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## On the solution of functional equations of Wilson's type on monoids

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### Abstract

*Let  $S$  be a monoid,  $\mathbf{C}$  be the set of complex numbers, and let  $\sigma, \tau \in \text{Antihom}(S, S)$  satisfy  $\tau \circ \tau = \sigma \circ \sigma = \text{id}$ . The aim of this paper is to describe the solution  $f, g : S \rightarrow \mathbf{C}$  of the functional equation*

$$f(x\sigma(y)) + f(\tau(y)x) = 2f(x)g(y), \quad x, y \in S,$$

*in terms of multiplicative and additive functions.*

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## 1. Notation and terminology

Throughout the paper we work in the following framework: A monoid is a semi-group  $S$  with an identity element that we denote  $e$ , that is an element such that  $ex = xe = x$  for all  $x \in S$  (a semi-group is an algebraic structure consisting of a set together with an associative binary operation) and  $\sigma, \tau : S \rightarrow S$  are two anti-homomorphisms (briefly  $\sigma, \tau \in \text{Antihom}(S, S)$ ) satisfying  $\tau \circ \tau = \sigma \circ \sigma = \text{id}$ .

For any function  $f : S \rightarrow \mathbf{C}$  we say that  $f$  is  $\sigma$ -even (resp.  $\tau$ -even) if  $f \circ \sigma = f$  (resp.  $f \circ \tau = f$ ), also we use the notation  $\check{f}(x) = f(x^{-1})$  in the case  $S$  is a group.

We say that a function  $\chi : S \rightarrow \mathbf{C}$  is multiplicative, if  $\chi(xy) = \chi(x)\chi(y)$  for all  $x, y \in S$ .

If  $\chi : S \rightarrow \mathbf{C}$  is multiplicative and  $\chi \neq 0$ , then

$$I_\chi := \{x \in S \mid \chi(x) = 0\} \text{ is either empty or a proper subset of } S.$$

If  $S$  is a topological space, then we let  $C(S)$  denote the algebra of continuous functions from  $S$  into  $\mathbf{C}$ .

## 2. Introduction

Wilson's functional equation on a group  $G$  is of the form

$$(2.1) \quad f(xy) + f(xy^{-1}) = 2f(x)g(y), \quad x, y \in G,$$

where  $f, g : G \rightarrow \mathbf{C}$  are two unknown functions.

Special cases of Wilson's functional equation are d'Alembert's functional equation

$$(2.2) \quad f(xy) + f(xy^{-1}) = 2f(x)f(y), \quad x, y \in G,$$

and Jensen's functional equation

$$f(xy) + f(xy^{-1}) = 2f(x), \quad x, y \in G.$$

In [3] Ebanks and Stetkær studied the solutions  $f, g : G \rightarrow \mathbf{C}$  of Wilson's functional equation (2.1) and the following variant of Wilson's functional (see [8])

$$(2.3) \quad f(xy) + f(y^{-1}x) = 2f(x)g(y), \quad x, y \in G.$$

They solve (2.3) and they obtained some new results about (2.1). We refer also to Wilson's first generalization of d'Alembert's functional equation

$$(2.4) \quad f(x+y) + f(x-y) = 2f(x)g(y), \quad x, y \in \mathbf{R}.$$

For more about the functional equation (2.4) see Aczél [1]. The solutions formulas of equation (2.4) for abelian groups are known.

In the same year Stetkær in [10] obtained the complex valued solution of the following variant of d'Alembert's functional equation

$$(2.5) \quad f(xy) + f(\sigma(y)x) = 2f(x)f(y), \quad x, y \in \mathcal{S},$$

where  $\mathcal{S}$  is a semi-group and  $\sigma$  is an involutive homomorphism of  $\mathcal{S}$ . The difference between d'Alembert's standard functional equation

$$f(xy) + f(\tau(y)x) = 2f(x)f(y), \quad x, y \in \mathcal{S},$$

and the variant (2.5) is that  $\tau$  is an anti-homomorphism (on a group typically the group inversion). Some information, applications and numerous references concerning (2.5) and their further generalizations can be found e.g. in [6, 9, 10].

Some general properties of the solutions of equation

$$(2.6) \quad f(xy) + f(\sigma(y)x) = 2f(x)g(y)$$

on a topological monoid  $M$  equipped with a continuous involution  $\sigma : M \rightarrow M$  can be found in [9, Chapter 11].

