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Existence of solutions for a nonlinear fractional system with nonlocal boundary conditions

A. Guezane-Lakoud
University Badji Mokhtar Annaba, Algeria
G. Rebiai
University of Guelma, Algeria and
R. Khaldi
University Badji Mokhtar Annaba, Algeria
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Abstract

In this paper, we use fixed point theorems to prove the existence and uniqueness of solution for a nonlinear fractional system with boundary conditions. At the end we present two examples illustrating the obtained results.

Keywords: *Fractional Rieman-Liouville derivative, Fractional differential equation, Fixed Point Theorem.*

Mathematics Subject Classification: *34B10, 26A33, 34B15.*

1. Introduction

In recent years, the theory of differential fractional equations has become an interesting field to explore. It is to be noted that such theory has many applications in several events existing in the real world, and also in many sciences such as: engineering, physics, chemistry, biology, etc ..., [13]. Moreover, the study of the systems of fractional differential equations has become more and more popular tool for controlling and modeling different systems [2,7,15-17]. Thus the fixed point theory is a powerful mathematical tool in the study of the existence, uniqueness, positivity and stability of solutions, see [1,3-6,9-14].

In this work, we consider the following system of fractional differential equations with boundary conditions:

$$(FS)\begin{cases} -D_{0^+}^{\alpha}u\left(t\right)=g\left(t\right)f\left(u\left(t\right)\right),0<t<1,\\ u\left(0\right)=u'\left(0\right)=0,\ au\left(1\right)+bu'\left(1\right)=0,\end{cases}$$

where $D_{0^+}^{\alpha}$ denotes the Reimann-Liouville fractional derivative, $2<\alpha<3$, $u=\left(u_1,u_2,...,u_n\right)^T$ is an unknown function with

$u_i:[0,1]\rightarrow\mathbf{R}, g:[0,1]\rightarrow\mathbf{R}$ is a given function, $f:\mathbf{R}^n\rightarrow\mathbf{R}^n$, $f\left(u\right)=\left(f_1\left(u_1,u_2,...,u_n\right),...,f_n\left(u_1,u_2,...,u_n\right)\right)^T$, $f_i:\mathbf{R}^n\rightarrow\mathbf{R}$.

This paper is organized as follows: in Section 2, some preliminary materials to be used later are stated. In Section 3, we present and prove our main results consisting of the existence and uniqueness of the solution of (FS). Finally our study is ended by an example illustrating the obtained results.

2. Preliminaries

In this section, we recall the basic definitions and lemmas from the fractional calculus theory, see [13].

Definition 1. *The Riemann-Liouville fractional integrals of order α of a function h is defined as*

$$I_{a^+}^{\alpha}h\left(t\right)=\frac{1}{\Gamma(\alpha)}\int_a^t\frac{h(s)}{(t-s)^{1-\alpha}}ds.$$

Definition 2. *The Riemann-Liouville derivative of fractional order $\alpha>0$ for a function h is defined as*

$$D_{0^+}^{\alpha}h\left(t\right)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_0^t\left(t-s\right)^{n-\alpha-1}h\left(s\right)ds=\left(\frac{d}{dt}\right)^nI^{n-\alpha}h\left(t\right),$$

where $n=[\alpha]+1$ ($[\alpha]$ denotes the integer part of the real number α).

Lemma 3. *For $\alpha > 0$, the general solution of the homogeneous equation*

$$D_{0^+}^{\alpha} u(t) = 0,$$

is given by

$$u(t) = c_0 t^{\alpha-n} + c_1 t^{\alpha-n-1} + + c_{n-2} t^{\alpha-2} + c_{n-1} t^{\alpha-1},$$

where c_i, $i = 1, 2, ..., n-1$, are arbitrary real constants.

