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New results on exponential stability of nonlinear Volterra integro-differential equations with constant time-lag

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Abstract

In the present work, we pay attention to a number of nonlinear Volterra integro-differential equations (VIDEs) with constant time-lag. We define three new Lyapunov functionals (LFs) and employ them to get specific conditions guaranteeing the uniform exponential asymptotic stability (UEAS) of the trivial solutions of the (VIDEs) considered. The results obtained generalize, compliment and improve the existing results in the literature from the cases of the without delay to the more general cases with time-lag.

Subjclass [2010] : *34D05, 34K20, 45J05.*

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1. Introduction

The Lyapunov's direct method containing an energy like function has come to be a powerful implement in the qualitative study of ordinary differential equations (ODEs). Over the last five decades years, this technique has also been applied to functional differential equations (FDEs) and (IDEs) by many researchers. It is well known that while Lyapunov functions are made use in the investigation of (ODEs), more generally (LFs) are used in studying (FDEs) and (IDEs) (see, for example, Burton [6, 7]. As (VIDEs) can also be treated as (FDEs), (LFs) have been constructed or defined exclusively for various (VIDEs) by Adıvar and Raffoul [1], Becker ([2],[3],[4]), Burton [5, 6, 7], Burton et al. [8], Burton and Mahfoud ([9],[10]), Corduneanu [11], Diamandescu [12], Eloe et al. [13], Furumochi and Matsuoka [14], Graef and Tunç [15], Graef et al. [16], Grimmer and Seifert [17], Gripenberg et al. [18], Hara et al. [19], Islam and Raffoul [20], Jordan [21], Levin [22], Mahfoud [23], Miller [24], Raffoul [25],[26],[27], Raffoul and Ren [28], Raffoul and Unal [29], Rama Mohana Rao and Raghavendra [30], Rama Mohana Rao and Srinivas [31], Staffans [32], Tunç ([33],[34],[35]), Tunç and Ayhan [38], Vanualailai and Nakagiri [39], Wang [40], Wazwaz [41], Zhang [42], Da Zhang [43] and many relative papers in their references in order to study the stability (S), instability (I), uniform stability (US), exponential stability (ES), boundedness (B), etc. properties that equations.

As differentiated from this line, in 2007, Raffoul [26] presents some basic theorems which provide a way of constructing (LFs) for the vector (VIDE)

$$x'(t) = Ax(t) + \int_0^t B(t,s)f(x(s))ds. \tag{1.1}$$

Using (LFs), the author obtains sufficient conditions guaranteeing the (EAS) of the trivial solution to system

$$x'(t) = G(t, x(s); 0 \leq s \leq t) := G(t, x(.)), \tag{1.2}$$

where x is an $n-$ dimensional vector, the function G is continuous function for $t \in [0, \infty)$ and $x \in \Re^n$ with $G(t, 0) = 0$. A typical example of (1.2) is (VIDE) (1.1) . The results of Raffoul [26] are applied to (VIDE) (1.1) with $f(x) = x^n$, n is positive and rational. Two examples are discussed in [26].

Then, Raffoul [26] applies the theory obtained for equation (1.2) to the real (VIDE)

$$x'(t) = \sigma(t)x(t) + \int_0^t B(t,s)f(s, x(s))ds + g(t, x(t)). \tag{1.3}$$

The author [26] describes specific conditions guaranteeing the trivial solution of (VIDE) (1.3) is (UEAS).

In addition, more recently, Raffoul and Ren [29] considered the n-dimensional stochastic differential equation (SDE) of the form

$$\begin{aligned} dx(t) = \quad & f\left(t, x(t), x(t-h), \int_{t-h}^{t} A(t,s)h(s,x(s))ds\right) dt \\ & + g(t, x(t), x(t-h))dB(t) \end{aligned}$$

for $t \in \Re^{+}$, $t \geq 0$. The authors provide general theorems about boundedness or bounded in probability of solutions to the former nonlinear delay stochastic differential equation. In [29], the analysis is based on the successful construction of suitable (LFs), and the authors also offer several good examples as application of their theorems.

In this paper, instead of (VIDE) (1.3) without time-lag and the (SDE) in [29], we pay attention to the below nonlinear (VIDE) with constant time-lag

$$\begin{aligned} x'(t) &= -\alpha(t)\varphi(x(t)) + \int_{t-\tau}^{t} C(t,s,x(s))\psi(s,x(s),x(s-\tau))ds \\ &\quad + h(t, x(t), x(t-\tau)), \end{aligned} \tag{1.4}$$

where $x(t) = \varphi_2(t)$ for $0 \leq t \leq t_0$, $t - \tau \geq 0$, τ is fixed constant time-lag, $x \in \Re$, $\alpha(t) : [0, \infty) \to (0, \infty)$, $\phi : \Re \to \Re$, $\psi, h : [0, \infty) \times \Re \times \Re \to \Re$ are continuous functions with $\phi(0) = \psi(t, 0, 0) = h(t, 0, 0) = 0$ and $C(t, s, x(s))$ is a continuous function for $0 \leq s \leq t < \infty$ and $x(s)$ with $C(t, s, 0) = 0$.

Let

$$\phi_1(x) = \begin{cases} \frac{\phi(x)}{x}, & x \neq 0 \\ \\ \phi'(0), & x = 0. \end{cases}$$

Then, it may obtained from (VIDE) (1.4) that

$$\begin{aligned} x'(t) &= -\alpha(t)\phi_1(x)x + \int_{t-\tau}^{t} C(t,s,x(s))\psi(s,x(s),x(s-\tau))ds \\ &\quad + h(t, x, x(t-\tau)), \end{aligned} \tag{1.5}$$

in which and throughout the paper, when we need x represents $x(t)$.