Stetkær[11] proved a natural interesting relation between Wilson's functional equation (2.6) and d'Alembert's functional equation (2.2) and for  $\sigma(x) = x^{-1}$ . That is if  $f \not\equiv 0$  is a solution of equation (2.6), then  $g$  is a solution of equation (2.2). In [2] Chahbi et al. give a generalization of the symmetrized multiplicative Cauchy equation.

Recently, EL-Fassi et al. [5] obtained the solution of following functional equation

$$(2.7) \quad f(x\sigma(y)) + f(\tau(y)x) = 2f(x)f(y), \quad x, y \in \mathcal{S},$$

where  $\mathcal{S}$  is a semi-group and  $\sigma, \tau$  are two anti-homomorphisms of  $\mathcal{S}$  such that  $\sigma \circ \sigma = \tau \circ \tau = id$ .

The main purpose of this paper is to solve the functional equation

$$(2.8) \quad f(x\sigma(y)) + f(\tau(y)x) = 2f(x)g(y), \quad x, y \in S,$$

where  $S$  is a monoid and  $\sigma, \tau \in \text{Antihom}(S, S)$  such that  $\sigma \circ \sigma = \tau \circ \tau = \text{id}$ . This equation is a natural generalization of (2.7) and of the following new functional equations

$$(2.9) \quad f(x\sigma(y)) + f(\sigma(y)x) = 2f(x)g(y), \quad x, y \in S,$$

$$(2.10) \quad f(x\sigma(y)) + f(\sigma(y)x) = 2f(x)f(y), \quad x, y \in S,$$

$$(2.11) \quad f(x\sigma(y)) + f(\tau(y)x) = 2f(x), \quad x, y \in S,$$

$$(2.12) \quad f(x\sigma(y)) + f(\sigma(y)x) = 2f(x), \quad x, y \in S,$$

where  $(S, \cdot)$  is a monoid and  $\sigma, \tau \in \text{Antihom}(S, S)$  such that  $\sigma \circ \sigma = \tau \circ \tau = \text{id}$ . Clearly, if  $S$  is a group and  $\tilde{f} = f$  with  $\sigma(x) = x^{-1}$ , then functional equation (2.10) becomes the symmetrized multiplicative Cauchy equation (see for instance [7] or [9, Theorem 3.21]). By elementary methods we find all solutions of (2.8) on monoid in terms of multiplicative functions. Finally, we note that the sine addition law on monoid given in [4, 9] is a key ingredient of the proof of our main result (Theorem 3.1).

### 3. Solution of the functional equation (2.8)

In this section we obtain the solution of the functional equation (2.8) on monoid. The following lemma will be used in the proof of Theorem 3.1.

**Lemma 3.1.** *Let  $S$  be a monoid and  $\sigma \in \text{Antihom}(S, S)$ . Let  $f, g : S \rightarrow \mathbf{C}$  be a solution of the functional equation*

$$(3.1) \quad f(x\sigma(y)) = f(x)g(y), \quad x, y \in S.$$

*Then  $g$  is a multiplicative function.*

**Proof.** For all  $x, y, z \in S$ , we have

$$\begin{aligned} f(x)g(yz) &= f(x\sigma(yz)) = f(x\sigma(z)\sigma(y)) \\ &= f(x\sigma(z))g(y) = f(x)g(z)g(y), \end{aligned}$$

then  $g(yz) = g(y)g(z)$  for all  $x, z \in S$ . This implies that  $g$  is a multiplicative function.  $\square$

**Theorem 3.1.** Let  $S$  be a monoid with identity element  $e$ , and  $\sigma, \tau \in \text{Antihom}(S, S)$  such that  $\sigma \circ \sigma = \tau \circ \tau = \text{id}$  (where  $\text{id}$  denotes the identity map). The solutions  $f, g : S \rightarrow \mathbf{C}$  of (2.8) are the following pairs of functions, where  $\chi : S \rightarrow \mathbf{C}$  denotes a multiplicative function such that  $\chi(e) = 1$ :

(1)  $f \equiv 0$  and  $g$  arbitrary.

