DOI: 10.4067/S0716-09172017000400545

Proyecciones Journal of Mathematics
Vol. 36, N^o 4, pp. 545-566, December 2017.
Universidad Católica del Norte
Antofagasta - Chile

The generalized Van Vleck's equation on locally compact groups

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Received : November 2016. Accepted : April 2017

Abstract

We determine the continuous solutions $f, g : G \to \mathbf{C}$ *of each of the two functional equations*

$$\int_G \{f(xyt) - f(\sigma(y)xt)\}d\mu(t) = f(x)g(y), \quad x, y \in G,$$

$$\int_G \{f(xyt) - f(\sigma(y)xt)\}d\mu(t) = g(x)f(y), \quad x, y \in G,$$

where G *is a locally compact group,* σ *is a continuous involutive automorphism on* G*, and* μ *is a compactly supported, complex-valued Borel measure on* G*.*

Subjclass [2010] : *Primary 39B32, 39B52.*

Keywords : *Functional equation, Van Vleck, involutive automorphism, character, additive map.*

1. Introduction

In the papers [8,9], Van Vleck studied the continuous solutions $f:\mathbf{R}\to\mathbf{R}$, $f\neq 0$, of the functional equation

$$f(x-y+z_0)-f(x+y+z_0)=2f(x)f(y),\quad x,y\in\mathbf{R}, \tag{1.1}$$

where $z_0>0$ is fixed. We shall in this paper study extensions of (1.1) and related functional equations from $\mathbf{R}$ to locally compact groups.

Let G be a group and $Z(G)$ be the center of G. In [4], Perkins and Sahoo extended the result of Van Vleck by determining the abelian solutions $f:G\to\mathbf{C}$ of the functional equation

$$f(x\tau(y)z_0)-f(xyz_0)=2f(x)f(y),\quad x,y\in G, \tag{1.2}$$

where $z_0\in Z(G)$ and $\tau:G\to G$ is an involution (that is, an anti-homomorphism such that $\tau(\tau(x))=x$ for all $x\in G$). As a very recent result, Stetkær extended the result of Perkins and Sahoo in [7] by solving the functional (1.2) on semigroups.

Van Vleck's functional equation (1.1) was generalized in another direction by the authors in [3], viz. to the functional equation

$$f(\sigma(y)xz_0)-f(xyz_0)=2f(x)f(y),\quad x,y\in G, \tag{1.3}$$

where $z_0\in G$ is a fixed element that need not belongs to $Z(G)$ and $\sigma:G\to G$ is an involutive automorphism (that is involutive means that $\sigma(\sigma(x))=x$ for all $x\in G$). Observe that (1.3) can be written as follows

$$\int_G\{f(\sigma(y)xt)-f(xyt)\}d\mu(t)=f(x)f(y),\quad x,y\in G,$$

where $\mu=\frac{1}{2}\delta_{z_0}$ (δ_{z_0} is the Dirac measure concentrated at z_0). Our aim is to generalize this equation by substituting the Dirac measure by an arbitrary compactly supported, complex-valued Borel measure and considering more unknown functions.

Let G be a locally compact group, σ be a continuous involutive automorphism on G, and μ be a compactly supported, complex-valued Borel measure on G. The purpose of the present paper is first to give an explicit description of the continuous solutions $f,g:G\to\mathbf{C}$ of each of the two integral-functional equations

$$\int_G \{f(xyt) - f(\sigma(y)xt)\}d\mu(t) = f(x)g(y), \quad x, y \in G, \tag{1.4}$$

$$\int_G \{f(xyt) - f(\sigma(y)xt)\}d\mu(t) = g(x)f(y), \quad x, y \in G, \tag{1.5}$$

and second to present several consequences of these results as well. In particular, using our main results (Theorem 3.3 and Theorem 4.2), we find the continuous solutions $f : G \to \mathbf{C}$ of the following Van Vleck's integral-functional equation

$$\int_G \{f(\sigma(y)xt) - f(xyt)\}d\mu(t) = f(x)f(y), \quad x, y \in G, \tag{1.6}$$

and the solutions $f, g : G \to \mathbf{C}$ of each of the equations

$$f(xyz_0) - f(\sigma(y)xz_0) = 2f(x)g(y), \quad x, y \in G, \tag{1.7}$$

$$f(xyz_0) - f(\sigma(y)xz_0) = 2g(x)f(y), \quad x, y \in G, \tag{1.8}$$

$$\sum_{i=0}^{n}\{f(xyz_i) - f(\sigma(y)xz_i)\} = 2f(x)g(y), \quad x, y \in G, \tag{1.9}$$

$$\sum_{i=0}^{n}\{f(xyz_i) - f(\sigma(y)xz_i)\} = 2g(x)f(y), \quad x, y \in G, \tag{1.10}$$

where $z_0, z_1, ..., z_n \in G$ are fixed elements. Note that each of the equations (1.7)-(1.10) results from (1.4) or (1.5) by replacing μ by a suitable discrete measure and that all these equations are, according to our knowledge, not in the literature even for abelian groups.