In [26], Raffoul also considers the following real (VIDEs)

$$x'(t) = \sigma(t)x(t) + \exp(-\delta t)\int_0^t B(t,s)x^{\frac{2}{3}}(s)ds \tag{1.6}$$

and

$$x'(t) = \sigma(t)x(t) + \exp(-k_1 t)\int_0^t B(t,s)x^{\frac{2}{3}}(s)ds. \tag{1.7}$$

The author [26] gives sufficient conditions under which the trivial solutions of (VIDE) (1.6) and (VIDE) (1.7) are (UEAS). It should be noted that (VIDE) (1.6) and (VIDE) (1.7) are equivalent when $\delta = k_1$. However, the results given in [26] with respect to that (VIDEs) are different since different (LFs) are used therein.

In this paper, instead of (VIDE) (1.6) and (VIDE) (1.7), we consider the nonlinear (VIDEs) with constant time-lag

$$x'(t) = -\gamma(t)h(x) + \exp(-\delta t)\int_{t-\tau}^t K(t,s)p^{\frac{2}{3}}(x(s))ds \tag{1.8}$$

and

$$x'(t) = -\gamma(t)f(x) + \exp(-k_1 t)\int_{t-\tau}^t K(t,s)q^{\frac{2}{3}}(x(s))ds, \tag{1.9}$$

respectively, in which $x(t) = \varphi_4(t)$, $0 \le t \le t_0$, $t-\tau \ge 0$, τ is fixed constant time-lag, $x \in \Re$, $\gamma(t) : [0,\infty) \to (0,\infty)$ and $h, f, p, q : \Re \to \Re$ are continuous functions with $h(0) = f(0) = p(0) = q(0) = 0$, and the function $K(t,s)$ is continuous for $0 \le s \le t < \infty$.

Let

$$h_1(x) = \begin{cases} \frac{h(x)}{x}, & x \neq 0 \\ h'(0), & x = 0 \end{cases}$$

and

$$f_1(x) = \begin{cases} \frac{f(x)}{x}, & x \neq 0 \\ f'(0), & x = 0. \end{cases}$$

Then, we have from (VIDE) (1.8) and (VIDE) (1.9) that

$$x'(t) = -\gamma(t)h_1(x)x + \exp(-\delta t)\int_{t-\tau}^t K(t,s)p^{\frac{2}{3}}(x(s))ds \tag{1.10}$$

and

$$x'(t) = -\gamma(t)f_1(x)x + \exp(-\delta t)\int_{t-\tau}^t K(t,s)q^{\frac{2}{3}}(x(s))ds, \tag{1.11}$$

respectively.

We investigate here the (UEAS) of the trivial solutions of (VIDEs) (1.4), (1.8) and (1.9) via new suitable (LFs).

It is notable form the mentioned information that Raffoul [26] considers certain non-linear (VIDEs) without time-lag. In spite of this case, here, we consider certain new non-linear (VIDEs) with time-lag. In fact, when we choose $\phi(x) = x$, $\psi(s, x(s), x(s-\tau)) = f(s, x(s))$, $g(t, x(t)) = h(t, x(t), x(t-\tau))$ and take zero instead of the term $t - \tau$, then our equation, equation (VIDE) (1.4) reduces to (VIDE) (1.3) discussed by Raffoul [26]. In addition, if take zero instead of the term $t - \tau$, $h(x) = x$, $p^{\frac{2}{3}}(x) = x^{\frac{2}{3}}$ and $f(x) = x$, $p^{\frac{2}{3}}(x) = x^{\frac{2}{3}}$, then (VIDEs)(1.8) and (1.9) reduce to (VIDEs) (1.6) and (1.7), respectively. That is, (VIDEs) (1.4), (1.8) and (1.9) include and extend (VIDEs) (1.3), (1.6) and (1.7) discussed by Raffoul [26]. Moreover, our results improve the related results of [26] from the cases of without delay to the more general cases with time-lag.

Finally, if we set the coefficient of the stochastic term in the (SDE) investigated in [29] equal to zero, then we have the following delay (VIDE) of the form

$$x'(t) = f(t, x(t), x(t-h), \int_{t-h}^{t} A(t,s)h(s, x(s))ds.$$

Hence, it is clear that the former delay (VIDE) includes the delay (VIDEs) (1.4), (1.6) and (1.7). However, the results established in [2] and that we are going to establish here, Theorems 2.1-2.3, are different from each other. Because the results proved in [29], their assumptions and the (LFs) constructed therein are different form that we are going to give and prove in this paper. By the way we would like to state that the results in [29] and those we are going to give here complement to each other.

All the mentioned information shows that the present paper has novel and original results and makes contribution to the literature.

We now consider the differential system given by (FDE) (1.2). Let $t_0 \geq 0$. Then, for each continuous function $\varphi : [0, t_0] \to \Re^n$, there exists in any case one function $x(t) = x(t, t_0, \phi)$, $x(.) \in C[t_0, I]$, appeasing (FDE) (1.2) for $0 \leq t_0 \leq t \leq I$ such that $x(t, t_0, \varphi) = \varphi$ for $0 \leq t \leq t_0$.

Definition 1.1. *The trivial solution of (FDE) (1.2) is (EAS) if for a positive constant λ any solution $x(t, t_0, \varphi)$ of (FDE) (1.2) fulfills*

$$\|x(t, t_0, \varphi)\| \leq K(|\varphi|, t_0) \exp(-\lambda(t - t_0)) \tag{1.12}$$

for all $t \geq t_0$, where $K(|\varphi|, t_0)$ is a positive constant depends on φ and t_0, and φ is a stated initial function, which is bounded and continuous.

If the constant K in (1.12) is independent of t_0, then it is said that the trivial solution of (FDE) (1.2) is (UEAS).

The following three basic theorems are needed for the (S) results of this paper.