Lemma 4. *Let p, $q \geq 0$, $f \in L_1[a, b]$. Then*

$$I_{0^+}^{p} I_{0^+}^{q} f(t) = I_{0^+}^{p+q} f(t) = I_{0^+}^{q} I_{0^+}^{p} f(t).$$

3. Main results

Lemma 5. Let $y \in C([0,1], \mathbf{R})$. Assume that $a, b \in \mathbf{R}$ such that $a - b(\alpha - 1) \neq 0$, then for $i \in \{1, .., n\}$, the linear nonhomogeneous problem

$$(S_i) = \begin{cases} -D_{0^+}^{\alpha} u_i(t) = y(t), \ 0 < t < 1, \\ u_i(0) = u_i'(0) = 0, \ a u_i(1) - b u_i'(1) = 0, \ i \in \{1, .., n\}, \end{cases}$$

(3.1)

has the following solution

$$u_i(t) = \int_0^1 G_i(t,s) y(s) ds, i \in \{1, .., n\} \tag{3.2}$$

where

$$G_i(t,s) = \begin{cases} -\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \\ \frac{t^{\alpha-1}}{a - b(\alpha-1)} \left(\frac{a}{\Gamma(\alpha)} (1-s)^{\alpha-1} - \frac{b}{\Gamma(\alpha-1)} (1-s)^{\alpha-2} \right), s \leq t, \\ \frac{t^{\alpha-1}}{a - b(\alpha-1)} \left(\frac{a}{\Gamma(\alpha)} (1-s)^{\alpha-1} - \frac{b}{\Gamma(\alpha-1)} (1-s)^{\alpha-2} \right), s \geq t. \end{cases}$$

(3.3)

Proof. Let u_i be a solution of the fractional boundary value problem (FS). Using Lemma 3, we obtain

$$u_i(t) = -I_{0+}^{\alpha} y(t) + At^{\alpha-1} + Bt^{\alpha-2} + Ct^{\alpha-3}, \tag{3.4}$$

then, by multiplying (3.4) by $t^{3-\alpha}$, it yields
$t^{3-\alpha} u_i(t) = -I_{0+}^{\alpha} y(t)\, t^{\alpha-3} + At^2 + Bt + C.$

According to the condition $u(0) = 0$, we obtain $C = 0$. Therefore, differentiating (3.4), we have

$$u_i'(t) = -I_{0+}^{\alpha-1} y(t) + (\alpha-1) At^{\alpha-2} + (\alpha-2) B. \tag{3.5}$$

Multiplying (3.5) by $t^{3-\alpha}$, we obtain

$$t^{3-\alpha} u_i'(t) = -I_{0+}^{\alpha-1} y(t)\, t^{3-\alpha} + (\alpha-1) At + (\alpha-2) B.$$

(3.6)

From condition $u_i'(0) = 0$, it follows $B = 0$, thus,

$$u_i(t) = -I_{0+}^{\alpha} y(t) + At^{\alpha-1}. \tag{3.7}$$

Since $au_i(1) - bu_i'(1) = 0$, then

$$A = \frac{a}{a - b(\alpha-1)} I_{0+}^{\alpha} y(1) - \frac{b}{a - b(\alpha-1)} I_{0+}^{\alpha-1} y(1). \tag{3.8}$$

By substituting A in (3.7), we get
$u_i(t) = \int_0^1 G_i(t,s)\, y(s)\, ds.$ □

Lemma 6. If $a > 0$ and $b < 0$, then the functions G_i are nonnegative, continuous
and

$$G_i(t,s) \leq \frac{1}{\Gamma(\alpha-1)}, \forall s,t \in [0,1], i \in \{1,..,n\}. \tag{3.9}$$

Proof. The proof is direct, we omit it. □

Let X be the Banach space of all functions $u \in C^n[0,1] = C[0,1] \times ... \times C[0,1]$ with the norm $\|.\|$ defined by $\|u\| = \sum_{i=1}^{i=n} \max_{t\in[0,1]} |u_i(t)|$. Define the integral operator $T : X \to X$ by $T(u) = (T_1 u, T_2 u, ..., T_n u)$, where

$$(T_i u)(t) = \int_0^1 G_i(t,s)\, g(s)\, f_i(u(s))\, ds, i = 1, ..., n. \tag{3.10}$$

Lemma 7. The function $u \in X$ is a solution of the system (FS) if and only if $T_i u(t) = u(t)$, for all $t \in [0,1]$, $\forall i \in \{1, ..., n\}$.