(2)  $g = \frac{\chi + \chi \circ \sigma \circ \tau}{2}$  and  $f = f(e)\chi \circ \sigma$ , where  $\chi \neq 0$  and  $f(e) \in \mathbf{C} \setminus \{0\}$ .

(3) If  $\chi \neq \chi \circ \sigma \circ \tau$ , then  $g = \frac{\chi + \chi \circ \sigma \circ \tau}{2}$  and

(i)  $f = f(e)\chi \circ \sigma$ , where  $f(e) \in \mathbf{C} \setminus \{0\}$ ,

or

(ii)  $f = \alpha\chi \circ \sigma + (f(e) - \alpha)\chi \circ \tau$  for some constant  $\alpha \in \mathbf{C} \setminus \{0, f(e)\}$ , where  $\chi \circ \sigma \circ \tau = \chi \circ \tau \circ \sigma$ .

(4) If  $\chi = \chi \circ \sigma \circ \tau$ , and  $S$  is generated by its squares, then

(i)  $g(x) = \chi(x)$  and  $f(x) = (A \circ \sigma(x) + f(e))\chi \circ \sigma(x)$  for  $x \in S \setminus I_\chi$ ,

(ii)  $g(x) = f(x) = 0$  for  $x \in I_\chi$ ,

where  $A : S \setminus I_\chi \rightarrow \mathbf{C}$  is a non-zero additive function such that  $A \circ \sigma \circ \tau = -A$ .

Furthermore, if  $S$  is a topological monoid, and  $f, g \in C(S)$ , then  $\chi, \chi \circ \sigma, \chi \circ \tau, \chi \circ \sigma \circ \tau \in C(S)$ , and  $A \circ \sigma \in C(S \setminus I_\chi)$ .

**Proof.** It is elementary to check that the cases stated in the Theorem define solutions, so it is left to show that any solution  $f, g : S \rightarrow \mathbf{C}$  of (2.8) falls into one of these case. We use in the proof similar Stetkaer's computations [10]. Let  $x, y, z \in S$  be arbitrary. If we replace  $x$  by  $x\sigma(y)$  and  $y$  by  $z$  in (2.8), we get

$$(3.2) \quad f(x\sigma(z)y) + f(\tau(z)x\sigma(y)) = 2f(x\sigma(y))g(z).$$

On the other hand if we replace  $x$  by  $\tau(z)x$  in (2.8), we infer that

$$\begin{aligned} f(\tau(z)x\sigma(y)) + f(\tau(z)y) &= 2f(\tau(z)x)g(y) \\ &= 2g(y)[2f(x)g(z) - f(x\sigma(z))]. \end{aligned}$$

$$(3.3)$$

Replacing  $y$  by  $zy$  in (2.8), we obtain

$$(3.4) \quad f(\tau(zy)x) = 2f(x)g(zy) - f(x\sigma(zy)).$$

It follows from (3.4) that (3.3) become

$$f(\tau(z)x\sigma(y)) + 2f(x)g(zy) - f(x\sigma(zy)) = 4g(y)f(x)g(z) - 2g(y)f(x\sigma(z)).$$

$$(3.5)$$

Subtracting this from (3.2) we get after some simplifications that

$$(3.6) \quad f(x\sigma(zy)) - f(x)g(zy) = g(y)[f(x\sigma(z)) - f(x)g(z)] + g(z)[f(x\sigma(y)) - f(x)g(y)].$$

With the notation

$$(3.7) \quad f_x(y) := f(x\sigma(y)) - f(x)g(y)$$

equation (3.6) can be written as follows

$$(3.8) \quad f_a(xy) = f_a(x)g(y) + f_a(y)g(x).$$

This shows that the pair  $(f_a, g)$  satisfies the sine addition law for any  $a \in S$ . From the Known solution of the sine addition formula (see for example [4, Lemma 3.4]), we have the following possibilities.