Theorem 1.1. *(Raffoul [26]) Assume that $D \subset \Re^n$contains the origin and there exists a (LF) $V : D \to [0, \infty)$ such that the below assumptions hold for all $(t, x) \in [0, \infty) \times D$:*

(A1) $\lambda_1 \|x\|^\rho \leq V(t, x(.))$,

(A2) $\dot{V}(t, x) \leq \lambda_3 V(t, x(.)) + L \exp(-\delta t)$,

where $\lambda_1 > 0$, $\lambda_3 > 0$, $\rho > 0$, $\delta > 0$, $L \geq 0$ are real constants and $0 < \epsilon \min\{\lambda_3, \delta\}$.
Then the trivial solution of (FDE) (1.2) is (UEAS).

Proof. See Raffoul [26]. □

Theorem 1.2. *(Raffoul [26]) Assume that $D \subset \Re^n$ contains the origin and there exists a continuously differentiable (LF) $V : [0, \infty) \times D \to [0, \infty)$ such that the below assumptions hold for all $(t, x) \in [0, \infty) \times D$:*

(B1) $\lambda_1 \|x\|^\rho \leq V(t, x(.)) \leq \lambda_2 W_2(|x|) + \lambda_2 \int_0^t \phi_1(t, s) W_3(|x(s)|) ds$,

(B2) $\dot{V}(t, x(.)) \leq -\lambda_3 W_4(|x|) - \lambda_3 \int_0^t \phi_2(t, s) W_5(|x(s)|) ds + L^* \exp(-\delta t)$,

where $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 > 0$, $\rho > 0$, $\delta > 0$ and $L^* > 0$ are real constants, and $\phi_1(t, s)$ and $\phi_2(t, s)$ are scalar valued and continuous functions for $0 \leq s \leq t < \infty$. If the below assumption

(B3) $W_2(|x|) - W_4(|x|) + \int_0^t [\phi_1(t, s) W_3 |x(s)| - \phi_2(t, s) W_3 |x(s)|] \, ds \leq L_1 \exp(-\delta t)$

holds for some positive constants L_1 and B with $\int_0^t \phi_1(t, s) ds \leq B$, then the trivial solution of (FDE) (1.2) is (UEAS).

Proof. See Raffoul [26]. □

Theorem 1.3. *(Raffoul [26]) Assume that $D \subset \Re^n$ contains the origin and there exists a continuously differentiable (LF) $V : [0,\infty) \times D \to [0,\infty)$ such that the following assumptions hold for all $(t,x) \in [0,\infty) \times D$:*

(C1) $\lambda_1(t) \|x\|^\rho \leq V(t,x(.)) \leq \lambda_2(t) W_2(|x|) + \lambda_2(t) \int_0^t \phi_1(t,s) W_3(|x(s)|) ds,$

(C2) $\dot{V}(t,x(.)) \leq -\lambda_3(t) W_4(|x|) - \lambda_3(t) \int_0^t \phi_2(t,s) W_5(|x(s)|) ds + \overline{L} \exp(-\delta t),$

where $\lambda_1(t)$, $\lambda_2(t)$, and $\lambda_3(t)$ are some positive and continuous functions, with $\lambda_1(t)$, is non-decreasing, $\rho > 0$, $\delta > 0$ and $\overline{L} > 0$ are some real constants, and $\phi_1(t,s) \geq 0$ and $\phi_2(t,s) \geq 0$ are scalar valued and continuous functions for $0 \leq s \leq t < \infty$. If the assumption

(C3) $W_2(|x|) - W_4(|x|) + \int_0^t [\phi_1(t,s) W_3 |x(s)| - \phi_2(t,s) W_3 |x(s)|] \, ds \leq L_2 \exp(-\delta t)$

holds for some positive constants L_2, B and N with $\int_0^t \phi_1(t,s) ds \leq B$ and $\lambda_2(t) \leq N$, then the trivial solution of (FDE) (1.2) is (UEAS).

Proof. See Raffoul [26]. □

2. Exponential stability

A. Assumptions

Let δ_0, δ_1, ϵ, β, τ, K_0 and K be positive constants. Assume that the following assumptions are true:

(D1) $\phi_1(x) \geq \delta_1$ for $x \in \Re$,

(D2) $|\psi(t,x,x(t-\tau))| \leq \sigma(t) \min(|x|, |x(t-\tau)|)$,

$|h(t,x,x(t-\tau))| \leq \theta(t) \min(|x|^{\frac{1}{2}}, |x(t-\tau)|^{\frac{1}{2}})$, where $\sigma(t)$ and $\theta(t)$ are positive, bounded and continuous functions,

(D3) $|C(t,s,x(s))| \geq K_0 \int\limits_{t-\tau}^{\infty} |C(u+\tau,s,x(s))| \, du$ with $0 \leq s \leq t \leq u < \infty$,

(D4) $|C(t+\tau,s,x(s))| \geq K_0 \int\limits_{t-\tau}^{\infty} |C(u+\tau,s,x(s))| \, du$, and

$\int\limits_0^{t_0} \int\limits_{t_0-\tau}^{\infty} |C(u+\tau,s,x(s))| \, du \sigma(s) ds \leq \delta_0 < \infty$ for all $t_0 - \tau \geq 0$.

(D5) $\alpha(t)\delta_1 - \frac{1}{2} - \beta\sigma(t)\int\limits_{t-\tau}^{\infty} |C(u+\tau, t, x(t))|\, du \geq K$ with $K_0 \geq \frac{\beta K}{\varepsilon} > 0$.

Theorem 2.1. *If assumptions (D1)-(D5) are true, then the trivial solution of (VIDE) (1.4) is (UEAS).*

Proof. Describe a (LF) $V_1 = V_1(t) = V_1(t, x(t))$ by

$$V_1 = |x| + \beta \int\limits_0^t \int\limits_{t-\tau}^{\infty} |C(u+\tau, s, x(s)|\, du\, |\psi(s, x(s), x(s-\tau))|\, ds, \tag{2.1}$$

where $\beta > 0$, $\beta \in \Re$, which is determined later in the proof.