The first main statement in this work is the uniqueness of solution of the boundary problem (FS).

Theorem 8. Assume that

i) $f_i \in C(\mathbf{R}^n, \mathbf{R})$, $g \in L^1([0,1], \mathbf{R})$

ii) There exists a constant $L > 0$ such that

$$|f_i(x_1, .., x_n) - f_i(y_1, ..., y_n)| \leq L \sum_{i=1}^{n} |x_i - y_i| \tag{3.11}$$

and

$$\frac{nL \|g\|_{L^1[0,1]}}{\Gamma(\alpha - 1)} < 1, \tag{3.12}$$

for all $t \in [0,1]$ and for all $x_i, y_i \in \mathbf{R}$, $i = 1, ...n$. Then, the boundary value problem (FS) has a unique solution in X.

Proof. We will use the Banach contraction principle to prove that the operator T has a fixed point. Using the properties of the function G_i, it yields

$$\begin{aligned} |T_i x(t) - T_i y(t)| &\leq \int_0^1 |G_i(t,s)|\, |g(s)|\, |f_i(x(s)) - f_i(y(s))|\, ds \\ &\leq \frac{L}{\Gamma(\alpha - 1)} \int_0^1 |g(s)| \sum_{i=1}^{n} |x_i(s) - y_i(s)|\, ds \\ &\leq \frac{L}{\Gamma(\alpha - 1)} \|g\|_{L^1[0,1]} \|x - y\|, \end{aligned}$$

then by taking the maximum over $t \in [0,1]$, it follows

$$t \in [0,1] \max |T_i x(t) - T_i y(t)| \leq \frac{L}{\Gamma(\alpha - 1)} \|g\|_{L^1[0,1]} \|x - y\|. \tag{3.13}$$

Summing the n inequalities in (3.13), it yields
$\|Tx - Ty\| \leq \frac{nL\|g\|_{L^1[0,1]}}{\Gamma(\alpha-1)} \|x-y\|.$

Since $\frac{nL\|g\|_{L^1[0,1]}}{\Gamma(\alpha-1)} < 1$, then T is a contraction. As a consequence of Banach fixed-point theorem, we deduce that T has a fixed point that is the unique solution of the (FS), this achieves the proof. □

The second mains statement of this work is an existence result for the boundary problem (FS).

Theorem 9. Assume that $f_i(0) \neq 0, i \in \{1,..,n\}$, there exist $\eta > 0$ and a nonnegative function $\Psi \in C(\mathbf{R}^n, (0,\infty))$ satisfying $\Psi(x_1,...,x_n) \leq \Psi(y_1,...,y_n)$ for $0 \leq x_i \leq y_i$, $i = 1,...,n$. If

$$|f_i(u)| \leq \Psi(|u|), \tag{3.14}$$

for all $t \in [0,1]$ and all $u \in \mathbf{R}^n$ and

$$\frac{n}{\Gamma(\alpha-1)} \Psi(\eta,...,\eta) \|g\|_{L^1[0,1]} \leq \eta, \tag{3.15}$$

then, the problem (FS) has at least one nontrivial solution $u^* \in X$.

For the proof of Theorem we need the nonlinear alternative of Leray-Schauder:

Lemma 10. Let F be a Banach space and Ω a bounded open subset of F, $0 \in \Omega$. Let $T : \Omega \to F$ be a completely continuous operator. Then, either there exists $x \in \partial\Omega$, $\lambda > 1$ such that $T(x) = \lambda x$, or there exists a fixed point $x \in \Omega$ of T.