Results of [1,2,3,5,10] have been an inspiration by their treatments of similar functional equations on groups.

2. Notation and terminology

To formulate our results we recall the following notation and assumptions that will be used throughout the paper:

Let G be a group with neutral element e. The map $\sigma : G \to G$ denote an involutive automorphism. That it is involutive means that $\sigma(\sigma(x)) = x$ for all $x \in G$. If $(G, +)$ is an abelian group, then the inversion $\sigma(x) := -x$ is an example of an involutive automorphism. Another example is the complex conjugation map on the multiplicative group of non-zero complex numbers.

For any complex-valued function F on G we use the notations

$$F_e = \frac{F + F \circ \sigma}{2} \quad \text{and} \quad F_o = \frac{F - F \circ \sigma}{2}.$$

We say that F is even if $F = F_e$, and odd if $F = F_o$.

A function $A : G \to \mathbf{C}$ is called additive, if it satisfies $A(xy) = A(x) + A(y)$ for all $x, y \in G$.

A character of G is a homomorphism from G into the multiplicative group of non-zero complex numbers.

By $N(G, \sigma)$ we mean the vector space of the solutions $\theta : G \to \mathbf{C}$ of the homogeneous equation

$$\theta(xy) - \theta(\sigma(y)x) = 0, \quad x, y \in G.$$

If G is a topological space, then we let $C(G)$ denote the algebra of all continuous functions from G into $\mathbf{C}$.

If G is a locally compact group, then we let $M_C(G)$ denote the space of all compactly supported, complex-valued Borel measures on G. For $\mu \in M_C(G)$, we say that μ is σ-invariant if $\mu(f \circ \sigma) = \mu(f)$ for all $f \in C(G)$, i.e.,

$$\int_G f(\sigma(t))d\mu(t) = \int_G f(t)d\mu(t) \quad \text{for all} \quad f \in C(G).$$

3. Solution of equation (1.4)

In this section, we solve the integral-functional equation (1.4), i.e.,

$$\int_G \{f(xyt) - f(\sigma(y)xt)\}d\mu(t) = f(x)g(y), \quad x, y \in G,$$

by expressing its continuous solutions in terms of continuous characters.

The following theorem is an immediate consequence of Theorem 4.2 in [1]. It will be used in the proof of Theorem 3.3. For the notation $N(G, \sigma)$ see section 2.

Theorem 3.1. *Let G be a group, let σ be an involutive automorphism on G, and let $f, g, h : G \to \mathbf{C}$ be solutions of the functional equation*

$$f(xy) - f(\sigma(y)x) = 2g(x)h(y), \quad x, y \in G. \tag{3.1}$$

Then we have the following possibilities:

a) $g = 0$, *h is arbitrary,* $f \in N(G, \sigma)$.

b) $h = 0$, *g is arbitrary,* $f \in N(G, \sigma)$.

c) There exist a character χ of G with $\chi \neq \chi \circ \sigma$, constants $\alpha, \beta \in \mathbf{C}$, $\gamma \in \mathbf{C} \backslash \{0\}$, and a function $\theta \in N(G, \sigma)$ such that

$$\begin{aligned} g &= \alpha\chi + \beta\chi \circ \sigma, \\ h &= \gamma(\chi - \chi \circ \sigma), \\ f &= 2\gamma(\alpha\chi - \beta\chi \circ \sigma) + \theta. \end{aligned}$$

d) There exist a character χ of G with $\chi = \chi \circ \sigma$, constants $\alpha, \beta \in \mathbf{C}$, an additive function $A : G \to \mathbf{C}$ with $A \circ \sigma = -A$, and a function $\theta \in N(G, \sigma)$ such that

$$\begin{aligned} g &= \alpha\chi + \beta\chi A, \\ h &= \chi A, \\ f &= \alpha\chi A + \frac{1}{2}\beta\chi A^2 + \theta. \end{aligned}$$

Conversely, the formulas above for f, g and h define solutions of (3.1). Moreover, if G is a topological group, $g \neq 0, h \neq 0$, and $f, g, h \in C(G)$, then $\chi, \chi \circ \sigma, A, \theta \in C(G)$.