Positive definiteness of the (LF) V_1 is clear. That is,

$$V_1(t, 0) = 0 \text{ and } V_1(t, x) \geq |x|\,.$$

Differentiating the (LF) V_1 with respect to t, we obtain from (2.1) that

$$\begin{aligned}
V_1' &= \frac{x}{|x|}x' + \beta \int\limits_{t-\tau}^{\infty} |C(u+\tau, t, x(t))|\, du\, |\psi(t, x, x(t-\tau))| \\
&\quad -\beta \int\limits_0^t |C(t, s, x(s))|\, |\psi(s, x(s), x(s-\tau))| ds \\
&\leq -\alpha(t)\phi_1(x)\, |x| + \int\limits_{t-\tau}^{t} |C(t, s, x(s))|\, |\psi(s, x(s), x(s-\tau))|\, ds \\
&\quad + |h(t, x, x(t-\tau))| \\
&\quad +\beta \int\limits_{t-\tau}^{\infty} |C(u+\tau, t, x(t))|\, du\, |\psi(t, x, x(t-\tau))| \\
&\quad -\beta \int\limits_0^t |C(t, s, x(s))|\, |\psi(s, x(s), x(s-\tau))|\, ds. \qquad (2.2)
\end{aligned}$$

Let

$$\beta = 1 + \varepsilon, \quad \varepsilon > 0.$$

Then, it follows from (2.2), the hypotheses of Theorem 2.1 and the reality $2\,|mn| \leq m^2 + n^2$ that

$$
\begin{aligned}
V_1' \leq\ & -\alpha(t)\phi_1(x)\,|x| + |h(t,x,x(t-\tau))| \\
& +\beta \int_{t-\tau}^{\infty} |C(u+\tau,t,x(t))|\,du\,|\psi(t,x,x(t-\tau))| \\
& -\varepsilon \int_0^t |C(t,s,x(s))|\,|\psi(s,x(s),x(s-\tau))|\,ds \\
\leq\ & -\alpha(t)\phi_1(x)\,|x| + \theta(t)\,|x|^{\frac{1}{2}} \\
& +\beta\sigma(t)\,|x| \int_{t-\tau}^{\infty} |C(u+\tau,t,x(t))|\,du \\
& -\varepsilon \int_0^t |C(t,s,x(s))|\,|\psi(s,x(s),x(s-\tau))|\,ds
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & -\left[\alpha(t)\phi_1(x) - \frac{1}{2} - \beta\sigma(t) \int_{t-\tau}^{\infty} |C(u+\tau,t,x(t))|\,du\right]|x| \\
& -\varepsilon \int_0^t |C(t,s,x(s))|\,|\psi(s,x(s),x(s-\tau))|\,ds + \frac{1}{2}\theta^2(t) \\
\leq\ & -K\,|x| - \varepsilon \int_0^t |C(t,s,x(s))|\,|\psi(s,x(s),x(s-\tau))|\,ds + \frac{1}{2}\theta^2(t)
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & -K\,|x| - \varepsilon K_0 \int_0^t \int_{t-\tau}^{\infty} |C(u+\tau,s,x(s))|\,du\,|\psi(s,x(s),x(s-\tau))|\,ds \\
& +\frac{1}{2}\theta^2(t) \\
\leq\ & -K\left[|x| + \beta \int_0^t \int_{t-\tau}^{\infty} |C(u+\tau,s,x(s))|\,du\,|\psi(s,x(s),x(s-\tau))|\,ds\right] \\
& +\frac{1}{2}\theta^2(t) \\
\leq\ & -KV_1(t,x(t)) + \frac{1}{2}\theta^2(t).
\end{aligned}
$$

If $\theta^2(t) \leq \exp(-\delta t)$ for a positive constant δ, $\lambda_1 = 1, \rho = 1$ and $L = 2^{-1}$, then we can show the verification of all the assumptions of Theorem 1.1 . That is, the trivial solution of (VIDE) (1.4) is (UEAS). This finishes the proof of Theorem 2.1. □

B. Assumptions

Let α_1, δ and τ be positive constants such that the following assumptions are correct:

(E1) $h(0) = p(0) = 0, h_1(x) \geq 1, x \in \Re, |p(x)| \leq \alpha_1^{1/2}|x|, 0 < \alpha_1 < 1,$

(E2) $2\gamma(t) - \exp(-\delta t) \int\limits_{t-\tau}^{t} |K(t,s)|\, ds - \int\limits_{t-\tau}^{\infty} \exp(-\delta(u+\tau))\, |K(u+\tau,t)|\, du \geq 1,$

where $\gamma(t)$ is a positive, bounded and continuous function,

(E3) $(-1 + \frac{2}{3}\alpha_1) \exp(-\delta t)\, |K(t,s)| \geq \int\limits_{t-\tau}^{\infty} \exp(-\delta(u+\tau)\, |K(u+\tau,s)|\, du$

with $0 \leq s \leq t \leq u < \infty$.

Theorem 2.2. *If assumptions (E1)-(E3) are true, then the null solution of (VIDE) (1.8) is (UEAS).*

Proof. We describe a (LF) $V_2 = V_2(t) = V_2(t, x(t))$ by

$$V_2 = x^2 + \lambda \int\limits_0^t \int\limits_{t-\tau}^{\infty} \exp(-\delta(u+\tau)\, |K(u+\tau,s)|\, du x^2(s) ds \tag{2.3}$$

where $\lambda > 0$, $\lambda \in \Re$, and we determine that constant later.

