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On Triple sequence space of Bernstein operator of Rough I - convergence pre-cauchy sequences

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Abstract

We introduce and study some basic properties of rough I - convergent pre-Cauchy sequences of triple sequence of Bernstein polynomials and also study the set of all rough I - limits of a pre-Cauchy sequence of triple sequence of Bernstein polynomials and relation between analytic ness and rough I - statistical convergence of pre-Cauchy sequence of a triple sequences of Bernstein polynomials .

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1. Introduction

The idea of rough convergence was first introduced by Phu [10-12] in finite dimensional normed spaces. He showed that the set LIM_x^r is bounded, closed and convex; and he introduced the notion of rough Cauchy sequence. He also investigated the relations between rough convergence and other convergence types and the dependence of LIM_x^r on the roughness of degree r .

Aytar [1] studied of rough statistical convergence and defined the set of rough statistical limit points of a sequence and obtained to statistical convergence criteria associated with this set and prove that this set is closed and convex. Also, Aytar [2] established that the r - limit set of the sequence is equal to intersection of these sets and that r - core of the sequence is equal to the union of these sets. Dündar and Cakan [9] investigated of rough ideal convergence and defined the set of rough ideal limit points of a sequence The notion of I - convergence of a triple sequence which is based on the structure of the ideal I of subsets of $\mathbf{N} \times \mathbf{N} \times \mathbf{N}$, where \mathbf{N} is the set of all natural numbers, is a natural generalization of the notion of convergence and statistical convergence.

In this paper we study some basic properties of rough I - convergence pre-Cauchy of a triple sequence of Bernstein polynomials in three dimensional cases which are not earlier. We study the set of all rough I - pre-Cauchy sequence of limits of a triple sequence of Bernstein polynomials and also the relation between analytic ness and rough I - convergence of pre-Cauchy sequence of a triple sequence of Bernstein polynomials.

Let K be a subset of the set of positive integers $\mathbf{N} \times \mathbf{N} \times \mathbf{N}$ and let us denote the set $K_{ik\ell} = \{(m, n, k) \in K : m \geq i, n \leq j, k \leq \ell\}$. Then the natural density of K is given by

$$\delta(K) = \lim_{i,j,\ell \rightarrow \infty} \frac{|K_{ij\ell}|}{ij\ell},$$

where $|K_{ij\ell}|$ denotes the number of elements in $K_{ij\ell}$.

The Bernstein operator of order (r, s, t) is given by

$$\begin{aligned} & B_{rst}(f, x) \\ &= \sum_{m=0}^r \sum_{n=0}^s \sum_{k=0}^t f\left(\frac{mnk}{rst}\right) (rm) (sn) (tk) x^{m+n+k} (1-x)^{(m-r)+(n-s)+(k-t)} \end{aligned}$$

where f is a continuous (real or complex valued) function defined on $[0, 1]$.

Throughout the paper, \mathbf{R} denotes the real of three dimensional space with metric (X, d) . Consider a triple sequence of Bernstein polynomials $(B_{mnk}(f, x))$ such that $(B_{mnk}(f, x)) \in \mathbf{R}, m, n, k \in \mathbf{N}$.

Let f be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of Bernstein polynomials $(B_{rst}(f, x))$ is said to be statistically convergent to $0 \in \mathbf{R}$, written as $st - \lim x = 0$, provided that the set

$$K_\epsilon := \{(m, n, k) \in \mathbf{N}^3 : |B_{mnk}(f, x) - f(x)| \geq \epsilon\}$$

has natural density zero for any $\epsilon > 0$. In this case, 0 is called the statistical limit of the triple sequence of Bernstein polynomials. i.e., $\delta(K_\epsilon) = 0$. That is,

$$\lim_{rst \rightarrow \infty} \frac{1}{rst} |\{(m, n, k) \leq (r, s, t) : |B_{mnk}(f, x) - f(x)| \geq \epsilon\}| = 0.$$

In this case, we write $\delta - \lim B_{mnk}(f, x) = f(x)$ or $B_{mnk}(f, x) \rightarrow^{SB} f(x)$.

Throughout the paper, \mathbf{N} denotes the set of all positive integers, χ_A - the characteristic function of $A \subset \mathbf{N}$, \mathbf{R} the set of all real numbers. A subset A of \mathbf{N} is said to have asymptotic density $d(A)$ if

$$d(A) = \lim_{ij\ell \rightarrow \infty} \frac{1}{ij\ell} \sum_{m=1}^i \sum_{n=1}^j \sum_{k=1}^\ell \chi_A(K).$$

A triple sequence (real or complex) can be defined as a function $x : \mathbf{N} \times \mathbf{N} \times \mathbf{N} \rightarrow \mathbf{R}(\mathbf{C})$, where \mathbf{N}, \mathbf{R} and \mathbf{C} denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by Sahiner et al. [13,14], Esi et al. [3-6], Datta et al. [7], Subramanian et al. [15], Debnath et al. [8], Tripathy and Goswami [16], [17], [18], [19]) and many others.

A very interesting notion was introduced, that of statistically pre-Cauchy sequences and it was shown that statistically convergent sequences are always statistically pre-Cauchy and on the other hand under certain general conditions statistical pre-Cauchy condition implies statistical convergence of a sequence (and so one do not have to guess the limit of the statistically convergent sequence before hand).

A triple sequence of Bernstein polynomials is said to be triple Bernstein polynomials pre-Cauchy of analytic if

$$\sup_{m,n,k} \frac{1}{r^6 s^6 t^6} |B_{mnk}(f, x) - B_{uvw}(f, x)|^{\frac{1}{m+n+k}} < \infty.$$

The space of all triple of Bernstein polynomials pre-Cauchy of analytic sequences are usually denoted by Λ_B^3 .