The following lemma will be used in the proof of Theorem 3.3 in which the integral-functional equation (1.4) will be solved.

Lemma 3.2 (3, Lemma 4.1). *Let G be a group and let σ be an involutive automorphism on G. Let χ be a character of G with $\chi \neq \chi \circ \sigma$, $A : G \to \mathbf{C}$ be an odd additive function, θ be a function in $N(G, \sigma)$, and α, β be complex numbers.*

1. *If $\alpha\chi + \beta\chi \circ \sigma + \theta = 0$, then $\alpha = \beta = 0$ and $\theta = 0$.*

2. *If $A^2 + \alpha A + \theta = 0$, then $A = \theta = 0$.*

By help of Theorem 3.1 and Lemma 3.2, we now describe the complete continuous solution of (1.4).

Theorem 3.3. *Let G be a locally compact group, let σ be a continuous involutive automorphism on G, and let $\mu \in M_C(G)$. Assume that the pair $f, g \in C(G)$ is a solution of the functional equation (1.4). Then we have the following possibilities:*

(a) $f = 0$, *g is arbitrary in $C(G)$.*

(b) $g = 0$, $f \in \{k \in C(G) \mid x \mapsto \int_G k(xt)d\mu(t) \in N(G,\sigma)\}$.

(c) There exist a non-even continuous character χ of G with $\mu(\chi) \neq 0$ and a non-zero complex number α such that

$$f = \alpha\chi \quad \text{and} \quad g = \mu(\chi)(\chi - \chi \circ \sigma).$$

(d) There exist a continuous character χ of G with $\mu(\chi) \neq 0$, $\mu(\chi \circ \sigma) = -\mu(\chi)$, and non-zero complex numbers α, β such that

$$f = \alpha\chi + \beta\chi \circ \sigma \quad \text{and} \quad g = \mu(\chi)(\chi - \chi \circ \sigma).$$

Conversely, the functions given with these properties satisfy the functional equation (1.4).

Proof. The first two cases are obvious, so we suppose that $f \neq 0$ and $g \neq 0$. Define $F : G \to \mathbf{C}$ by

$$F(x) = 2\int_G f(xt)d\mu(t) \quad \text{for all} \quad x \in G. \tag{3.2}$$

Since $\mu \in M_C(G)$ and $f \in C(G)$, we have $F \in C(G)$. Using (3.2), the equation (1.4) becomes

$$F(xy) - F(\sigma(y)x) = 2f(x)g(y), \quad x, y \in G.$$

Since $f, g \in C(G) \setminus \{0\}$, we know from Theorem 3.1 that there are only the following two cases:

Case 1: There exist a continuous character χ of G with $\chi \neq \chi\circ\sigma$, constants $\alpha, \beta \in \mathbf{C}$, $\gamma \in \mathbf{C}\setminus\{0\}$, and a continuous function $\theta \in N(G,\sigma)$ such that

$$\begin{aligned} f &= \alpha\chi + \beta\chi \circ \sigma, \\ g &= \gamma(\chi - \chi \circ \sigma), \\ F &= 2\gamma(\alpha\chi - \beta\chi \circ \sigma) + \theta. \end{aligned}$$

Using the expression $F(x) = 2\int_G f(xt)d\mu(t)$ for all $x \in G$, we get that

$$2\gamma[\alpha\chi(x) - \beta\chi \circ \sigma(x)] + \theta(x) = 2\alpha\chi(x)\mu(\chi) + 2\beta\chi \circ \sigma(x)\mu(\chi \circ \sigma),$$

for all $x \in G$. We reformulate the last equality as follows

$$2\alpha[\gamma - \mu(\chi)]\chi(x) - 2\beta[\gamma + \mu(\chi \circ \sigma)]\chi \circ \sigma(x) + \theta(x) = 0,$$

for all $x \in G$. According to Lemma 3.2(a), we obtain

$$\begin{cases} \alpha[\gamma - \mu(\chi)] = 0 \\ \beta[\gamma + \mu(\chi \circ \sigma)] = 0 \end{cases}$$

Since $f \neq 0$, then at least one of α and β is non-zero.

Subcase 1.1: Suppose that $\beta = 0$. Then $\alpha \neq 0$ and hence $\gamma = \mu(\chi)$, which implies that $f = \alpha\chi$ and $g = \mu(\chi)(\chi - \chi \circ \sigma)$. This is case ($c$) in our statement.