Obviously, it follows that the (LF) V_2 is positive definite, since

$$V_2(t,0) = 0 \quad \text{and} \quad V_2(t,x) \geq x^2.$$

Differentiating the (LF) V_2, it follows from (LF) (2.3) and (VIDE) (1.8) that

$$V_2' \;=\; 2xx' + \lambda x^2 \int\limits_{t-\tau}^{\infty} \exp(-\delta(u+\tau)\, |K(u+\tau,t)|\, du$$

$$-\lambda \int\limits_0^t \exp(-\delta t)\, |K(t,s)|\, x^2(s) ds$$

$$=\text{-}2\gamma(t)g_1(x)x^2+2\exp(-\delta t)x\int\limits_{t-\tau}^{t}K(t,s)p^{\frac{2}{3}}(x(s))ds$$

$$+\lambda x^2\int\limits_{t-\tau}^{\infty}\exp(-\delta(u+\tau))\left|K(u+\tau,t)\right|du$$

$$\text{-}\lambda\int\limits_{0}^{t}\exp(-\delta t)\left|K(t,s)\right|x^2(s)ds.$$

By the disparity $2\left|mn\right|\leq m^2+n^2$ and the assumption $g_1(x)\geq 1$, we reach that

$$
\begin{aligned}
V_2' \leq\ & -2\gamma(t)x^2+\exp(-)\int\limits_{t-\tau}^{t}\left|K(t,s)\right|[x^2(t)+p^{\frac{4}{3}}(x(s))]ds \\
& +\lambda x^2\int\limits_{t-\tau}^{\infty}\exp(-\delta(u+\tau))\left|K(u+\tau,t)\right|du \\
& -\lambda\int\limits_{0}^{t}\exp(-\delta t)\left|K(t,s)\right|x^2(s)ds \\
=\ & -2\gamma(t)x^2+\exp(-\delta t)x^2\int\limits_{t-\tau}^{t}\left|K(t,s)\right|ds \\
& +\exp(-\delta t)\int\limits_{t-\tau}^{t}(t,s)|\,p^{\frac{4}{3}}(x(s))ds \\
& +\lambda x^2\int\limits_{t-\tau}^{\infty}\exp(-\delta(u+\tau))\left|K(u+\tau,t)\right|du \\
& -\lambda\int\limits_{0}^{t}\exp(-\delta t)\left|K(t,s)\right|x^2(s)ds.
\end{aligned}
\tag{2.4}
$$

Let $a=\frac{3}{2}$ and $b=3$. By using the Young's inequality,

$$mn\leq\frac{1}{a}m^a+\frac{1}{b}n^b,\quad \frac{1}{a}+\frac{1}{b}=1,$$

and assumption (E1), we have from (2.4) that

$$\begin{aligned}
\int\limits_{t-\tau}^{t} |K(t,s)|\, p^{\frac{4}{3}}(x(s))ds &= \int\limits_{t-\tau}^{t} |K(t,s)|^{\frac{1}{3}} |K(t,s)|^{\frac{2}{3}} p^{\frac{4}{3}}(x(s))ds \\
&\le \int\limits_{t-\tau}^{t} \left[\frac{1}{3}|K(t,s)| + \frac{2}{3}|K(t,s)|\, p^2(x(s))\right] ds \\
&\le \frac{1}{3}\int\limits_{t-\tau}^{t} (t,s)|\, ds + \frac{2}{3}\alpha_1 \int\limits_{t-\tau}^{t} |K(t,s)|\, x^2(s)ds.
\end{aligned}$$

Setting former inequality into (2.4), we have

$$\begin{aligned}
V_2' \le\ & -2\gamma(t)x^2 + \exp(-\delta t)x^2 \int\limits_{t-\tau}^{t} |K(t,s)|\, ds \\
& + \frac{1}{3}\exp(-\delta t)\int\limits_{t-\tau}^{t} |K(t,s)|\, ds \\
& + \frac{2}{3}\alpha_1 \exp(-\delta t)\int\limits_{t-\tau}^{t} |K(t,s)|\, x^2(s)ds \\
& + \lambda x^2 \int\limits_{t-\tau}^{\infty} \exp(-\delta(u+\tau))\, |K(u+\tau,t)|\, du \\
& - \lambda \int\limits_{0}^{t} \exp(-\delta t)\, |K(t,s)|\, x^2(s)ds \\
=\ & -[2\gamma(t) - \exp(-\delta t)\int\limits_{t-\tau}^{t} |K(t,s)|\, ds \\
& - \lambda \int\limits_{t-\tau}^{\infty} \exp(-\delta(u+\tau))\, |K(u+\tau,t)|\, du]x^2 \\
& + \frac{1}{3}\exp(-\delta t)\int\limits_{t-\tau}^{t} |K(t,s)|\, ds
\end{aligned}$$

$$+\frac{2}{3}\alpha_1 \exp(-\delta t)\int\limits_{t-\tau}^{t} |K(t,s)|\, x^2(s)ds$$

$$-\lambda\int\limits_{0}^{t} \exp(-\delta t)\, |K(t,s)|\, x^2(s)ds.$$

Let $\lambda = 1$ and $L^* = \frac{1}{3}\int\limits_{t-\tau}^{t} |K(t,s)|\, ds$. Then

$$\begin{aligned} V_2' \quad \leq \quad & -[2\gamma(t) - \exp(-\delta t)\int\limits_{t-\tau}^{t} |K(t,s)|\, ds \\ & -\int\limits_{t-\tau}^{\infty} \exp(-\delta(u+\tau))\, |K(u+\tau,t)|\, du]x^2 \end{aligned}$$

$$+(-1+\frac{2}{3}\alpha_1)\int\limits_{0}^{t} \exp(-\delta t)\, |K(t,s)|\, x^2(s)ds + L^*\exp(-\delta t).$$

