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On fractional powers of double band matrices

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Abstract

In the present article, we determine the explicit formula for finding the fractional powers of a double band matrix and in particular, we establish the formula for finding the nth root of the matrix. Some examples are also given for supporting the new formulas.

Keywords: *Forward difference operator; Backward difference operator; Double band matrix.*

1. Introduction

Let $\mathbf{R}$ be the set of all real numbers and $\mathbf{N}$ denote the set of all positive integers. Let w be the space of all real valued sequences. Let X and Y be two subspaces of w, then we define a matrix mapping $A : X \to Y$, as

$$(Ax)_n := \sum_k a_{nk}x_k, (n \in \mathbf{N}). \tag{1.1}$$

In fact, for $x = (x_k) \in X$, Ax is called as the $A-$transform of x provided the series in the right hand side of (1.1) converges for each $n \in \mathbf{N}$. Moreover, the matrix $A = (a_{ij})(i, j \in \mathbf{N})$ is also regarded as a linear operator. Let $x = (x_k)$ be any sequence in w and $r(\neq 0), s$ be two real numbers, then we define the generalized first forward and backward difference operators as

$$(B_+(r, s)x)_k = rx_k + sx_{k+1}, (k \in \mathbf{N})$$

and

$$(B_-(r, s)x)_k = rx_k + sx_{k-1,}(k \in \mathbf{N}).$$

Clearly, the operators $B_+(r, s)$ and $B_-(r, s)$ represent an upper and a lower double band matrices, respectively as follows:

$$B_+(r,s) = \begin{pmatrix} r & s & 0 & 0 & \dots \\ 0 & r & s & 0 & \dots \\ 0 & 0 & r & s & \dots \\ 0 & 0 & 0 & r & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, B_-(r,s) = \begin{pmatrix} r & 0 & 0 & 0 & \dots \\ s & r & 0 & 0 & \dots \\ 0 & s & r & 0 & \dots \\ 0 & 0 & s & r & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The forward difference operator Δ was initially studied by Kızmaz [9] by defining the difference sequence $(\Delta x_k) = (x_k - x_{k+1}), k \in \mathbf{N}$. In matrix notation, it can also be expressed as double band matrix where main and super diagonal elements are 1 and -1, respectively. Ahmad and Mursaleen [1] have studied the backward difference operator $\Delta^{(1)}$, defined by $(\Delta^{(1)}x_k) = (x_k - x_{k-1}), k \in \mathbf{N}$ and the corresponding double band matrix contains 1 and -1, respectively in main and sub diagonal positions. It is noted that any integral power of the difference matrices Δ and $\Delta^{(1)}$ can be directly found out by taking the difference operators Δ^m and $\Delta^{(m)}, m \in \mathbf{N}$ introduced by Et and Colak [7] and Ahmad and Mursaleen [1], respectively. Subsequently, the fractional powers of these matrices are directly evaluated by using the operator Δ^α and $\Delta^{(\alpha)}$ introduced by Baliarsingh [4](see also [5, 6]). The double band matrix $B(r, s)$ was studied by Altay and Başar

[3](see also [8]). The primary goal of this work is to evaluate the integral and fractional powers of the matrix $B(r,s)$.

Since both of the operators $B_+(r,s)$ and $B_-(r,s)$ are linear and now taking α times successive transformations, we write

$$(1.2)\qquad (B_+^{\alpha}(r,s)x)_k = \sum_{i=0}^{\infty} \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} r^{\alpha-i} s^i x_{k+i}, (k \in \mathbf{N})$$

$$(1.3)\qquad (B_-^{\alpha}(r,s)x)_k = \sum_{i=0}^{\infty} \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} r^{\alpha-i} s^i x_{k-i}, (k \in \mathbf{N}),$$

where $\Gamma(\alpha)$ denotes the well known Gamma function of a real number α and $\alpha \notin \{0,-1,-2,-3\ldots\}$. For any integral value of α, Eqns.(1.2) and (1.3) reduce to finite sums. It is noticed that operators $B_+^{\alpha}(r,s)$ and $B_-^{\alpha}(r,s)$ includes several difference operators such as

(i) Forward and backward difference operators Δ and $\Delta^{(1)}$ for $\alpha = 1, r = 1$ and $s = -1$ (see [9, 2]).

(ii) Forward and backward difference operators Δ^m and $\Delta^{(m)}$ for $\alpha = m \in \mathbf{N}, r = 1$ and $s = -1$ (see [7, 1]).

(iii) Forward and backward difference operators Δ^{α} and $\Delta^{(\alpha)}$ for $r = 1$ and $s = -1$ (see [4, 5, 6]).

(iv) Difference operator $B(r,s)$ for $\alpha = 1$ (see [3]).