Subcase 1.2: Suppose that $\alpha = 0$. Then $\beta \neq 0$ and hence $\gamma = -\mu(\chi \circ \sigma)$, which implies that $f = \beta\chi \circ \sigma$ and $g = -\mu(\chi \circ \sigma)(\chi - \chi \circ \sigma) = \mu(\chi \circ \sigma)(\chi \circ \sigma - \chi)$. So we are in case ($c$) with the continuous character $\chi \circ \sigma$ replacing χ.

Subcase 1.3: We now suppose that $\alpha \neq 0$ and $\beta \neq 0$. Then $\gamma = \mu(\chi) = -\mu(\chi \circ \sigma)$ which implies that $g = \mu(\chi)(\chi - \chi \circ \sigma)$. Then we arrive at the solution in case (d). This completes case 1.

Case 2: There exist a continuous character χ of G with $\chi = \chi \circ \sigma$, constants $\alpha, \beta \in \mathbf{C}$, a continuous additive function $A : G \to \mathbf{C}$ with $A \circ \sigma = -A$, and a continuous function $\theta \in N(G, \sigma)$ such that

$$\begin{aligned} f &= \alpha\chi + \beta\chi A, \\ g &= \chi A, \\ F &= \alpha\chi A + \frac{1}{2}\beta\chi A^2 + \theta. \end{aligned}$$

A small computation based on (3.2) shows that

$$\begin{aligned} &\alpha\chi(x)A(x) + \frac{1}{2}\beta\chi(x)A^2(x) + \theta(x) \\ &\quad = 2\alpha\mu(\chi)\chi(x) + 2\beta\mu(\chi)A(x)\chi(x) + 2\beta\chi(x)\mu(A\chi), \end{aligned}$$

for all $x \in G$. We reformulate the last equation as follows

$$\beta A^2 + 2[\alpha - 2\beta\mu(\chi)]A + \theta_1 = 0, \tag{3.3}$$

where $\theta_1 := 2(\frac{\theta}{\chi}) - 4\alpha\mu(\chi) - 4\beta\mu(A\chi)$. Since χ is even we have $\theta_1 \in N(G, \sigma)$.

Subcase 2.1: Suppose that $\beta = 0$. From (3.3) we see that $2\alpha A + \theta_1 = 0$. Then $\alpha A \in N(G, \sigma)$, i.e., for all $x, y \in G$ we have

$$\alpha A(xy) - \alpha A(\sigma(y)x) = 0,$$

which implies that $2\alpha A(y) = 0$ for all $y \in G$, i.e., $\alpha A = 0$. Since $f = \alpha\chi$ and $f \neq 0$, then $\alpha \neq 0$ and hence $A = 0$. Therefore $g = 0$. This subcase does not apply, because $g \neq 0$ by assumption.

Subcase 2.2: We now suppose that $\beta \neq 0$. Dividing (3.3) by β and using Lemma 3.2(b), we get that $A = 0$. Hence $g = 0$. Also this subcase does not apply, because $g \neq 0$. This finishes the necessity assertion.

Conversely, simple computations prove that the formulas above for (f, g) define solutions of (1.4). $\square$

As a consequence of Theorem 3.3 one can obtain the following corollaries.

Corollary 3.4. *Let G be a locally compact group, let σ be a continuous involutive automorphism on G, and let μ be a σ-invariant measure in $M_C(G)$. Then a pair $f, g \in C(G) \setminus \{0\}$ satisfies the functional equation (1.4) if and only if there exist a non-even continuous character χ of G with $\mu(\chi) \neq 0$ and a non-zero complex number α such that*

$$f = \alpha\chi \quad \text{and} \quad g = \mu(\chi)(\chi - \chi \circ \sigma).$$

Proof. Let χ be a continuous character of G such that $\mu(\chi) \neq 0$. Since μ is σ-invariant we have $\mu(\chi \circ \sigma) = \mu(\chi)$. This implies that $\mu(\chi \circ \sigma) \neq -\mu(\chi)$. Indeed, $\mu(\chi \circ \sigma) = -\mu(\chi)$ would entail $\mu(\chi) = 0$, contradicting our assumption. So a pair $f, g \in C(G) \setminus \{0\}$ satisfies the equation (1.4) if and only if it has the form stated in case (c) of Theorem 3.3. $\square$