By assumption (E2), we have

$$V_2' \leq -x^2 + (-1+\frac{2}{3}\alpha_1)\int\limits_{0}^{t} \exp(-\delta t)\, |K(t,s)|\, x^2(s)ds + L^*\exp(-\delta t), \tag{2.5}$$

Let $W_2(.) = W_4(.) = x^2(t)$, $W_3(.) = W_5(.) = x^2(s)$, $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 1$,

$$\phi_1(t,s) = \int\limits_{t-\tau}^{\infty} \exp(-\delta(u+\tau))\, |K(u+\tau,t)|\, du$$

and

$$\phi_2(t,s) = (-1+\frac{2}{3}\alpha_1)\exp(-\delta t)\, |K(t,s)|\, .$$

Then, it follows from (2.3) and (2.5), respectively, that

$$\begin{aligned} x^2 \quad \leq \quad & V_2 = x^2 + \lambda\int\limits_{0}^{t}\int\limits_{t-\tau}^{\infty} \exp(-\delta(u+\tau))\, |K(u+\tau,s)|\, du x^2(s)ds \\ \leq \quad & \lambda_2 W_2(|x|) + \lambda_2\int\limits_{0}^{t} \phi_1(t,s)W_3(|x(s)|)ds \end{aligned}$$

and

$$\begin{aligned} V_2' &\leq -x^2 + (-1 + \frac{2}{3}\alpha_1) \int_0^t \exp(-\delta t) \left| K(t,s) \right| x^2(s) ds + L^* \exp(-\delta t) \\ &= -\lambda_3 W_4(|x|) - \lambda_3 \int_0^t \phi_2(t,s) W_5(|x(s)|) ds + L^* \exp(-\delta t). \end{aligned}$$

Hence, we can conclude that assumptions (B1) and (B2) of Theorem 1.2 hold.

Then, in view of the assumption

$$\frac{1}{3} \exp(-\delta t) \left| K(t,s) \right| \geq \int_{t-\tau}^{\infty} \exp(-\delta(u+\tau)) \left| K(u+\tau, s) \right| du,$$

we get

$$\begin{aligned} & W_2(|x|) - W_4(|x|) + \int_0^t \left[\phi_1(t,s) W_3 \left| x(s) \right| - \phi_2(t,s) W_3 \left| x(s) \right| \right] ds \\ &= x^2 - x^2 + \int_0^t \left[\int_{t-\tau}^{\infty} \exp(-\delta(u+\tau)) \left| K(u+\tau, t) \right| du + \right. \\ & \qquad \left. (-1 + \frac{2}{3}\alpha_1) \exp(-\delta t) \left| K(t,s) \right| \right] x^2(s) ds \\ &= \int_0^t \left[\int_{t-\tau}^{\infty} \exp(-\delta(u+\tau)) \left| K(u+\tau, t) \right| du + \right. \\ & \qquad \left. (-1 + \frac{2}{3}\alpha_1) \exp(-\delta t) \left| K(t,s) \right| \right] x^2(s) ds \leq 0. \end{aligned}$$

Therefore, the last estimate shows that assumption (B3) of Theorem 1.2 holds for $L_1 = 0$. That is, the trivial solution of (VIDE) (1.8) is (UEAS). Thus, the desirable result is available for the proof of Theorem 2.2. □

C.Assumptions

Let β and k_2 be positive constants. Assume that the below assumptions hold:

(F1) $f(0) = q(0) = 0, f_1(x) \geq 1, (x \neq 0), \left| q(x) \right| \leq \beta^{\frac{1}{2}} \left| x \right|, 0 < \beta < 1,$

(F2) $k_2+2\gamma(t)-\int\limits_{t-\tau}^{t}|K(t,s)|\,ds-\int\limits_{t-\tau}^{\infty}|K(u+\tau,t)|\,du \geq 1$ with $0 \leq s \leq t \leq u < \infty$, where $\gamma(t)$ is a positive, bounded and continuous function,

(F3) $\int\limits_{t-\tau}^{t}|K(t,s)|\,ds < \infty$ and $\int\limits_{0}^{t}\int\limits_{t-\tau}^{\infty}|K(u+\tau,s)|\,duds < \infty$with $0 \leq s \leq t \leq u < \infty$ and $(1-\frac{2}{3}\beta)\exp(-\delta t)\,|K(t,s)| \geq \int\limits_{t-\tau}^{\infty}\exp(-\delta(u+\tau))\,|K(u+\tau,s)|\,du$.

Theorem 2.3. *Let δ, k_1, k_2 be positive constants such that $1 < \delta = k_1+k_2$. If assumptions (F1)-(F3) are correct, then the null solution of (VIDE) (1.9) is (UEAS).*

Proof. We define a (LF) $V_3 = V_3(t) = V_3(t,x(t))$ by

$$V_3 = \exp(-k_2t)\left[x^2+\mu\int\limits_{0}^{t}\int\limits_{t-\tau}^{\infty}|K(u+\tau,s)|\,dux^2(s)ds\right], \tag{2.6}$$

where $\mu > 0$, $\mu \in \Re$, and it is determined later.