Keeping in view Eqns.(1.2) and (1.3), the inverse operators of $B_+^{\alpha}(r,s)$ and $B_-^{\alpha}(r,s)$ are $B_+^{-\alpha}(r,s)$ and $B_-^{-\alpha}(r,s)$, respectively,
where

$$(1.4)\qquad (B_+^{-\alpha}(r,s)x)_k = \sum_{i=0}^{\infty} \frac{\Gamma(-\alpha+1)}{i!\Gamma(-\alpha-i+1)} r^{-\alpha-i} s^i x_{k+i}, (k \in \mathbf{N})$$

$$(1.5)\qquad (B_-^{-\alpha}(r,s)x)_k = \sum_{i=0}^{\infty} \frac{\Gamma(-\alpha+1)}{i!\Gamma(-\alpha-i+1)} r^{-\alpha-i} s^i x_{k-i}, (k \in \mathbf{N}),$$

It is remarked that for $\alpha = 1$, Eqns. (1.4) and (1.5) are bing simplified by using formula $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ and finally we obtain

$$(1.6)\qquad (B_+^{-1}(r,s)x)_k = \sum_{i=0}^{\infty} (-1)^i \frac{s^i}{r^{i+1}} x_{k+i}, (k \in \mathbf{N})$$

$$(1.7)\qquad (B_-^{-1}(r,s)x)_k = \sum_{i=0}^{\infty} (-1)^i \frac{s^i}{r^{i+1}} x_{k-i}, (k \in \mathbf{N}),$$

2. Main results

In the present section, we establish some new results on the operators $B_+^\alpha(r,s)$ and $B_-^\alpha(r,s)$ and their application to the matrix theory involving the integral and non integral powers of double band matrices.

Theorem 1. *For any reals α, β and $k \in \mathbf{N}$,*

(i) $\left(B_+^\alpha(r,s)\left(B_+^\beta(r,s)x\right)\right)_k = \left(B_+^\beta(r,s)\left(B_+^\alpha(r,s)x\right)\right)_k = \left(B_+^{\alpha+\beta}(r,s)x\right)_k,$

(ii) $\left(B_-^\alpha(r,s)\left(B_-^\beta(r,s)x\right)\right)_k = \left(B_-^\beta(r,s)\left(B_-^\alpha(r,s)x\right)\right)_k = \left(B_-^{\alpha+\beta}(r,s)x\right)_k,$

(iii) $\|B_+^\alpha(r,s)\| = \|B_-^\alpha(r,s)\| = (|r|+|s|)^\alpha.$

Proof. The proof is a routine verification, hence omitted. $\square$

Corollary 1. *For any reals α, β and $k \in \mathbf{N}$,*

(i) $\left(B_+^\alpha(r,s)\left(B_+^{-\alpha}(r,s)x\right)\right)_k = \left(B_+^{-\alpha}(r,s)\left(B_+^\alpha(r,s)x\right)\right)_k = x_k,$

(ii) $\left(B_-^\alpha(r,s)\left(B_-^{-\alpha}(r,s)x\right)\right)_k = \left(B_-^{-\alpha}(r,s)\left(B_-^\alpha(r,s)x\right)\right)_k = x_k,$

(iii) $(B_\pm^{-2}(r,s)x)_k = \sum_{i=0}^\infty (-1)^i (i+1)\frac{s^i}{r^{i+1}} x_{k\pm i},$

(iv) $(B_\pm^{-3}(r,s)x)_k = \sum_{i=0}^\infty (-1)^i (i+1)(i+2)\frac{s^i}{2r^{i+1}} x_{k\pm i}.$

Proof. This is a direct application of Theorem 1. $\square$

Theorem 2. *Let the lower double band matrix $A = (a_{nk})$ be defined by*

$$a_{nk} = \begin{cases} r, & (k=n) \\ s, & (k=n-1) \\ 0, & (\textit{otherwise}) \end{cases},$$

then for $\alpha \in \mathbf{R}$, $A^\alpha = (a_{nk}^\alpha)$ is given by

$$a_{nk}^\alpha = \begin{cases} r^\alpha, & (k=n) \\ \frac{\Gamma(\alpha+1)}{(n-k)!\Gamma(\alpha-n+k+1)} r^{\alpha-n+k} s^{n-k}, & (0 \le k < n) \\ 0, & (k>n) \end{cases},$$

Proof. We prove this theorem by induction. Since $(Ax)_k = (B_-(r,s)x)_k = rx_k + sx_{k-1}$, hence we obtain that

$$(A^2x)_k = A(rx_k + sx_{k-1}) = r^2x_k + 2rsx_{k-1} + s^2x_{k-2},$$

$$(A^3x)_k = A(r^2x_k+2rsx_{k-1}+s^2x_{k-2}) = r^3x_k+3r^2sx_{k-1}+3rs^2x_{k-2}+s^3x_{k-3},$$