2. Definitions and Preliminaries

Throughout the paper \mathbf{R}^3 denotes the real three dimensional case with the metric. Consider a triple sequence $x = (x_{mnk})$ such that $x_{mnk} \in \mathbf{R}^3$; $m, n, k \in \mathbf{N}$. The following definition are obtained:

2.1. Definition

Let f be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of Bernstein polynomials $(B_{mnk}(f, x))$ is said to be statistically pre-Cauchy if, for any $\epsilon > 0$ and $\delta > 0$,

$$\left\{ (r, s, t) \in \mathbf{N} : \frac{1}{r^6 s^6 t^6} |\{ |B_{mnk}(f, x) - B_{uvw}(f, x)| \geq \epsilon \}| \geq \delta \right\} \in I.$$

2.2. Definition

Let f be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of Bernstein polynomials $(B_{mnk}(f, x))$ is said to be statistically pre-Cauchy convergent to $B_{uvw}(f, x)$ denoted by $B_{mnk}(f, x) \xrightarrow{st-limx} B_{uvw}(f, x)$, provided that the set, for any $\epsilon > 0$ and $\delta > 0$

$$\left\{ (r, s, t) \in \mathbf{N} : \frac{1}{r^6 s^6 t^6} |\{ |B_{mnk}(f, x) - B_{uvw}(f, x)| \geq \epsilon \}| \geq \delta \right\} \in I.$$

has natural density zero for every $\epsilon > 0$.

In this case, $B_{uvw}(f, x)$ is called the statistical pre-Cauchy limit of the sequence of Bernstein polynomials.

2.3. Definition

Let f be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of Bernstein polynomials $(B_{mnk}(f, x))$ in a metric space $(X, |., .|)$ and r be a non-negative real number is said to be pre-Cauchy r -convergent to $B_{uvw}(f, x)$, denoted by $B_{mnk}(f, x) \xrightarrow{r} B_{uvw}(f, x)$, if for any $\epsilon > 0$ and $\delta > 0$ there exists $N_\epsilon \in \mathbf{N}$ such that for all $m, n, k \geq N_\epsilon$ we have

$$\left\{ (r, s, t) \in \mathbf{N} : \frac{1}{r^6 s^6 t^6} |\{ |B_{mnk}(f, x) - B_{uvw}(f, x)| < r + \epsilon \}| < \delta \right\} \in I.$$

In this case $B_{mnk}(f, x)$ is called an pre-Cauchy r - limit of $B_{uvw}(f, x)$.

2.4. Remark

We consider r - limit set $B_{mnk}(f, x)$ which is denoted by $LIM_{B_{mnk}(f,x)}^r$ and is defined by

$$LIM_{B_{mnk}(f,x)}^r = \{B_{mnk}(f, x) \in X : B_{mnk}(f, x) \rightarrow^r B_{uvw}(f, x)\}.$$

2.5. Definition

Let f be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of Bernstein polynomials $(B_{mnk}(f, x))$ is said to be pre-Cauchy r - convergent if $LIM_{B_{mnk}(f,x)}^r \neq \phi$ and r is called a pre-Cauchy rough convergence degree of $B_{mnk}(f, x)$. If $r = 0$ then it is ordinary convergence of triple sequence of Bernstein polynomials.

2.6. Definition

Let f be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of Bernstein polynomials $(B_{mnk}(f, x))$ in a metric space $(X, |., .|)$ and r be a non-negative real number is said to be r - statistically pre-Cauchy convergent to $B_{uvw}(f, x)$, denoted by $B_{mnk}f(x) \xrightarrow{r-st3} B_{uvw}(f, x)$, if for any $\epsilon > 0$ and $\delta > 0$

$$\left\{ (r, s, t) \in \mathbf{N} : \frac{1}{r^6 s^6 t^6} |\{ |B_{mnk}(f, x) - B_{uvw}(f, x)| \geq r + \epsilon \}| < \delta \right\} \in I.$$

In this case $B_{uvw}(f, x)$ is called r - statistical pre-Cauchy limit of $B_{mnk}(f, x)$. If $r = 0$ then it is ordinary statistical convergent of triple sequence of Bernstein polynomials.

2.7. Definition

A class I of subsets of a nonempty set X is said to be an ideal in X provided

- (i) $\phi \in I$
 - (ii) $A, B \in I$ implies $A \cup B \in I$.
 - (iii) $A \in I, B \subset A$ implies $B \in I$.
- I is called a nontrivial ideal if $X \notin I$.

2.8. Definition

A nonempty class F of subsets of a nonempty set X is said to be a filter in X . Provided

- (i) $\phi \in F$.
- (ii) $A, B \in F$ implies $A \cap B \in F$.
- (iii) $A \in F, A \subset B$ implies $B \in F$.

2.9. Definition

I is a non trivial ideal in X , $X \neq \phi$, then the class

$$F(I) = \{M \subset X : M = X \setminus A \text{ for some } A \in I\}$$

is a filter on X , called the filter associated with I .

2.10. Definition

A non trivial ideal I in X is called admissible if $\{x\} \in I$ for each $x \in X$.

2.11. Note

If I is an admissible ideal, then usual convergence in X implies I convergence in X .

2.12. Remark

If I is an admissible ideal, then usual rough convergence implies rough I -convergence.

2.13. Definition

Let f be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of Bernstein polynomials $(B_{mnk}(f, x))$ in a metric space $(X, |., .|)$ and r be a non-negative real number is said to be rough ideal convergent pre-Cauchy or rI -convergent to $B_{uvw}(f, x)$, denoted by $B_{mnk} \xrightarrow{rI} B_{uvw}(f, x)$, if for any $\epsilon > 0$ and $\delta > 0$ we have

$$\left\{ \left\{ (r, s, t) \in \mathbf{N} : \frac{1}{r\delta s^6 t^6} ||B_{mnk}(f, x) - B_{uvw}(f, x)| \geq r + \epsilon \right\} \geq \delta \right\} \in I.$$

In this case $(B_{mnk}(f, x))$ is called rI -pre-Cauchy sequence of limit of $B_{uvw}(f, x)$ and a triple sequence of Bernstein polynomials $(B_{mnk}(f, x))$ is called rough I -convergent of pre-Cauchy sequence of $B_{uvw}(f, x)$ with r as roughness pre-Cauchy sequences of degree. If $r = 0$ then it is ordinary I -convergent of pre-Cauchy sequences.