Corollary 3.5. *Let G be a locally compact group, let σ be a continuous involutive automorphism on G, and let $\mu \in M_C(G)$. Then a function $f \in C(G) \setminus \{0\}$ satisfies the functional equation (1.4) if and only if there exists a continuous character χ of G with $\mu(\chi) \neq 0$ and $\mu(\chi \circ \sigma) = -\mu(\chi)$ such that*

$$f = -\mu(\chi)(\chi - \chi \circ \sigma).$$

Proof. The proof follows on putting $g = -f$ in Theorem 3.3. $\square$

In the following corollaries let G be a group, $n \in \mathbf{N}$, $z_0, z_1, ..., z_n \in G$ be arbitrarily fixed elements, and let σ be an involutive automorphism on G. To illustrate our theory, we continue by discussing the solutions of Eq. (1.4), but now for the case of μ being supported by a finite set. We can of course get all solutions, continuous or not, by considering the special instance of the discrete topology on G.

Corollary 3.6. *The non-zero functions $f, g : G \to \mathbf{C}$ satisfying the functional equation*

$$f(xyz_0) - f(\sigma(y)xz_0) = 2f(x)g(y), \quad x, y \in G,$$

are the ones of the forms:

(a)

$$f = \alpha\chi \quad \text{and} \quad g = \frac{\chi(z_0)}{2}(\chi - \chi \circ \sigma),$$

where α is a non-zero complex number and χ is a non-even character of G.

(b)

$$f = \alpha\chi + \beta\chi \circ \sigma \quad \text{and} \quad g = \frac{\chi(z_0)}{2}(\chi - \chi \circ \sigma),$$

α, β are non-zero complex numbers and χ is a character of G such that $\chi \circ \sigma(z_0) = -\chi(z_0)$.

Proof. The proof follows on putting $\mu = \frac{1}{2}\delta_{z_0}$ in Theorem 3.3. $\square$

As a consequence of Corollary 3.5 (or Corollary 3.6) we have the following result which is a natural extension of Van Vleck's equation (1.1).

Corollary 3.7. *The non-zero solutions $f : G \to \mathbf{C}$ of the functional equation*

$$f(\sigma(y)xz_0) - f(xyz_0) = 2f(x)f(y), \quad x, y \in G,$$

are the functions of the form $f = -\frac{\chi(z_0)}{2}(\chi - \chi \circ \sigma)$, where χ is a character of G such that $\chi \circ \sigma(z_0) = -\chi(z_0)$.

Proof. The proof follows on putting $\mu = \frac{1}{2}\delta_{z_0}$ in Corollary 3.5. $\square$

Corollary 3.8. *The non-zero functions $f, g : G \to \mathbf{C}$ satisfying the functional equation*

$$\sum_{i=0}^{n}\{f(xyz_i) - f(\sigma(y)xz_i)\} = 2f(x)g(y), \quad x, y \in G,$$

are the ones of the forms:

(a)

$$f = \alpha\chi \quad \text{and} \quad g = \sum_{i=0}^{n} \frac{\chi(z_i)}{2}(\chi - \chi \circ \sigma),$$

where α is a non-zero complex number and χ is a non-even character of G such that $\sum_{i=0}^{n} \chi(z_i) \neq 0$.

(b)

$$f = \alpha\chi + \beta\chi \circ \sigma \quad \text{and} \quad g = \sum_{i=0}^{n} \frac{\chi(z_i)}{2}(\chi - \chi \circ \sigma),$$

where α, β are non-zero complex numbers and χ is a character of G such that $\sum_{i=0}^{n} \chi(z_i) \neq 0$ and

$$\sum_{i=0}^{n} \chi \circ \sigma(z_i) = -\sum_{i=0}^{n} \chi(z_i).$$

Proof. The proof follows on putting $\mu = \frac{1}{2}\sum_{i=0}^{n} \delta_{z_i}$ in Theorem 3.3. $\square$

In view of Corollary 3.8, we have the following.

Corollary 3.9. *The non-zero solutions $f : G \to \mathbf{C}$ of the functional equation*

$$\sum_{i=0}^{n}\{f(\sigma(y)xz_i) - f(xyz_i)\} = 2f(x)f(y), \quad x, y \in G,$$

are the functions of the form

$$f = -\sum_{i=0}^{n} \frac{\chi(z_i)}{2}(\chi - \chi \circ \sigma),$$

where χ is a character of G such that $\sum_{i=0}^{n} \chi(z_i) \neq 0$ and $\sum_{i=0}^{n} \chi \circ \sigma(z_i) = -\sum_{i=0}^{n} \chi(z_i)$.