Let the theorem be assumed for any arbitrary β and hence

$$(A^\beta x)_k = \sum_{i=0}^{\infty} \frac{\Gamma(\beta+1)}{i!\Gamma(\beta-i+1)} r^{\beta-i}s^i x_{k-i}.$$

Now,

$$\begin{aligned}
&(A^{\beta+1}x)_k \\
&= A\left(\textstyle\sum_{i=0}^{\infty} \frac{\Gamma(\beta+1)}{i!\Gamma(\beta-i+1)} r^{\beta-i}s^i x_{k-i}\right), \\
&= A\left(r^\beta x_k + \beta r^{\beta-1}sx_{k-1} + \frac{\beta(\beta-1)}{2!}r^{\beta-2}s^2x_{k-2} + \ldots\right) \\
&= r^\beta(rx_k + sx_{k-1}) + \beta r^{\beta-1}s(rx_{k-1} + sx_{k-2}) \\
&+\frac{\beta(\beta-1)}{2!}r^{\beta-2}s^2(rx_{k-2} + sx_{k-3}) + \ldots) \\
&= r^{\beta+1}x_k + (\beta+1)r^\beta sx_{k-1} + \frac{(\beta+1)\beta}{2!}r^{\beta-1}s^2x_{k-2} + \ldots \\
&= \textstyle\sum_{i=0}^{\infty} \frac{\Gamma(\beta+2)}{i!\Gamma(\beta-i+2)} r^{\beta+1-i}s^i x_{k-i}
\end{aligned}$$

This completes the proof. □

Theorem 3. *Let the upper double band matrix $A = (a_{nk})$ be defined by*

$$a_{nk} = \begin{cases} r, & (k=n) \\ s, & (k=n+1) \\ 0, & (otherwise) \end{cases},$$

then for $\alpha \in \mathbf{R}$, $A^\alpha = (a^\alpha_{nk})$ is given by

$$a^\alpha_{nk} = \begin{cases} r^\alpha, & (k=n) \\ \frac{\Gamma(\alpha+1)}{(k-n)!\Gamma(\alpha-k+n+1)} r^{\alpha-k+n}s^{k-n}, & (k>n) \\ 0, & (0 \le k < n) \end{cases},$$

Proof. The proof is analogous to that of Theorem 2. □

Corollary 2. *The explicit formula for mth root of the double band matrix A, defined in Theorem 2 is given by*

$$a^{\frac{1}{m}}_{nk} = \begin{cases} r^{1/m}, & (k=n) \\ \dfrac{(1-m)(1-2m)\ldots(1-m(n-k-1))}{(n-k)!m^{n-k}} r^{\frac{1}{m}-n+k}s^{n-k}, & (0 \le k < n) \\ 0, & (k>n) \end{cases},$$

Proof. Suppose $m \in \mathbf{N}$ and mth root of A is denoted by $A^{1/m} = (a_{nk}^{\frac{1}{m}})$. From Theorem 2, one may write

$$a_{nk}^{1/m} = \begin{cases} r^{1/m}, & (k = n) \\ \frac{\Gamma(1/m+1)}{(n-k)!\Gamma(\frac{1}{m}-n+k+1)} r^{\frac{1}{m}-n+k} s^{n-k}, & (0 \leq k < n) \\ 0, & (k > n) \end{cases},$$

However, for $(0 \leq k < n)$ one may easily calculate

$$\begin{aligned} a_{nk}^{\frac{1}{m}} &= \frac{\Gamma(1/m+1)}{(n-k)!\Gamma(\frac{1}{m}-n+k+1)} r^{\frac{1}{m}-n+k} s^{n-k} \\ &= \frac{\frac{1}{m}\left(\frac{1}{m}-1\right)\left(\frac{1}{m}-2\right)\ldots\left(\frac{1}{m}-n+k+1\right)}{(n-k)!} r^{\frac{1}{m}-n+k} s^{n-k} \\ &= \frac{(1-m)(1-2m)\ldots(1-m(n-k-1))}{(n-k)!m^{n-k}} r^{\frac{1}{m}-n+k} s^{n-k}. \end{aligned}$$

Since A is a triangle, for $k = n$ and $k > n$, $a_{nk}^{\frac{1}{m}}$ takes the values as $r^{1/m}$ and 0, respectively. □

Corollary 3. *The explicit formula for mth root of the double band matrix A, defined in Theorem 3 is given by*

$$a_{nk}^{\frac{1}{m}} = \begin{cases} r^{1/m}, & (k = n) \\ \dfrac{(1-m)(1-2m)\ldots(1-m(k-n-1))}{(k-n)!m^{k-n}} r^{\frac{1}{m}-k+n} s^{k-n}, & (k > n) \\ 0, & (0 \leq k < n) \end{cases},$$