2.14. Note

Generally, Let f be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of Bernstein polynomials $(B_{mnk}(f, y))$ is not I -convergent pre-Cauchy sequences in usual sense and $|B_{mnk}(f, x) - B_{mnk}(f, y)| \leq r$ for all $(r, s, t) \in \mathbf{N}$ or

$$\left\{ (r, s, t) \in \mathbf{N} : \frac{1}{r^6 s^6 t^6} ||B_{mnk}(f, x) - B_{mnk}(f, y)| \geq r| \geq \delta \right\} \in I.$$

for some $r > 0$. Then the triple sequence of Bernstein polynomials $(B_{mnk}(f, x))$ is rI -convergent pre-Cauchy sequences.

2.15. Note

It is clear that rI -pre-Cauchy sequence of limit of $B_{uvw}(f, x)$ is not necessarily unique.

2.16. Definition

Consider rI -pre-Cauchy sequence limit set of $B_{uvw}(f, x)$, which is denoted by

$$I - LIM_{B_{mnk}(f,x)}^r = \left\{ L \in X : B_{mnk}(f, x) \rightarrow^{rI} B_{uvw}(f, x) \right\},$$

then the triple sequence of Bernstein polynomials $(B_{mnk}(f, x))$ is said to be rI -convergent pre-Cauchy sequences, if $I - LIM_{B_{mnk}(f,x)}^r \neq \phi$ and r is called a rough I -convergence pre-Cauchy sequences of degree of $B_{mnk}(f, x)$.

2.17. Definition

Let f be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of Bernstein polynomials $(B_{mnk}(f, x))$ is said to be I -pre-Cauchy sequences of analytic if there exists a positive real number M such that

$$\left\{ (m, n, k) \in \mathbf{N} : |B_{mnk}(f, x)|^{1/m+n+k} \geq M \right\} \in I.$$

2.18. Definition

A point $L \in X$ is said to be an I -pre-Cauchy sequences of accumulation point and Let f be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of Bernstein polynomials $(B_{mnk}(f, x))$ in a metric space (X, d) if and only if for each $\epsilon > 0$ and $\delta > 0$, the set

$$\{(r, s, t) \in \mathbf{N} : |d(B_{mnk}(f, x), B_{uvw}(f, x))| = |B_{mnk}(f, x) - B_{uvw}(f, x)| < \epsilon < \delta \notin I\}.$$

We denote the set of all I -pre-Cauchy sequence of accumulation points of $(B_{mnk}(f, x))$ by $I(\Gamma_{B_{mnk}(f, x)})$.

2.19. Definition

Let f be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of Bernstein polynomials $(B_{mnk}(f, x))$ of real numbers, the notions of ideal pre-Cauchy sequences of limit superior and ideal pre-Cauchy sequences of limit inferior are defined as follows:

$$I - \limsup B_{mnk}(f, x) = \left\{ \begin{array}{ll} \sup B_x, & \text{if } B_{mnk}(f, x) \neq \phi, \\ -\infty, & \text{if } B_{mnk}(f, x) = \phi \end{array} \right\},$$

and

$$I - \liminf (B_{mnk}(f, x)) = \left\{ \begin{array}{ll} \inf A_{B_{mnk}(f, x)}, & \text{if } A_{B_{mnk}(f, x)} \neq \phi, \\ +\infty, & \text{if } A_{B_{mnk}(f, x)} = \phi \end{array} \right\},$$

where $A_{B_{mnk}(f, x)} = \{a \in \mathbf{R} : \{(m, n, k) \in \mathbf{N} : B_{mnk}(f, x) < a\} \notin I\}$ and $B_{B_{mnk}(f, x)} = \{b \in \mathbf{R} : \{(m, n, k) \in \mathbf{N}^3 : B_{mnk}(f, x) > b\} \notin I\}$.

2.20. Definition

Let f be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of Bernstein polynomials $(B_{mnk}(f, x))$ is said to be rough I -convergent pre-Cauchy sequences if $I - LIM^r B_{mnk}(f, x) \neq \phi$. It is clear that if $I - LIM^r B_{mnk}(f, x) \neq \phi$ for a triple sequence of Bernstein polynomials $(B_{mnk}(f, x))$ of real numbers, then we have

$$I-LIM^r B_{mnk}(f, x) = [I - \limsup B_{mnk}(f, x) - r, I - \liminf B_{mnk}(f, x) + r].$$

3. Main Results

3.1. Theorem

Let f be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of Bernstein polynomials $(B_{mnk}(f, x))$ of real numbers and $I \subset 2^{\mathbf{N}}$ be an admissible ideal of pre-Cauchy sequences, we have $\text{diam}(I - LIM^r B_{mnk}(f, x)) \leq 2r$. In general, $\text{diam}(I - LIM^r B_{mnk}(f, x))$ has an upper bound.