If  $f \equiv 0$  we deal with **case (1)** in the Theorem. So during the rest of the proof we will assume that  $f \not\equiv 0$ . If we replace  $(x, y)$  by  $(e, \sigma(x))$  in (3.7), we get

$$(3.9) \quad f(x) = f_e \circ \sigma(x) + f(e)g \circ \sigma(x), \quad x \in S.$$

• Suppose that  $f_x = 0$  for all  $x \in S$ , then  $f_e = 0$ , i.e.,  $f(x) = f(e)g \circ \sigma(x)$  for all  $x \in S$ , and hence  $f(e) \neq 0$ . Indeed,  $f(e) = 0$  would entail  $f \equiv 0$ , contradicting our assumption. From Lemma 3.1, we see that  $g$  is a

multiplication function. Substituting  $f = f(e)g \circ \sigma$  into (2.8), we infer that  $g = g \circ \sigma \circ \tau$ . We may thus write  $g = (g + g \circ \sigma \circ \tau)/2$  which is the form claimed in the **case (2)**.

• Now suppose that  $f_x \neq 0$  for some  $x \in S$ .

If  $f_e \neq 0$  then, from [4, Lemma 3.4], we see that there exist two multiplicative functions  $\chi_1, \chi_2 : S \rightarrow \mathbf{C}$  such that

$$g = \frac{\chi_1 + \chi_2}{2}.$$

**Case (3):** If  $\chi_1 \neq \chi_2$ , then  $f_e = c(\chi_1 - \chi_2)$  for some constant  $c \in \mathbf{C} \setminus \{0\}$ . From equality (3.9), we find after a reduction that

$$f = \alpha\chi_1 \circ \sigma + \beta\chi_2 \circ \sigma$$

where  $\alpha = (2c + f(e))/2$  and  $\beta = f(e) - \alpha$ . Substituting  $f$  and  $g$  in (2.8), we get after some simplification that

$$\alpha\chi_1 \circ \sigma(x)[\chi_1 \circ \sigma \circ \tau(y) - \chi_2(y)] + \beta\chi_2 \circ \sigma(x)[\chi_2 \circ \sigma \circ \tau(y) - \chi_1(y)] = 0$$

for all  $x, y \in S$ . Since  $\chi_1 \neq \chi_2$  we get from the theory of multiplicative functions (see for instance [9, Theorem 3.18]) that both terms are 0, so

$$(3.10) \quad \begin{cases} \alpha\chi_1 \circ \sigma(x)[\chi_1 \circ \sigma \circ \tau(y) - \chi_2(y)] = 0 \\ \beta\chi_2 \circ \sigma(x)[\chi_2 \circ \sigma \circ \tau(y) - \chi_1(y)] = 0 \end{cases}$$

for all  $x, y \in S$ . Since  $f \not\equiv 0$  at least one of  $\alpha$  and  $\beta$  is not zero.

**Subcase (3.i):** If  $\alpha = 0$  and  $\beta \neq 0$ , by (3.10) and  $f \not\equiv 0$ , for this to be the case we must have  $\chi_1 \neq 0, \chi_2 \neq 0$  and  $\chi_2 \circ \sigma \circ \tau(y) = \chi_1(y)$  for all  $y \in S$ , then  $f$  and  $g$  have the desired form (3.i) with  $\chi_2 := \chi$ .

If  $\alpha \neq 0$  and  $\beta = 0$ , by (3.10) and  $f \not\equiv 0$ , for this to be the case we must have  $\chi_1 \neq 0, \chi_2 \neq 0$  and  $\chi_1 \circ \sigma \circ \tau(y) = \chi_2(y)$  for all  $y \in S$ , then  $f$  and  $g$  have the desired form (3.i) with  $\chi_1 := \chi$ .

**Subcase (3.ii):** If  $\alpha \neq 0$  and  $\beta \neq 0$ , by (3.10) and  $f \not\equiv 0$ , for this to be the case we must have  $\chi_1 \neq 0, \chi_2 \neq 0$ ,  $\chi_1 \circ \sigma \circ \tau(y) = \chi_2(y)$  and  $\chi_2 \circ \sigma \circ \tau(y) = \chi_1(y)$  for all  $y \in S$ , with  $\chi_1 = \chi$ , after some simplification, we obtain  $\chi \circ \sigma \circ \tau = \chi \circ \tau \circ \sigma$  and the desired form (3.ii) of  $f$  and  $g$ .