Proof. This follows from Theorem 3. □

Now, as applications of Corollaries 2 and 3, we illustrate following counter examples:

Example 1 : Consider a matrix A of order 5, where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 3 & 1 \end{pmatrix}$$

Using Corollary 2, we calculate the square root of A as $a_{nk}^{\frac{1}{2}} = 1$ for all $k = n$ and $a_{21}^{\frac{1}{2}} = a_{32}^{\frac{1}{2}} = a_{43}^{\frac{1}{2}} = a_{54}^{\frac{1}{2}} = 3/2$,
$a_{31}^{\frac{1}{2}} = a_{42}^{\frac{1}{2}} = a_{53}^{\frac{1}{2}} = -9/8$,
$a_{41}^{\frac{1}{2}} = a_{52}^{\frac{1}{2}} = 27/16$,
$a_{51}^{\frac{1}{2}} = -405/128$, Therefore, one can write

$$A^{\frac{1}{2}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 3/2 & 1 & 0 & 0 & 0 \\ -9/8 & 3/2 & 1 & 0 & 0 \\ 27/16 & -9/8 & 3/2 & 1 & 0 \\ -405/128 & 27/16 & -9/8 & 3/2 & 1 \end{pmatrix}$$

Example 2 : Consider a matrix B of order 5, where

$$B = \begin{pmatrix} 8 & 5 & 0 & 0 & 0 \\ 0 & 8 & 5 & 0 & 0 \\ 0 & 0 & 8 & 5 & 0 \\ 0 & 0 & 0 & 8 & 5 \\ 0 & 0 & 0 & 0 & 8 \end{pmatrix}$$

Using Corollary 3, we calculate the cube root of B as $b_{nk}^{\frac{1}{3}} = 2$ for all $k = n$ and

$b_{12}^{\frac{1}{3}} = b_{23}^{\frac{1}{3}} = b_{34}^{\frac{1}{3}} = b_{45}^{\frac{1}{3}} = 5/12,$
$b_{13}^{\frac{1}{3}} = b_{24}^{\frac{1}{3}} = b_{35}^{\frac{1}{3}} = -28/288,$
$b_{14}^{\frac{1}{3}} = b_{25}^{\frac{1}{3}} = 625/20736,$
$b_{15}^{\frac{1}{3}} = -6250/497664,$

Therefore, we write

$$B^{\frac{1}{3}} = \begin{pmatrix} 2 & 5/12 & -28/288 & 625/20736 & -6250/497664 \\ 0 & 2 & 5/12 & -28/288 & 625/20736 \\ 0 & 0 & 2 & 5/12 & -28/288 \\ 0 & 0 & 0 & 2 & 5/12 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Conclusion:

In this study, we derive an explicit formula for finding any arbitrary power of the double band matrix. Being an application of these results, we have also determine the nth root and inverse of the double band matrix of finite order.

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References

[1] Z. U. Ahmad, M. Mursaleen, Köthe-Toeplitz duals of some new sequence spaces and their martix maps, Publ. Inst. Math. (Beograd), **42 (56)**, pp. 57-61, (1987).

[2] B. Altay, F. Başar, On the fine spectrum of the difference operator Δ on c_0 and c, Inform. Sci., **168**, pp. 217-224, (2004).

[3] B. Altay, F. Başar, On the fine spectrum of the generalized difference operator $B(r, s)$ over the sequence spaces c_0 and c, Int. J. Math. Math. Sci. **18**, pp. 3005-3013, (2005).

[4] P. Baliarsingh, Some new difference sequence spaces of fractional order and their dual spaces, Appl. Math. Comput. **219 (18)**, pp. 9737-9742, (2013).

[5] P. Baliarsingh, S. Dutta, On the classes of fractional order difference sequence spaces and their matrix transformations, Appl. Math. Comput. **250**, pp. 665–674, (2015).

[6] P. Baliarsingh, S. Dutta, A unifying approach to the difference operators and their applications, Bol. Soc. Paran. Mat.,**33 (1)**, pp. 49-57, (2015).

[7] M. Et, R. Çolak, On some generalized difference sequence spaces, Soochow J. Math., **21** (4), pp. 377-386, (1995).

[8] H. Furkan, H. Bilgiç, H. Furkan, On the fine spectrum of the generalized difference operator $B(r, s)$ over the sequence spaces ℓ_p and $bv_p, (1 < p < \infty)$, Nonlinear Anal., **68(3)** pp. 499–506, (2008).

[9] H. Kızmaz, On Certain Sequence spaces, Canad. Math. Bull., **24**(2), pp. 169-176, (1981).

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