It can be seen the positive definiteness of the (LF) V_3, since

$$V_3(t,0) = 0 \quad \text{and} \quad V_3(t,x) \geq x^2\exp(-k_2t).$$

Differentiating the (LF) V_3 with respect to t, along solutions of (VIDE) (1.9), we obtain from (LF) (2.6) and (VIDE) (1.9) that

$$\begin{aligned}
V_3' &= -k_2\exp(-k_2t)\left[x^2+\mu\int\limits_{0}^{t}\int\limits_{t-\tau}^{\infty}|K(u+\tau,s)|\,dux^2(s)ds\right] \\
&\quad +\exp(-k_2t)\left[2xx'+\mu x^2\int\limits_{t-\tau}^{\infty}|K(u+\tau,t)|\,du\right] \\
&\quad -\exp(-k_2t)\left[\mu\int\limits_{0}^{t}|K(t,s)|\,x^2(s)ds\right]
\end{aligned}$$

$$= -k_2\exp(-k_2t)\left[x^2+\mu\int\limits_{0}^{t}\int\limits_{t-\tau}^{\infty}|K(u+\tau,s)|\,dux^2(s)ds\right]$$

$$
\begin{aligned}
&-2\exp(-k_2t)\left[\gamma(t)f_1(x)x^2\right] \\
&+2\exp(-(k_1+k_2)t)\left[x\int_{t-\tau}^{t}K(t,s)q^{\frac{2}{3}}(x(s))ds\right] \\
&+\mu\exp(-k_2t)\left[x^2\int_{t-\tau}^{\infty}|K(u+\tau,t)|\,du\right] \\
&-\mu\exp(-k_2t)\left[\int_{0}^{t}|K(t,s)|\,x^2(s)ds\right].
\end{aligned}
$$

In view of the inequality $2|mn|\leq m^2+n^2$ and the assumption $f_1(x)\geq 1$, we have

$$
\begin{aligned}
V_3' \;\leq\;& -k_2\exp(-k_2t)x^2-k_2\mu\exp(-k_2t)\int_{0}^{t}\int_{t-\tau}^{\infty}|K(u+\tau,s)|\,du x^2(s)ds \\
&+2\exp(-(k_1+k_2)t)\left[|x(t)|\int_{t-\tau}^{t}|K(t,s)|\,q^{\frac{2}{3}}(x(s))ds\right] \\
&-2\exp(-k_2t)\gamma(t)x^2+\mu\exp(-k_2t)\left[x^2\int_{t-\tau}^{\infty}|K(u+\tau,t)|\,du\right] \\
&-\mu\exp(-k_2t)\left[\int_{0}^{t}|K(t,s)|\,x^2(s)ds\right] \\
\leq\;& -[k_2+2\gamma(t)]\exp(-k_2t)x^2 \\
&-k_2\mu\exp(-k_2t)\int_{0}^{t}\int_{t-\tau}^{\infty}|K(u+\tau,s)|\,du x^2(s)ds \\
&+\exp(-(k_1+k_2))x^2\left[\int_{t-\tau}^{t}|K(t,s)|\,ds\right] \\
&+\exp(-(k_1+k_2)t)\left[\int_{t-\tau}^{t}|K(t,s)|\,q^{\frac{4}{3}}(x(s))ds\right]
\end{aligned}
$$

$$+\mu \exp(-k_2 t)x^2 \left[\int_{t-\tau}^{\infty} |K(u+\tau,t)|\, du \right]$$

$$-\mu \exp(-k_2 t) \left[\int_{0}^{t} |K(t,s)|\, x^2(s) ds \right]. \tag{2.7}$$

Let $a = \frac{3}{2}$ and $b = 3$. By using the Young's inequality,

$$mn \leq \frac{1}{a} m^a + \frac{1}{b} n^b, \quad \frac{1}{a} + \frac{1}{b} = 1,$$

and assumption (F1), we arrive at

$$\begin{aligned}
&\exp(-(k_1+k_2)t) \int_{t-\tau}^{t} |K(t,s)|\, q^{\frac{4}{3}}(x(s)) ds \\
= {} & \exp(-(k_1+k_2)t) \int_{t-\tau}^{t} |K(t,s)|^{\frac{1}{3}} |K(t,s)|^{\frac{2}{3}} q^{\frac{4}{3}}(x(s)) ds \\
\leq {} & \exp(-(k_1+k_2)t) \int_{t-\tau}^{t} [\frac{1}{3} |K(t,s)| + \frac{2}{3} |K(t,s)|\, q^2(x(s))] ds \\
\leq {} & \frac{1}{3} \exp(-(k_1+k_2)t) \int_{t-\tau}^{t} |K(t,s)|\, ds \\
& + \frac{2}{3} \beta \exp(-(k_1+k_2)t) \int_{t-\tau}^{t} |K(t,s)|\, x^2(s) ds.
\end{aligned}$$

Placing the former inequality into (2.7), we obtain

$$\begin{aligned}
V_3' \leq {} & -\exp(-k_2 t) \left[k_2 + 2\gamma(t) - \int_{t-\tau}^{t} |K(t,s)|\, ds - \mu \int_{t-\tau}^{\infty} |K(u+\tau,t)|\, du \right] x^2 \\
& - k_2 \mu \exp(-k_2 t) \int_{0}^{t} \int_{t-\tau}^{\infty} |K(u+\tau,s)|\, du x^2(s) ds \\
& + \frac{1}{3} \exp(-(k_1+k_2)t) \int_{t-\tau}^{t} |K(t,s)|\, ds
\end{aligned}$$

$$+\frac{2}{3}\beta\exp(-(k_1+k_2)t)\int\limits_{t-\tau}^{t}|K(t,s)|\,x^2(s)ds$$
$$-\mu\exp(-k_2t)\left[\int\limits_{0}^{t}|K(t,s)|\,x^2(s)ds\right].$$