Proof. The proof follows on putting $g = -f$ in Corollary 3.8. □

4. Solution of equation (1.5)

In this section, we solve the integral-functional equation (1.5), namely

$$\int_G \{f(xyt) - f(\sigma(y)xt)\}d\mu(t) = g(x)f(y), \quad x, y \in G,$$

by expressing its continuous solutions in terms of continuous characters and continuous additive functions.

Lemma 4.1. *Let G be a locally compact group, let σ be a continuous involutive automorphism on G, and let $\mu \in M_C(G)$. The following pairs of functions $f, g : G \to \mathbf{C}$ are continuous solutions of the functional equation (1.5):*

(a)

$$f = c(\chi - \chi \circ \sigma) \quad \text{and} \quad g = \mu(\chi)\chi + \mu(\chi \circ \sigma)\chi \circ \sigma,$$

where $c \in \mathbf{C}$ and χ is a continuous character of G, and

(b)

$$f = \chi A \quad \text{and} \quad g = 2\mu(\chi)\chi,$$

where χ is a continuous character of G such that $\chi \circ \sigma = \chi$ and where A is a continuous additive function on G such that $A \circ \sigma = -A$.

Proof. (a) Assume that there exist a continuous character χ of G and a complex number $c \in \mathbf{C}$ such that $f = c(\chi - \chi \circ \sigma)$ and $g = \mu(\chi)\chi + \mu(\chi \circ \sigma)\chi \circ \sigma$. Then $f, g \in C(G)$ and a small computation shows that

$$f(xyt) - f(\sigma(y)xt) = c[\chi(x)\chi(t) + \chi \circ \sigma(x)\chi \circ \sigma(t)][\chi(y) - \chi \circ \sigma(y)],$$

(4.1)

for all $x, y, t \in G$. Using (4.1) at the first equality sign:

$$\begin{aligned}
&\int_G \{f(xyt) - f(\sigma(y)xt)\} d\mu(t) \\
&\quad = c[\chi(y) - \chi \circ \sigma(y)] \int_G \{\chi(x)\chi(t) + \chi \circ \sigma(x)\chi \circ \sigma(t)\} d\mu(t) \\
&\quad = c[\chi(y) - \chi \circ \sigma(y)][\mu(\chi)\chi(x) + \mu(\chi \circ \sigma)\chi \circ \sigma(x)] \\
&\quad = g(x)f(y).
\end{aligned}$$

So the pair (f, g) is a solution of (1.5).

(b) Assume that there exist a continuous character χ of G with $\chi \circ \sigma = \chi$ and a continuous additive function A on G with $A \circ \sigma = -A$ such that $f = \chi A$ and $g = 2\mu(\chi)\chi$. Then $f, g \in C(G)$ and a small computation shows that

$$f(xyt) - f(\sigma(y)xt) = 2\chi(x)\chi(y)A(y)\chi(t), \quad \text{for all} \quad x, y, t \in G. \tag{4.2}$$

Using (4.2) at the first equality sign:

$$\begin{aligned}
\int_G \{f(xyt) - f(\sigma(y)xt)\} d\mu(t) &= 2\chi(x)\chi(y)A(y) \int_G \chi(t) d\mu(t) \\
&= 2\chi(x)\chi(y)A(y)\mu(\chi) \\
&= f(x)g(y).
\end{aligned}$$

So the pair (f, g) is a solution of (1.5). □

The second main theorem of the present paper asserts that the converse result to Lemma 4.1 is also valid. It reads as follows:

Theorem 4.2. *Let G be a locally compact group, let σ be a continuous involutive automorphism on G, and let $\mu \in M_C(G)$. Let the pair $f, g \in C(G) \setminus \{0\}$ be a solution of the functional equation (1.5). Then there exists a continuous character χ of G such that*

$$g = \mu(\chi)\chi + \mu(\chi \circ \sigma)\chi \circ \sigma.$$

Furthermore, we have the following possibilities:

(a) If $\chi \neq \chi \circ \sigma$, then there exists a constant $c \in \mathbf{C} \setminus \{0\}$ such that

$$f = c(\chi - \chi \circ \sigma).$$

(b) If $\chi = \chi \circ \sigma$, then there exists a continuous additive function $A : G \to \mathbf{C}$ with $A \circ \sigma = -A$ such that

$$f = \chi A.$$

Conversely, the formulas above for g and f define continuous solutions of (1.5).