Proof. of Theorem 9. The continuity of the operator T follows from the continuity of f. Set $B_\eta = \{u \in X : \|u\| \leq \eta\}$. Let us prove that $T : B_\eta \to X$ is a completely continuous operator. From (3.14), we have for each $t \in [0,1]$

$$|T_i u(t)| \leq \int_0^1 |G_i(t,s)| |g(s)| |f_i(u(s))| ds \tag{3.16}$$

$$\leq \frac{1}{\Gamma(\alpha-1)} \int_0^1 |g(s)| \Psi(|u(s)|) ds \tag{3.17}$$

$$= \frac{1}{\Gamma(\alpha-1)}\int_0^1 |g(s)|\,\Psi(|u_1(s)|,...,|u_2(s)|)\,ds \tag{3.18}$$

$$\leq \frac{1}{\Gamma(\alpha-1)}\Psi(\eta,...,\eta)\,\|g\|_{L^1[0,1]} \tag{3.19}$$

Taking the supremum over $[0,1]$, then summing the obtained inequalities according to i from 1 to n, we get

$$\|Tu\| \leq \frac{n\Psi(\eta,...,\eta)\,\|g\|_{L^1[0,1]}}{\Gamma(\alpha-1)} \tag{3.20}$$

which implies that $T(B_\eta)$ is uniformly bounded.
Let us show that (Tu) is equicontinuous, $u \in B_\eta$. Let $t_1, t_2 \in [0,1]$, $t_1 < t_2$, then

$$\begin{aligned}
|T_iu(t_1) - T_iu(t_2)| &\leq \int_0^1 |G_i(t_1,s) - G_i(t_2,s)|\,|g(s)|\,|f_i(u(s))|\,ds \\
&\leq \int_0^{t_1} |G_i(t_1,s) - G_i(t_2,s)|\,|g(s)|\,|f_i(u(s))|\,ds \\
&\quad + \int_{t_1}^{t_2} |G_i(t_1,s) - G_i(t_2,s)|\,|g(s)|\,|f_i(u(s))|\,ds \\
&\quad + \int_{t_2}^{1} |G_i(t_1,s) - G_i(t_2,s)|\,|g(s)|\,|f_i(u(s))|\,ds
\end{aligned}$$

then

$|T_iu(t_1) - T_iu(t_2)| \leq$

$$\begin{aligned}
&\frac{\Psi(\eta,...,\eta)}{\Gamma(\alpha)}\left[\int_0^{t_1}\left[\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right) + (t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right]|g(s)|\,ds\right. \\
&+ \int_{t_1}^{t_2}\left[\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right) + (t_2-s)^{\alpha-1}\right]|g(s)|\,ds \\
&\left.+ \int_{t_2}^{1}\left[\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right)\right]|g(s)|\,ds\right]
\end{aligned}$$

$$\begin{aligned}
\leq\ & \frac{\Psi(\eta,...,\eta)}{\Gamma(\alpha)}\left[\left[\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right) + (t_2-t_1)^{\alpha-1}\right]\int_0^{t_1}|g(s)|\,ds\right. \\
&+ \left[\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right) + (t_2-t_1)^{\alpha-1}\right]\int_{t_1}^{t_2}|g(s)|\,ds
\end{aligned}$$

$$
\begin{aligned}
& +\left[\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)\right]\int_{t_2}^{1}|g(s)|\,ds \\
\leq \quad & \frac{\Psi(\eta,...,\eta)}{\Gamma(\alpha)}\left[3\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)+2(t_2-t_1)^{\alpha-1}\right]\int_0^1|g(s)|\,ds.
\end{aligned}
$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. By Ascoli-Arzela theorem, we conclude that the operator $T : X \to X$ is completely continuous.

Now we apply the nonlinear alternative of Leray-Schauder. Let $u \in \partial B_\eta$, such that $u = \lambda Tu$ for some $0 < \lambda < 1$. We have

$$
\begin{aligned}
u_i(t) &= \lambda T_i u(t) \leq t \in [0,1]\max |T_i u(t)| \\
&\leq \frac{1}{\Gamma(\alpha-1)}\Psi(\eta,...,\eta)\,\|g\|_{L^1[0,1]}.
\end{aligned}
$$

Taking the supremum over $[0,1]$, then summing the obtained inequalities according to i from 1 to n, we get

$$\|u\| \leq \frac{n}{\Gamma(\alpha-1)}\Psi(\eta,...,\eta)\,\|g\|_{L^1[0,1]}.$$

taking into account (3.15) we conclude

$$\|u\| < \eta, \tag{3.21}$$

that contradicts the fact that $u \in \partial B_\eta$. So, we conclude that T has at least one fixed point $u^* \in B_\eta$ and then the (FS) has a nontrivial solution $u^* \in B_\eta$.