**Case 4:** If  $\chi_1 = \chi_2$  then letting  $\chi := \chi_1$  we have  $g = \chi$ . If  $S$  is generated by its squares, then there exists an additive function  $A : S \setminus I_\chi \rightarrow \mathbf{C}$  for which

$$(3.11) \quad f_e(x) = \begin{cases} A(x)\chi(x) & \text{if } x \in S \setminus I_\chi \\ 0 & \text{if } x \in I_\chi. \end{cases}$$

**Subcase (4.i):** If  $x \in S \setminus I_\chi$ , then by (3.9) and (3.11), we get

$$f(x) = (A \circ \sigma(x) + f(e))\chi \circ (x)$$

for all  $x \in S \setminus I_\chi$ . Substituting  $f$  and  $g$  in (2.8), we get after some simplification that

$$(A \circ \sigma(x) + f(e))(\chi \circ \sigma \circ \tau(y) - \chi(y)) + A \circ \sigma \circ \tau(y)\chi \circ \sigma \circ \tau(y) + A(y)\chi(y) = 0, \quad (3.12)$$

for all  $x, y \in S \setminus I_\chi$ . Suppose that  $\chi \circ \sigma \circ \tau \neq \chi$ . From (3.12) we infer that  $A \equiv 0$ , this contradicts with  $f_e \neq 0$  on  $S$ . So  $\chi \circ \sigma \circ \tau = \chi$  and  $A \circ \sigma \circ \tau = -A$ .

**Subcase (4.ii):** If  $x \in I_\chi$ , then  $g(x) = f(x) = 0$ .

If  $f_e = 0$ , then  $f(x) = f(e)g \circ \sigma(x)$  for all  $x \in S$ , and hence  $f(e) \neq 0$ . Replacing  $(x, y)$  by  $(e, x)$  in (2.8), we get

$$(3.13) \quad f(\sigma(x)) + f(\tau(x)) = 2f(e)g(x), \quad x \in S.$$

From (3.13) and  $f(x) = f(e)g \circ \sigma(x)$  for all  $x \in S$ , we obtain  $f(x) = f(e)g \circ \tau(x)$  for all  $x \in S$ . So,  $g$  is a solution of the functional equation

$$(3.14) \quad g(xy) + g(\tau \circ \sigma(y)x) = 2g(x)g(y), \quad x, y \in S.$$

Similar to the proofs of [10, Theorem 2.1], we find that  $g = (\chi + \chi \circ \sigma \circ \tau)/2$ , where  $\chi : S \rightarrow \mathbf{C}$  is multiplicative and  $\chi \circ \sigma \circ \tau = \chi \circ \tau \circ \sigma$ . Hence we are in case (2) or (3).

The continuity statement follows from [9, Theorem 3.18 (d)]. This completes the proof of Theorem.  $\square$

#### 4. Some consequences

As immediate consequences of Theorem 3.1, we have the following corollaries.

**Corollary 4.1.** *Let  $S$  be a monoid with identity element  $e$ , and  $\sigma, \tau \in \text{Antihom}(S, S)$  such that  $\sigma \circ \sigma = \tau \circ \tau = \text{id}$ . The solutions  $f, g : S \rightarrow \mathbf{C}$  of the functional equation*

$$f(x\sigma(y)) + f(\sigma(y)x) = 2f(x)g(y), \quad x, y \in S$$

*are the following pairs of functions, where  $\chi : S \rightarrow \mathbf{C}$  denotes a multiplicative function such that  $\chi(e) = 1$ :*



- (1)  $f \equiv 0$  and  $g$  arbitrary.
- (2)  $g = \chi$  and  $f = f(e)\chi \circ \sigma$ , where  $\chi \neq 0$  and  $f(e) \in \mathbf{C} \setminus \{0\}$ .

Furthermore, if  $S$  is a topological monoid and  $f, g \in C(S)$ , then  $\chi, \chi \circ \sigma \in C(S)$ .

**Proof.** It suffices to take  $\tau(x) = \sigma(x)$  for all  $x \in S$  in Theorem 3.1.  $\square$

**Corollary 4.2 ([5]).** Let  $S$  be a monoid with identity element  $e$ , and  $\sigma, \tau \in \text{Antihom}(S, S)$  such that  $\sigma \circ \sigma = \tau \circ \tau = \text{id}$ . The solutions  $f : S \rightarrow \mathbf{C}$  of the functional equation

$$f(x\sigma(y)) + f(\tau(y)x) = 2f(x)f(y), \quad x, y \in S$$

are the functions of the form  $f = (\chi + \chi \circ \sigma \circ \tau)/2$ , where  $\chi : S \rightarrow \mathbf{C}$  is a multiplicative such that:

- (i)  $\chi \circ \sigma \circ \tau = \chi \circ \tau \circ \sigma$ , and
- (ii)  $\chi$  is  $\sigma$ -even or/and  $\tau$ -even.