Proof: Assume that $\text{diam}(LIM^r B_{mnk}(f, x)) > 2r$. Then, $\exists p, q \in LIM^r B_{mnk}(f, x) \ni: |p - q| > 2r$. Take $\epsilon \in (0, \frac{|p-q|}{2} - r)$. Because $p, q \in I - LIM^r B_{mnk}(f, x)$, we have $A_1(\epsilon) \in I$ and $A_2(\epsilon) \in I$ for every $\epsilon > 0$, where

$$A_1(\epsilon) = \left\{ (r, s, t) \in \mathbf{N} : \frac{1}{r^6 s^6 t^6} ||B_{mnk}(f, x) - p| \geq r + \epsilon| \geq \delta \right\}$$

$$A_2(\epsilon) = \left\{ (r, s, t) \in \mathbf{N} : \frac{1}{r^6 s^6 t^6} ||B_{mnk}(f, x) - q| \geq r + \epsilon| \geq \delta \right\}.$$

Using the properties $F(I)$, we get

$$(A_1(\epsilon)^c \cap A_2(\epsilon)^c) \in F(I).$$

Thus we write, $|p - q| \leq |B_{mnk}(f, x) - p| + |B_{mnk}(f, x) - q| < (r + \epsilon) + (r + \epsilon) < 2\delta < 2(r + \epsilon)$, for all $(m, n, k) \in A_1(\epsilon)^c \cap A_2(\epsilon)^c$ which is a contradiction. Hence $\text{diam}(LIM^r B_{mnk}(f, x)) \leq 2r$.

Now, consider a triple sequence of Bernstein polynomials of $(B_{mnk}(f, x))$ of real numbers such that $I - \lim_{m,n,k \rightarrow \infty} B_{mnk}(f, x) = B_{uvw}(f, x)$.

Let $\epsilon > 0$. Then we can write

$$\left\{ (r, s, t) \in \mathbf{N} : \frac{1}{r^6 s^6 t^6} |\{|B_{mnk}(f, x) - B_{uvw}(f, x)| \geq \epsilon\}| \geq \delta \right\} \in I.$$

Thus, we have

$$\begin{aligned} |B_{mnk}(f, x) - p| &\leq |B_{mnk}(f, x) - B_{uvw}(f, x)| + |B_{uvw}(f, x) - p| \\ &\leq |B_{mnk}(f, x) - B_{uvw}(f, x)| + r + \epsilon, \end{aligned}$$

for each $p \in \bar{B}_r(B_{uvw}(f, x)) := \{p \in \mathbf{R}^3 : |p - B_{uvw}(f, x)| \leq r\}$. Then, we get

$$|B_{mnk}(f, x) - p| < r + \epsilon$$

for each $(r, s, t) \in \left\{ (r, s, t) \in \mathbf{N} : \frac{1}{r^6 s^6 t^6} \|B_{mnk}(f, x) - B_{uvw}(f, x)\| < \epsilon < \delta \right\}$. Because the triple sequence of Bernstein polynomials of $B_{mnk}(f, x)$ is I -convergent pre-Cauchy sequences to $B_{uvw}(f, x)$, we have

$$\left\{ (r, s, t) \in \mathbf{N} : \frac{1}{r^6 s^6 t^6} \|B_{mnk}(f, x) - B_{uvw}(f, x)\| < \epsilon < \delta \right\} \in F(I).$$

Therefore, we get $p \in I-LIM^r B_{mnk}(f, x)$. Consequently, we can write

$$(3.1) \quad I-LIM^r B_{mnk}(f, x) = \bar{B}_r(B_{uvw}(f, x)).$$

Because $diam(\bar{B}_r(B_{uvw}(f, x))) = 2r$, this shows that in general, the upper bound $2r$ of the diameter of the set $I-LIM^r B_{mnk}(f, x)$ is not lower bound.

3.2. Theorem

Let $I \subset 3^{\mathbf{N}}$ be an admissible ideal and Let f be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of Bernstein polynomials of $(B_{mnk}(f, x))$ is pre-Cauchy I -analytic if and only if there exists a non-negative real number r such that $I-LIM^r B_{mnk}(f, x) \neq \phi$, for all $r > 0$, an I -analytic pre-Cauchy triple sequence of Bernstein polynomials always contains a sub sequence $(B_{m_i n_j k_\ell}(f, x))$ with $I-LIM^{(B_{m_i n_j k_\ell}(f, x), r)} \neq \phi$.

Proof: Because the triple sequence of Bernstein polynomials of $B_{mnk}(f, x)$ is Pre-Cauchy I -analytic then there exists a positive real number M such that

$$\left\{ (r, s, t) \in \mathbf{N} : \frac{1}{r^6 s^6 t^6} \left| \left\{ |B_{mnk}(f, x)|^{1/m+n+k} \geq M \right\} \right| \geq \delta \right\} \in I.$$

Define $r' = \sup \left\{ \mathbf{N}^3 : |B_{mnk}(f, x)|^{1/m+n+k} \geq M : (m, n, k) \in K^c \right\}$, where $K = \left\{ (m, n, k) \in \mathbf{N}^3 : |B_{mnk}(f, x)|^{1/m+n+k} \geq M \right\}$. Then the set $I-LIM^{r'} B_{mnk}(f, x)$ contains the origin of \mathbf{R}^3 . So we have $I-LIM^{r'} B_{mnk}(f, x) \neq \phi$.

If $I-LIM^r B_{mnk}(f, x) \neq \phi$ for some $r \geq 0$, then there exists $B_{uvw}(f, x)$ such that $L \in I-LIM^r B_{mnk}(f, x)$, i.e.,

$$\left\{ (r, s, t) \in \mathbf{N} : \frac{1}{r^6 s^6 t^6} \left| \left\{ |B_{mnk}(f, x) - B_{uvw}(f, x)|^{1/m+n+k} \geq r + \epsilon \right\} \right| \geq \delta \right\} \in I.$$

for each $\epsilon > 0$. Then we say that almost all $B_{mnk}(f, x)$ are contained in some ball with any radius greater than r . So the triple sequence space of Bernstein polynomials of $B_{mnk}(f, x)$ is I - analytic.