□

4. Examples

In this section, we give examples to illustrate the usefulness of our main results.

Example 1. Consider the following two-dimensional fractional order system

$$
(S_i) = \begin{cases}
D_{0^+}^{\frac{5}{2}}u_1(t) = 2t\frac{e^{-\left(u_1^2+u_2^2\right)}}{1+u_1^2+u_2^2}, \quad D_{0^+}^{\frac{5}{2}}u_2(t) = 2t\frac{e^{-u_1^2}}{1+u_1^2+u_2^2}, \\
u_1(0)=0,\ u_1'(0)=0, \quad u_2(0)=0,\ u_2'(0)=0, \\
au_1(1)-bu_1'(0)=0, \quad au_2(1)-bu_2'(0)=0.
\end{cases} \tag{4.1}
$$

We have $\alpha = \frac{5}{2}$, $g(t) = 2t$, $f_1(u_1, u_2) = \frac{e^{-\left(u_1^2+u_2^2\right)}}{1+u_1^2+u_2^2}$, $f_2(u_1, u_2) = \frac{e^{-u_1^2}}{1+u_1^2+u_2^2}$, $f_i \in C(\mathbf{R}^2, \mathbf{R})$, $f_i(0) \neq 0$. If we choose $\Psi(u_1, u_2) = \frac{1}{1+u_1^2+u_2^2}$, then

$|f_i(u_1, u_2)| \leq \frac{1}{1+u_1^2+u_2^2} = \Psi(|u_1|, |u_2|)$.

For $\eta = 2$, we get

$$\frac{n}{\Gamma(\alpha-1)} \Psi(\eta, \eta) \|g\|_{L^1[0,1]} \leq \frac{2}{\Gamma\left(\frac{3}{2}\right)(1+2\eta^2)} = 0.250\,75 \leq \eta.$$

Then, according to the Theorem 9, the boundary value problem (4.1) has at least one fixed point $u^* \in B_2$.

Example 2. Consider the following two-dimensional fractional order system

$$(4.2)(S_i) = \begin{cases} D_{0^+}^{\frac{5}{2}} u_1(t) = \frac{e^{-t}}{10}(u_1 - u_2), & D_{0^+}^{\frac{5}{2}} u_2(t) = \frac{e^{-t}}{10}(u_1 + 1), \\ u_1(0) = 0,\ u_1'(0) = 0, & u_2(0) = 0,\ u_2'(0) = 0, \\ au_1(1) - bu_1'(0) = 0, & au_2(1) - bu_2'(0) = 0. \end{cases}$$

We have $\alpha = \frac{5}{2}$, $g(t) = \frac{e^{-t}}{10}$, $f_1(u_1, u_2) = \frac{e^{-t}}{10}(u_1 - u_2)$, $f_2(u_1, u_2) = \frac{e^{-t}}{10}(u_1 + 1)$,$f_i \in C(\mathbf{R}^2, \mathbf{R})$, then

$|f_i(x_1, x_2) - f_i(y_1, y_2)| \leq L \sum_{i=1}^{2} |x_i - y_i|$

with $L = 1$ and $K = \dfrac{2\left(1-e^{-1}\right)}{10\Gamma\left(\frac{3}{2}\right)} = 0.142\,65 < 1$, then hypotheses of Theorem 8 are satisfied. So, the boundary value problem (4.2) has a unique solution $u \in X$.

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A. Guezane-Lakoud
Laboratory of Advanced Materials
University of Badji Mokhtar Annaba
P. O. Box 12, 23000 Annaba,
Algeria
e-mail : a_guezane@yahoo.fr

G. Rebiai
Department of Mathematics
University of Guelma
Algeria
e-mail : rebeai-ghania@yahoo.fr,

and

R. Khaldi
Laboratory of Advanced Materials
University of Badji Mokhtar Annaba
P. O. Box 12, 23000 Annaba,
Algeria
e-mail : rkhadi@yahoo.fr