Furthermore, if  $S$  is a topological monoid and  $f \in C(S)$ , then  $\chi, \chi \circ \sigma \circ \tau \in C(S)$ .

**Proof.** It suffices to take  $g(x) = f(x)$  for all  $x \in S$  in Theorem 3.1.  $\square$

**Corollary 4.3.** Let  $S$  be a monoid with identity element  $e$ , and  $\sigma, \tau \in \text{Antihom}(S, S)$  such that  $\sigma \circ \sigma = \tau \circ \tau = \text{id}$ . The solutions  $f : S \rightarrow \mathbf{C}$  of the functional equation

$$f(x\sigma(y)) + f(\tau(y)x) = 2f(x), \quad x, y \in S$$

are the functions of the form:

- (1)  $f \equiv 0$ .
- (2)  $f = f(e)$ , where  $f(e) \in \mathbf{C} \setminus \{0\}$ .
- (3) If  $S$  is generated by its squares, then

- (i)  $f(x) = A \circ \sigma(x) + f(e)$  for  $x \in S \setminus I_\chi$ ,
- (ii)  $f(x) = 0$  for  $x \in I_\chi$ ,

where  $A : S \setminus I_\chi \rightarrow \mathbf{C}$  is an additive function such that  $A \circ \sigma \circ \tau = -A \neq 0$ . Furthermore, if  $S$  is a topological monoid, and  $f \in C(S)$ , then  $A \circ \sigma \in C(S \setminus I_\chi)$ .

**Proof.** It suffices to take  $g(x) = 1$  for all  $x \in S$  in Theorem 3.1.  $\square$

**Corollary 4.4.** *Let  $S$  be a monoid with identity element  $e$ , and  $\sigma \in \text{Antihom}(S, S)$  such that  $\sigma \circ \sigma = \text{id}$ . The solutions  $f : S \rightarrow \mathbf{C}$  of the functional equation*

$$f(x\sigma(y)) + f(\sigma(y)x) = 2f(x)f(y), \quad x, y \in S$$

*are the functions of the form  $f = \chi$ , where  $\chi : S \rightarrow \mathbf{C}$  is a multiplicative such that  $\chi$  is  $\sigma$ -even.*

**Proof.** It suffices to take  $g(x) = f(x)$  and  $\tau(x) = \sigma(x)$  for all  $x \in S$  in Theorem 3.1.  $\square$

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## References

- [1] J. Aczél and J. Dhombres, *Functional equations in several variables*, Cambridge University Press, New York, (1989).
- [2] A. Chahbi, B. Fadli and S. Kabbaj, *A generalization of the symmetrized multiplicative Cauchy equation*, ActaMath. Hungar., **149**, pp. 1–7, (2016).
- [3] B. R. Ebanks and H. Stetkær, *On Wilson's functional equations*, Aequat. Math., **89**, pp. 339–354, (2015).
- [4] B. R. Ebanks and H. Stetkær, *d'Alembert's other functional equation on monoids with an involution*, Aequationes Math. **89**, pp. 187–206, (2015).
- [5] Iz. EL-Fassi, A. Chahbi and S. Kabbaj, *The Solution of a class functional equations on semi-groups*, Filomat, to appear.

- [6] P.L. Kannappan, *Functional equations and inequalities with applications*, Springer, New York, (2009).
- [7] H. Stetkær, *On multiplicative maps*, semi-group Forum, **63** (3), pp. 466-468, (2001).
- [8] H. Stetkær, *On a variant of Wilson's functional equation on groups*, Aequat. Math., **68**, pp. 160–176, (2004).
- [9] H. Stetkær, *Functional equations on groups*, World Scientific Publishing Co, Singapore, (2013).
- [10] H. Stetkær, *A variant of d'Alembert's functional equation*, Aequationes Math. **89**, pp. 657-662, (2015).
- [11] H. Stetkær, *A link between Wilson's and d'Alembert's functional equations*, Aequat. Math. **90**, pp. 407–409, (2015).

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