As $B_{mnk}(f, x)$ is a I - analytic triple sequence of Bernstein polynomials in a three-dimensional metric space, it certainly contains a I - convergent sub sequence $(B_{m_i n_j k_\ell}(f, x))$. Let $f(x)$ be its I - limit point, then $I - LIM^r B_{m_i n_j k_\ell}(f, x) = \bar{B}_r(f(x))$ and, for $r > 0$,

$$I - LIM^{(B_{m_i n_j k_\ell}(f, x))^r} B_{m_i n_j k_\ell}(f, x) \neq \phi.$$

3.3. Theorem

Let $I \subset 3^{\mathbf{N}}$ be an pre-Cauchy admissible ideal. If $(B_{m_i n_j k_\ell}(f, x))$ is a pre-Cauchy sub sequence of $(B_{mnk}(f, x))$, then

$$I - LIM^r B_{mnk}(f, x) \subseteq I - LIM^r B_{m_i n_j k_\ell}(f, x).$$

Proof: The proof is trivial (See [10] , Proposition 2.3).

3.4. Theorem

Let f be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of Bernstein polynomials of $(B_{mnk}(f, x))$ and $I \subset 3^{\mathbf{N}}$ be an pre-Cauchy admissible ideal is closed.

Proof: The result is true for $I - LIM^r B_{mnk}(f, x) = \phi$. Assume that $I - LIM^r B_{mnk}(f, x) \neq \phi$.

Then, we can choose a triple sequence of Bernstein polynomials of $B_{mnk}(f, y) \subseteq I - LIM^r B_{mnk}(f, x)$ such that $B_{mnk}(f, y) \xrightarrow{r} B_{uvw}(f, y)$ for $m, n, k \rightarrow \infty$. To prove $f(x) \in I - LIM^r B_{mnk}(f, x)$.

Let $\epsilon > 0$ be given. Because $B_{mnk}(f, y) \rightarrow B_{uvw}(f, y)$, $\exists i, j, \ell = i_{\frac{\epsilon}{2}}, j_{\frac{\epsilon}{2}}, \ell_{\frac{\epsilon}{2}} \in \mathbf{N}^3$ such that

$$|B_{mnk}(f, y) - B_{uvw}(f, y)| < \frac{\epsilon}{2}, \forall m \geq i_{\frac{\epsilon}{2}}, n \geq j_{\frac{\epsilon}{2}}, k \geq \ell_{\frac{\epsilon}{2}}.$$

Now choose an $m_0 n_0 k_0 \in \mathbf{N}^3$ such that $m_0 \geq i_{\frac{\epsilon}{2}}, n_0 \geq j_{\frac{\epsilon}{2}}, k_0 \geq \ell_{\frac{\epsilon}{2}}$.

Then we can write

$$|B_{m_0 n_0 k_0}(f, y) - B_{u_0 v_0 w_0}(f, y)| < \frac{\epsilon}{2}.$$

On the other hand, because $B_{mnk}(f, y) \subseteq I - LIM^r B_{mnk}(f, x)$, we have $B_{m_0n_0k_0}(f, y) \in I - LIM^r B_{mnk}(f, x)$, namely,

$$(3.2) \quad \left\{ (r, s, t) \in \mathbf{N} : \frac{1}{r^6 s^6 t^6} \left| \left\{ |B_{mnk}(f, x) - B_{m_0n_0k_0}(f, y)| \geq r + \frac{\epsilon}{2} \right\} \right| \geq \delta \right\} \in I.$$

Now let us prove that the inclusion

$$(3.3) \quad A^c\left(\frac{\epsilon}{2}\right) \subseteq A^c(\epsilon)$$

holds, where

$$A(\epsilon) = \left\{ (r, s, t) \in \mathbf{N} : \frac{1}{r^6 s^6 t^6} \left| \left\{ |B_{mnk}(f, x) - B_{uvw}(f, x)| \geq r + \epsilon \right\} \right| \geq \delta \right\} \in I.$$

Take $(u, v, w) \in A^c\left(\frac{\epsilon}{2}\right)$. Then we have

$$|B_{uvw}(f, x) - B_{m_0n_0k_0}(f, y)| < r + \frac{\epsilon}{2}$$

and hence

$$\begin{aligned} & |B_{uvw}(f, x) - B_{uvw}(f, x)| \leq \\ & |B_{uvw}(f, x) - B_{m_0n_0k_0}(f, y)| + |B_{m_0n_0k_0}(f, y) - B_{uvw}(f, x)| < r + \epsilon, \end{aligned}$$

i.e., $(u, v, w) \in A^c(\epsilon)$, which proves (3.3), we get $A(\epsilon) \in I$

(i.e., $B_{uvw}(f, x) \in I - LIM^r B_{mnk}(f, x)$).

3.5. Theorem

Let f be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of Bernstein polynomials of $(B_{mnk}(f, x))$ of real numbers and $I \subset 2^{\mathbf{N}}$ be an pre-Cauchy admissible ideal. The pre-Cauchy rough I -limit set of triple sequence of Bernstein polynomials of $B_{mnk}(f, x)$ is convex.

Proof: Let $y_1, y_2 \in B_{mnk}(f, x) \in I - LIM^r B_{mnk}(f, x)$ for triple sequence of Bernstein polynomials of $B_{mnk}(f, x)$ and let $\epsilon > 0$ be given. Define

$$A_1(\epsilon) = \left\{ (r, s, t) \in \mathbf{N} : \frac{1}{r^6 s^6 t^6} \left| \left\{ |B_{mnk}(f, x) - y_1| \geq r + \epsilon \right\} \right| \geq \delta \right\} \in I.$$

and

$$A_2(\epsilon) = \left\{ (r, s, t) \in \mathbf{N} : \frac{1}{r^6 s^6 t^6} |\{ |B_{mnk}(f, x) - y_2| \geq r + \epsilon \}| \geq \delta \right\} \in I.$$