Proof. It remains to prove the necessity assertion. Similarly to the proof of Theorem 3.3 we define $F : G \to \mathbf{C}$ by

$$F(x) = 2\int_G f(xt)d\mu(t) \quad \text{for all} \quad x \in G.$$

So $F \in C(G)$ and the equation (1.5) becomes

$$F(xy) - F(\sigma(y)x) = 2g(x)f(y), \quad x, y \in G.$$

Since $g, f \in C(G) \setminus \{0\}$, we know from Theorem 3.1 that there are only the following two cases:

Case 1: There exist a continuous character χ of G with $\chi \neq \chi \circ \sigma$, constants $\alpha, \beta \in \mathbf{C}$, $\gamma \in \mathbf{C} \setminus \{0\}$, and a continuous function $\theta \in N(G, \sigma)$ such that

$$\begin{aligned} g &= \alpha\chi + \beta\chi \circ \sigma, \\ f &= \gamma(\chi - \chi \circ \sigma), \\ F &= 2\gamma(\alpha\chi - \beta\chi \circ \sigma) + \theta. \end{aligned}$$

Since $F(x) = 2\int_G f(xt)d\mu(t)$ for all $x \in G$, we have

$$2\gamma[\alpha\chi(x) - \beta\chi \circ \sigma(x)] + \theta(x) = 2\gamma[\chi(x)\mu(\chi) - \chi \circ \sigma(x)\mu(\chi \circ \sigma)],$$

for all $x \in G$. Since $\gamma \neq 0$, we can write the last equation as follows

$$[\alpha - \mu(\chi)]\chi(x) + [\mu(\chi \circ \sigma) - \beta]\chi \circ \sigma(x) + \theta(x) = 0,$$

for all $x \in G$. Using Lemma 3.2(a), we get that $\alpha = \mu(\chi)$ and $\beta = \mu(\chi \circ \sigma)$. From this, we see that $g = \mu(\chi)\chi + \mu(\chi \circ \sigma)\chi \circ \sigma$ and arrive at the solution in case (a) with $c = \gamma$.

Case 2: There exist a continuous character χ of G with $\chi = \chi\circ\sigma$, constants $\alpha, \beta \in \mathbf{C}$, a continuous additive function $A : G \to \mathbf{C}$ with $A \circ \sigma = -A$, and a continuous function $\theta \in N(G, \sigma)$ such that

$$\begin{aligned} g &= \alpha\chi + \beta\chi A, \\ f &= \chi A, \\ F &= \alpha\chi A + \frac{1}{2}\beta\chi A^2 + \theta. \end{aligned}$$

Since $F(x) = 2\int_G f(xt)d\mu(t)$ for all $x \in G$, then a small computation shows that

$$\alpha\chi(x)A(x) + \frac{1}{2}\beta\chi(x)A^2(x) + \theta(x) = 2\mu(\chi)\chi(x)A(x) + 2\chi(x)\mu(A\chi),$$

for all $x \in G$. We reformulate the last equation as follows

$$\beta A^2 + 2[\alpha - 2\mu(\chi)]A + \theta_1 = 0,$$

where $\theta_1 := 2(\frac{\theta}{\chi}) - 4\mu(A\chi)$. Since χ is even we have $\theta_1 \in N(G, \sigma)$.

Subcase 2.1: Suppose that $\beta = 0$. Hence $[\alpha - 2\mu(\chi)]A \in N(G, \sigma)$, which implies that $[\alpha - 2\mu(\chi)]A = 0$. Since $f \neq 0$ we have $A \neq 0$, and hence $\alpha - 2\mu(\chi) = 0$, i.e., $\alpha = 2\mu(\chi)$. Therefore $g = 2\mu(\chi)\chi$ and we arrive at the solution in case (b).

Subcase 2.2: We now suppose that $\beta \neq 0$. According to Lemma 3.2(b), we get that $A = 0$. Hence $f = 0$. This subcase does not apply, because $f \neq 0$ by assumption. This finishes the proof. □

As a consequence of Theorem 4.2, we have the following result on the solution of the functional equation

$$f(xyz_0) - f(\sigma(y)xz_0) = 2g(x)f(y), \quad x, y \in G, \tag{4.3}$$

which contains the solution of Van Vleck's equation on abelian groups.

Corollary 4.3. *Let G be a group, $z_0 \in G$ be a fixed element, and σ be an involutive automorphism on G. Let $f, g : G \to \mathbf{C}$ be non-zero functions satisfying the functional equation (4.3). Then there exists a character χ of G such that*

$$g = \frac{\chi(z_0)}{2}\chi + \frac{\chi\circ\sigma(z_0)}{2}\chi\circ\sigma.$$

Furthermore, we have the following possibilities:

(a) *If $\chi \neq \chi \circ \sigma$, then there exists a constant $c \in \mathbf{C} \setminus \{0\}$ such that*

$$f = c(\chi - \chi \circ \sigma).$$

(b) *If $\chi = \chi \circ \sigma$, then there exists an additive function $A : G \to \mathbf{C}$ with $A \circ \sigma = -A$ such that*

$$f = \chi A.$$

Conversely, the formulas above for g and f define solutions of (4.3).