Let $\mu = 1$ and $\overline{L} = \frac{1}{3}\int\limits_{t-\tau}^{t}|K(t,s)|\,ds$. Then, using assumption (F2) of Theorem 2.3, we conclude that

$$\begin{aligned}
V_3' \leq\ & -\exp(-k_2t)\left[k_2+2\gamma(t)-\int\limits_{t-\tau}^{t}|K(t,s)|\,ds-\int\limits_{t-\tau}^{\infty}|K(u+\tau,t)|\,du\right]x^2\\
& +\frac{1}{3}\exp(-(k_1+k_2)t)\int\limits_{t-\tau}^{t}|K(t,s)|\,ds\\
& +\frac{2}{3}\beta\exp(-(k_1+k_2)t)\int\limits_{t-\tau}^{t}|K(t,s)|\,x^2(s)ds\\
& -\exp(-k_2t)\left[\int\limits_{0}^{t}|K(t,s)|\,x^2(s)ds\right]\\
\leq\ & -\exp(-k_2t)\left[k_2+2\gamma(t)-\int\limits_{t-\tau}^{t}|K(t,s)|\,ds-\int\limits_{t-\tau}^{\infty}|K(u+\tau,t)|\,du\right]x^2\\
& +(-1+\frac{2}{3}\beta)\exp(-k_2t)\int\limits_{0}^{t}|K(t,s)|\,x^2(s)ds+\overline{L}\exp(-(k_1+k_2)t)
\end{aligned}$$

$$\begin{aligned}
\leq\ & -\exp(-k_2t)x^2+(-1+\frac{2}{3}\beta)\exp(-k_2t)\int\limits_{0}^{t}|K(t,s)|\,x^2(s)ds\\
& +\overline{L}\exp(-(k_1+k_2)t).
\end{aligned}$$

Let

$$W_2(.)=W_4(.)=x^2(t),\quad W_3(.)=W_5(.)=x^2(s),$$

$$\lambda_1 = \exp(-k_2 t), \lambda_2 = \exp(-k_2 t), \lambda_3 = \exp(-k_2 t)$$

$$\phi_1(t,s) = \int_{t-\tau}^{\infty} \exp(-\delta(u+\tau)) \left| K(u+\tau,t) \right| du$$

and

$$\phi_2(t,s) = (1 - \frac{2}{3}\beta) \left| K(t,s) \right|.$$

Then, it follows from (2.6) and the last estimate that

$$\begin{aligned} x^2 \exp(-k_2 t) &\leq V_3 = \exp(-k_2 t) \left[x^2 + \mu \int_0^t \int_{t-\tau}^{\infty} \left| K(u+\tau,s) \right| du x^2(s) ds \right] \\ &\leq \lambda_2 W_2(|x|) + \lambda_2 \int_0^t \phi_1(t,s) W_3(|x(s)| \, ds \end{aligned}$$

and

$$\begin{aligned} V_3' &\leq -x^2 + (-1 + \frac{2}{3}\beta) \int_0^t \exp(-\delta t) \left| K(t,s) \right| x^2(s) ds + \overline{L} \exp(-k_2 t) \\ &= -\lambda_3 W_4(|x|) - \lambda_3 \int_0^t \phi_2(t,s) W_5(|x(s)|) ds + \overline{L} \exp(-k_2 t). \end{aligned}$$

Hence, we can conclude that assumptions (B1) and (B2) of Theorem 1.3 hold.

Then, in view of the assumption

$$(1 - \frac{2}{3}\beta) \exp(-\delta t) \left| K(t,s) \right| \geq \int_{t-\tau}^{\infty} \exp(-\delta(u+\tau)) \left| K(u+\tau,s) \right| du,$$

we get

$$
\begin{aligned}
W_2(|x|) - W_4(|x|) + \int_0^t \left[\phi_1(t,s)W_3\,|x(s)| - \phi_2(t,s)W_3\,|x(s)|\right] ds \\
= x^2 - x^2 + \int_0^t \Bigg[\int_{t-\tau}^{\infty} \exp(-\delta(u+\tau))\,|K(u+\tau,t)|\,du + \\
(-1+\frac{2}{3}\beta)\exp(-\delta t)\,|K(t,s)| \Bigg] x^2(s)ds \\
= \int_0^t \Bigg[\int_{t-\tau}^{\infty} \exp(-\delta(u+\tau))\,|K(u+\tau,t)|\,du + \\
(-1+\frac{2}{3}\beta)\exp(-\delta t)\,|K(t,s)| \Bigg] x^2(s)ds \le 0.
\end{aligned}
$$

Therefore, the last estimate shows that assumption $(C2)$ of Theorem 1.3 holds for $\overline{L} = 0$. That is, the trivial solution of (VIDE) (1.9) is (UEAS). This brings to an end the proof of Theorem 2.3. □

Remark 2.1. *By Theorems 2.1, 2.2 and 2.3, we improve and extend the stability results of Raffoul [26] obtained for (VIDEs) without time-lag to their more general and time-lag forms. In addition, Theorems 2.1, 2.2 and 2.3 complement to the results in the references like Adıvar and Raffoul [1], Becker ([2],[3],[4]), Burton ([5],[6],[7]), Burton et al. [8], Burton and Mahfoud ([9],[10]), Eloe et al. [13], Furumochi and Matsuoka [14], Graef and Tunç [15], Graef et al. [16], Grimmer and Seifert [17], Gripenberg et al. [18], Hara et al. [19], Islam and Raffoul [20], Jordan [21], Levin [22], Mahfoud [23], Miller [24], Raffoul ([25],[26],[27],[28]), Raffoul and Ren [29], Raffoul and Unal [29], Rama Mohana Rao and Raghavendra [30], Rama Mohana Rao and Srinivas [31], Staffans [32], Tunç ([33],[34],[35]) Tunç and Ayhan [38], Vanualailai and Nakagiri [39], Wang [40], Wazwaz [41], Zhang [42], Da Zhang [43].*

By this way, we also mean that the non-linear (VIDEs) thought about and the appointed assumptions here are separate from that in the sources mentioned above. It is noticeable that investigators working on the qualitative properties of (VIDEs) or (IDEs) may derive advantages from that results obtained here. These cases show the newness and contribution of the current paper.

3. Conclusion

Here, we consider three non-linear (VIDEs) of first order with constant time-lag. The (UEAS) of the trivial solutions of that (VIDEs) is discussed by the (LF) approach. The results obtained generalize, compliment and improve the existing results in the literature.

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