Because $y_1, y_2 \in I-LIM^r B_{mnk}(f, x)$, we have $A_1(\epsilon), A_2(\epsilon) \in I$. Thus we have

$$\begin{aligned} |B_{mnk}(f, x) - [(1 - \lambda)y_1 + \lambda y_2]| &= |(1 - \lambda)(B_{mnk}(f, x) - y_1) + \lambda(B_{mnk}(f, x) - y_2)| \\ &\leq (1 - \lambda)|B_{mnk}(f, x) - y_1| + \lambda|B_{mnk}(f, x) - y_2| \\ &< (1 - \lambda)(r + \epsilon) + \lambda(r + \epsilon) < r + \epsilon \end{aligned}$$

for each $(m, n, k) \in A_1^c(\epsilon) \cap A_2^c(\epsilon)$ and each $\lambda \in [0, 1]$. Because $(A_1^c(\epsilon) \cap A_2^c(\epsilon)) \in F(I)$ by definition of $F(I)$, we get

$$\left\{ (r, s, t) \in \mathbf{N} : \frac{1}{r^6 s^6 t^6} |\{ |B_{mnk}(f, x) - [(1 - \lambda)y_1 + \lambda y_2]| \geq r + \epsilon \}| \geq \delta \right\} \in I.$$

that is

$$[(1 - \lambda)y_1 + \lambda y_2] \in I - LIM^r B_{mnk}(f, x),$$

which proves the convexity of the set $I - LIM^r B_{mnk}(f, x)$.

3.6. Theorem

Let f be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of Bernstein polynomials of $(B_{mnk}(f, x))$ of real numbers $r > 0$ and $I \subset 3^{\mathbf{N}}$ be an pre-Cauchy admissible ideal is rough I -convergent to $B_{uvw}(f, x)$ if and only if there exists a triple sequence of Bernstein polynomials of $B_{mnk}(f, y)$ such that

$$(3.4) \quad \begin{aligned} I - \lim B_{mnk}(f, y) &= B_{uvw}(f, y) \text{ and } |B_{mnk}(f, x) - B_{mnk}(f, y)| \leq r, \\ &\text{for each } m, n, k \in \mathbf{N}^3. \end{aligned}$$

Proof: Assume that the triple sequence of Bernstein polynomials of $B_{mnk}(f, x)$ is rough I -convergent to $B_{uvw}(f, x)$. Then we have

$$(3.5) \quad I - \limsup |B_{mnk}(f, x) - B_{uvw}(f, x)| \leq r.$$

Now, define

$$\begin{aligned} B_{mnk}(f, y) &= \{ B_{uvw}(f, x), \text{ if } |B_{mnk}(f, x) - B_{uvw}(f, x)| \leq r, \\ &B_{mnk}(f, x) + r \left(\frac{B_{uvw}(f, x) - B_{mnk}(f, x)}{|B_{mnk}(f, x) - B_{uvw}(f, x)|} \right), \text{ otherwise,} \end{aligned}$$

Then, we we write

$$\begin{aligned} & |B_{mnk}(f, x) - B_{uvw}(f, x)| = \\ \{ & |B_{uvw}(f, x) - B_{uvw}(f, x)|, \text{ if } |B_{mnk}(f, x) - B_{uvw}(f, x)| \leq r, \\ & r \left(\frac{|B_{uvw}(f, x) - B_{uvw}(f, x)| - |B_{mnk}(f, x) - B_{uvw}(f, x)|}{|B_{mnk}(f, x) - B_{uvw}(f, x)|} \right), \text{ otherwise,} \end{aligned}$$

$$\text{(i.e) } |B_{mnk}(f, y) - B_{uvw}(f, y)| = \{ 0, \text{ if } |B_{mnk}(f, x) - B_{uvw}(f, x)| \leq r, \\ |B_{mnk}(f, x) - B_{uvw}(f, x)| - r \left(\frac{|B_{mnk}(f, x) - B_{uvw}(f, x)|}{|B_{mnk}(f, x) - B_{uvw}(f, x)|} \right), \text{ otherwise,}$$

$$\text{(i.e) } |B_{mnk}(f, y) - f(y)| = \{ 0, \text{ if } |B_{mnk}(f, x) - B_{uvw}(f, x)| \leq r, \\ |B_{mnk}(f, x) - B_{uvw}(f, x)| - r, \text{ otherwise.}$$

We have $|B_{mnk}(f, y) - B_{uvw}(f, y)| \geq |B_{mnk}(f, x) - B_{uvw}(f, x)| - r \implies$
 $|B_{mnk}(f, x) - B_{uvw}(f, x) - B_{mnk}(f, y) + B_{uvw}(f, y)| \leq r$

$$(3.6) \quad |B_{mnk}(f, x) - B_{mnk}(f, y)| \leq r$$

for all $m, n, k \in \mathbf{N}^3$. By equation (3.5) and by definition of $B_{mnk}(f, y)$, we get $I - \limsup |B_{mnk}(f, x) - B_{uvw}(f, y)| = 0$.

$$\implies I - B_{mnk}(f, y) \xrightarrow{r} B_{uvw}(f, y).$$

Assume that (3.4) holds. Because $I - \lim B_{mnk}(f, y) = B_{uvw}(f, y)$, we have

$$A(\epsilon) = \{(m, n, k) \in \mathbf{N}^3 : |B_{mnk}(f, y) - B_{uvw}(f, y)| \geq r + \epsilon\} \in I,$$

for each $\epsilon > 0$. Now, define the set

$$B(\epsilon) = \{(m, n, k) \in \mathbf{N}^3 : |B_{mnk}(f, x) - B_{uvw}(f, x)| \geq r + \epsilon\} \in I.$$

We have

$$B(\epsilon) \subseteq A(\epsilon)$$

holds. Since $A(\epsilon) \in I \implies B(\epsilon) \in I$. Hence, $B_{mnk}(f, x)$ is rough I -convergent to $B_{uvw}(f, x)$.

3.7. Note

If we replace the condition $|B_{mnk}(f, x) - B_{mnk}(f, y)| \leq r$ for all $(m, n, k) \in \mathbf{N}^3$ in the hypothesis of the above theorem with the condition

$\{(m, n, k) \in \mathbf{N}^3 : |B_{mnk}(f, x) - B_{mnk}(f, y)| > r\} \in I$ then the theorem will also be valid.

