thus we have,

Theorem 4.5.

Let the desired spectrum in the problem $M(S_n)$ be such that $\lambda_i < \lambda_{i+1}$, $i = 1,2,\ldots,n-1$ and $r = \dfrac{\|\lambda\|_\infty(\|A\|_\infty - 1)}{2 - \|A\|_\infty}$. If $\|A\|_\infty < 2$, and

$$\min_i |\lambda_{i+1} - \lambda_i| > 4r, \tag{4.11}$$

then the problem $M(S_n)$ is solvable.

The inequality (4.11) is equivalent to that given by Biegler-Konig [1]. We note from the definition of r and since $1 \leq \|A\|_\infty < 2$, that the best situation is when $\|A\|_\infty = 1$, that is, the case A equals the identity matrix, which always has a solution. If $\|A\|_\infty$ is close to 2, then the condition (4.11) is not useful any more. We also note that we may choose a different value for r. For example we can set $r = \bar{\mu} = \max_i\{\mu_i\}$, which is positive since $\Sigma \mu_i = n$, where the μ_i's are the eigenvalues of A. In this case the problem $M(S_n)$ has a solution provided that

$$\min_i |\lambda_{i+1} - \lambda_i| > 2\bar{\mu} + 2(\|\lambda\|_\infty + \bar{\mu})(\|A\|_\infty - 1) \tag{4.12}$$

and

$$\|\lambda\|_\infty \leq \frac{\bar{\mu}(2 - \|A\|_\infty)}{\|A\|_\infty - 1}. \tag{4.13}$$

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