3.8. Theorem

Let f be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of Bernstein polynomials of $(B_{mnk}(f, x))$ of real numbers and $I \subset \mathbb{3}^{\mathbf{N}}$ be an pre-Cauchy admissible ideal. For an arbitrary $c \in I(\Gamma_x)$, we have $|f(x) - c| \leq r$ for all $f(x) \in I - LIM^r B_{mnk}(f, x)$.

Proof: Assume on the contrary that there exist a point $c \in I(\Gamma_x)$ and $f(x) \in I - LIM^r B_{mnk}(f, x)$ such that $|B_{uvw}(f, x) - c| > r$. Define $\epsilon := \frac{|B_{uvw}(f, x) - c| - r}{3}$. Then

$$\begin{aligned} & \left\{ (r, s, t) \in \mathbf{N} : \frac{1}{r^6 s^6 t^6} |\{|B_{mnk}(f, x) - c| < \epsilon\}| < \delta \right\} \subseteq \\ & \left\{ (r, s, t) \in \mathbf{N} : \frac{1}{r^6 s^6 t^6} |\{|B_{mnk}(f, x) - B_{uvw}(f, x)| \geq r + \epsilon\}| \geq \delta \right\} \in I. \dots \\ & \dots \dots \dots (1) \end{aligned}$$

Since $c \in I(\Gamma_x)$, we have

$$\left\{ (r, s, t) \in \mathbf{N} : \frac{1}{r^6 s^6 t^6} |\{|B_{mnk}(f, x) - c| < \epsilon\}| < \delta \right\} \notin I.$$

But from definition of I - convergence, since

$$\left\{ (r, s, t) \in \mathbf{N} : \frac{1}{r^6 s^6 t^6} |\{|B_{mnk}(f, x) - B_{uvw}(f, x)| \geq r + \epsilon\}| \geq \delta \right\} \in I.$$

so by (1) we have

$$\left\{ (r, s, t) \in \mathbf{N} : \frac{1}{r^6 s^6 t^6} |\{|B_{mnk}(f, x) - c| < \epsilon\}| < \delta \right\} \in I.$$

which contradicts the fact $c \in I(\Gamma_x)$. On the other hand, if $c \in I(\Gamma_x)$ i.e.,

$$\left\{ (r, s, t) \in \mathbf{N} : \frac{1}{r^6 s^6 t^6} |\{|B_{mnk}(f, x) - c| < \epsilon\}| < \delta \right\} \notin I.$$

then

$$\left\{ (r, s, t) \in \mathbf{N} : \frac{1}{r^6 s^6 t^6} |\{|B_{mnk}(f, x) - B_{uvw}(f, x)| \geq r + \epsilon\}| \geq \delta \right\} \notin I,$$

which contradicts the fact $B_{uvw}(f, x) \in I - LIM^r B_{mnk}(f, x)$.

3.9. Theorem

Let f be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of Bernstein polynomials of $(B_{mnk}(f, x))$ of real numbers and $I \subset \mathfrak{Z}^{\mathbf{N}}$ be an pre-Cauchy admissible ideal, $(\mathbf{R}^3, |., .|)$ be a strictly convex, if there exist $y_1, y_2, y_3, y_4, y_5, y_6 \in I - LIM^r B_{mnk}(f, x)$ such that $|y_1 - y_2| < 2r, |y_3 - y_4| < 2r$ and $|y_5 - y_6| < 2r$, then this triple sequence space of Bernstein polynomials is I -convergent to

$$\frac{1}{6}(y_1 + y_2 + y_3 + y_4 + y_5 + y_6).$$

Proof: Let $c \in I(\Gamma_x)$. Then since $y_1, y_2, y_3, y_4, y_5, y_6 \in I - LIM^r B_{mnk}(f, x)$, we have

$$|y_1 - c| \leq r, |y_2 - c| \leq r, |y_3 - c| \leq r, |y_4 - c| \leq r, |y_5 - c| \leq r \text{ and } |y_6 - c| \leq r \quad (3.7)$$

By Theorem (3.9). On the other hand, we have

$$6r = |y_1 - y_6| \leq |y_1 - c| + |y_2 - c| + |y_3 - c| + |y_4 - c| + |y_5 - c| + |y_6 - c| \quad (3.8)$$

Therefore, we get $|y_1 - c| = \dots = |y_6 - c| = r$ by inequalities (3.7) and (3.8). Since

$$\frac{1}{6}(y_6 - y_1) = \frac{1}{6}[(c - y_1) + (c - y_2)(c - y_3)(c - y_4) + (c - y_5) + (c - y_6)] \quad (3.9)$$

and $|y_1 - y_6| = 6r,$

we get $\left| \frac{1}{6}(y_6 - y_1) \right| = r$. By the strict convexity of the space from the equality (3.9), we get

$$\frac{1}{6}(y_6 - y_1) = c - y_1 = \dots = c - y_6,$$

which implies that $c = \frac{1}{6}(y_1 + y_2 + y_3 + y_4 + y_5 + y_6)$. Hence c is the unique I -cluster point of the triple sequence space of Bernstein polynomials of $(B_{mnk}(f, x))$.

On the other hand, the assumption $y_1, y_2, y_3, y_4, y_5, y_6 \in I - LIM^r B_{mnk}(f, x) \implies I - LIM^r B_{mnk}(f, x) \neq \phi$.

By theorem (3.3), the triple sequence space of Bernstein polynomials of $B_{mnk}(f, x)$ is I -analytic. Consequently, the triple sequence space of Bernstein polynomials of $B_{mnk}(f, x)$ must I -convergent to $\frac{1}{6}(y_1 + y_2 + y_3 + y_4 + y_5 + y_6)$, i.e., $I\text{-lim} B_{mnk}(f, x) = \frac{1}{6}(y_1 + y_2 + y_3 + y_4 + y_5 + y_6)$.

3.10. Theorem

Let f be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of Bernstein polynomials of $(B_{mnk}(f, x))$ of real numbers and $I \subset 3^{\mathbf{N}}$ be an pre-Cauchy admissible ideal.

(i) If $c \in I(\Gamma_x)$ then

$$(3.10) \quad I - LIM^r B_{mnk}(f, x) \subseteq \bar{B}_r(c).$$

(ii)

$$(3.11) \quad I - LIM^r B_{mnk}(f, x) = \bigcap_{c \in I(\Gamma_x)} \bar{B}_r(c) = \left\{ B_{uvw}(f, x) \in \mathbf{R}^3 : I(\Gamma_x) \subseteq \bar{B}_r(L) \right\}.$$

Proof:

(i) If $c \in I(\Gamma_x)$ then by Theorem (3.9), we have $|B_{uvw}(f, x) - c| \leq r$ for all $B_{uvw}(f, x) \in I - LIM^r B_{mnk}(f, x)$, other wise we get

$$\left\{ (r, s, t) \in \mathbf{N} : \frac{1}{r^6 s^6 t^6} |\{ |B_{mnk}(f, x) - B_{uvw}(f, x)| \geq r + \epsilon \}| \geq \delta \right\} \notin I.$$

for $\epsilon := \frac{|B_{uvw}(f, x) - c| - r}{3}$. Because c is an I -cluster point of $B_{mnk}(f, x)$, this contradicts with the fact that $B_{uvw}(f, x) \in I - LIM^r B_{mnk}(f, x)$.

(ii) From (3.11) we have

$$(3.12) \quad I - LIM^r B_{mnk}(f, x) \subseteq \bigcap_{c \in I(\Gamma_x)} \bar{B}_r(c).$$

Now, let $B_{mnk}(f, x) \in \bigcap_{c \in I(\Gamma_x)} \bar{B}_r(c)$. Then we have

$$\left\{ (r, s, t) \in \mathbf{N} : \frac{1}{r^6 s^6 t^6} |\{ |B_{mnk}(f, x) - c| \leq r \}| \leq \delta \right\} \in I.$$

for all $c \in I(\Gamma_x)$, which is equivalent to $I(\Gamma_x) \subseteq \bar{B}_r(B_{mnk}(f, y))$, i.e.,

$$(3.13) \quad \bigcap_{c \in I(\Gamma_x)} \bar{B}_r(c) = \left\{ f(x) \in \mathbf{R}^3 : I(\Gamma_x) \subseteq \bar{B}_r(f(x)) \right\}.$$

Now, let $B_{mnk}(f, y) \notin I - LIM^r B_{mnk}(f, x)$. Then, there exists an $\epsilon > 0$ such that

$$\left\{ (r, s, t) \in \mathbf{N} : \frac{1}{r^6 s^6 t^6} |\{|B_{mnk}(f, x) - B_{uvw}(f, y)| \geq r + \epsilon\}| \geq \delta \right\} \notin I,$$

\implies the existence of an I - cluster point c of the sequence $B_{mnk}(f, x)$ with

$$\left\{ (r, s, t) \in \mathbf{N} : \frac{1}{r^6 s^6 t^6} |\{|B_{mnk}(f, x) - c| \geq r + \epsilon\}| \geq \delta \right\} \in I,$$

i.e.,

$$I(\Gamma_x) \bar{B}_r(B_{mnk}(f, y)) \text{ and } B_{mnk}(f, y) \notin \{f(x) \in \mathbf{R}^3 : I(\Gamma_x) \subseteq \bar{B}_r(L)\}.$$

Hence $B_{mnk}(f, y) \in I - LIM^r B_{mnk}(f, x)$ follows from $B_{mnk}(f, y) \in \{B_{uvw}(f, x) \in \mathbf{R}^3 : I(\Gamma_x) \subseteq \bar{B}_r(B_{uvw}(f, x))\}$, i.e.,

$$(3.14) \quad \left\{ B_{uvw}(f, x) \in \mathbf{R}^3 : I(\Gamma_x) \subseteq \bar{B}_r(B_{uvw}(f, x)) \right\} \subseteq I - LIM^r B_{mnk}(f, x).$$

Therefore, the inclusions (3.12)-(3.14) ensure that (3.11) holds i.e.,

$$I - LIM^r B_{mnk}(f, x) \cap_{c \in I(\Gamma_x)} \bar{B}_r(c) = \{B_{uvw}(f, x) \in \mathbf{R}^3 : I(\Gamma_x) \subseteq \bar{B}_r(B_{uvw}(f, x))\}.$$

3.11. Theorem

Let $I \subset 3^{\mathbf{N}}$ be an pre-Cauchy admissible ideal and Let f be a continuous function defined on the closed interval $[0, 1]$. A triple I - analytic sequence of Bernstein polynomials of $(B_{mnk}(f, x))$ of real numbers $r \geq \text{diam}(I(\Gamma_x))$, then we have have $I(\Gamma_x) \subseteq I - LIM^r B_{mnk}(f, x)$.

Proof: Let $c \notin I-LIM^r B_{mnk}(f, x)$. Then there exists an $\epsilon > 0$ such that

$$(3.15) \quad \left\{ (r, s, t) \in \mathbf{N} : \frac{1}{r^6 s^6 t^6} \left| \left\{ |B_{mnk}(f, x) - c|^{1/m+n+k} \geq r + \epsilon \right\} \right| \geq \delta \right\} \notin I.$$

Since $B_{mnk}(f, x)$ is I -analytic and from the inequality (3.15), there exists an I -cluster point c_1 such that

$$|c - c_1| > r + \epsilon_1, \text{ where } \epsilon_1 := \frac{\epsilon}{2}.$$

So we get

$$\text{diam}(I(\Gamma_x)) > r + \epsilon_1.$$

The converse of this theorem is also true, i.e., if $I(\Gamma_x) \subseteq I-LIM^r B_{mnk}(f, x)$, then we have $r \geq \text{diam}(I(\Gamma_x))$.

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