Proof. The proof follows on putting $\mu = \frac{1}{2}\delta_{z_0}$ in Theorem 4.2. $\square$

As another consequence of Theorem 4.2, we have the following result on the solution of the functional equation

$$\sum_{i=0}^{n} \{f(xyz_i) - f(\sigma(y)xz_i)\} = 2g(x)f(y), \quad x, y \in G, \tag{4.4}$$

which generalizes the equation (4.3).

Corollary 4.4. *Let G be a group, $z_0, z_1, ..., z_n \in G$ be fixed elements, and σ be an involutive automorphism on G. Let $f, g : G \to \mathbf{C}$ be non-zero functions satisfying the functional equation (4.4). Then there exists a character χ of G such that*

$$g = \sum_{i=0}^{n} \left[\frac{\chi(z_i)}{2}\chi + \frac{\chi \circ \sigma(z_i)}{2}\chi \circ \sigma \right].$$

Furthermore, we have the following possibilities:

(a) *If $\chi \neq \chi \circ \sigma$, then there exists a constant $c \in \mathbf{C} \setminus \{0\}$ such that*

$$f = c(\chi - \chi \circ \sigma).$$

(b) *If $\chi = \chi \circ \sigma$, then there exists an additive function $A : G \to \mathbf{C}$ with $A \circ \sigma = -A$ such that*

$$f = \chi A.$$

Conversely, the formulas above for g and f define solutions of (4.4).

Proof. The proof follows on putting $\mu = \frac{1}{2}\sum_{i=0}^{n} \delta_{z_i}$ in Theorem 4.2. $\square$

5. Examples

Example 5.1. *Let $G = (\mathbf{R}, +)$, $\sigma(x) = -x$ for all $x \in \mathbf{R}$, $z_0 \in \mathbf{R}\backslash\{0\}$ be a fixed element, and let $\mu = \frac{1}{2}\delta_{z_0}$.*

We indicate here the corresponding continuous solutions of Eqs. (1.4), (1.5) and (1.6) by the help of Theorems 3.3 and 4.2 and Corollary 3.5.

The continuous characters on $\mathbf{R}$ are known to be $\chi(x) = e^{\lambda x}$, $x \in \mathbf{R}$, where λ ranges over $\mathbf{C}$ (see for instance [6, Example 3.7(a)]). The condition $\mu(\chi\circ\sigma) = -\mu(\chi)$, i.e. $\chi(2z_0) = -1$, of Theorem 3.3 (d) becomes $e^{2\lambda z_0} = -1$, which reduces to $\lambda = i\frac{(2n+1)\pi}{2z_0}$, where $n \in \mathbf{Z}$. The relevant characters are thus

$$\chi_n(x) := e^{i\frac{(2n+1)\pi}{2z_0}x}, \quad x \in \mathbf{R}, \text{ and } n \in \mathbf{Z}.$$

The continuous additive functions on $\mathbf{R}$ are the functions of the form $A(x) = \alpha x$, $x \in \mathbf{R}$, where the constant α ranges over $\mathbf{C}$ (see for instance [6, Corollary 2.4]).

That $\theta \in N(\mathbf{R}, \sigma)$ is equivalent to $\theta(x + y) = \theta(x - y)$ for all $x, y \in \mathbf{R}$, i.e., $\theta(x) = \theta(x + 2y)$ for all $x, y \in \mathbf{R}$. Since $\mathbf{R}$ is 2-divisible, then each function in $N(\mathbf{R}, \sigma)$ is a constant. From this we infer that a function $f \in \{k \in C(\mathbf{R}) \mid x \mapsto \frac{1}{2}k(x + z_0) \in N(\mathbf{R}, \sigma)\}$ if and only if f is a constant function.

In conclusion, by help of Theorem 3.3 we find that the continuous solutions $f, g : \mathbf{R} \to \mathbf{C}$ of the functional equation (1.4), which is here

$$f(x + y + z_0) - f(x - y + z_0) = 2f(x)g(x), \quad x, y \in \mathbf{R},$$

are

(a) $f = 0$ and g is arbitrary in $C(\mathbf{R})$.

(b) f is a non-zero constant and $g = 0$.

(c) $f(x) = \alpha e^{\lambda x}$ and

$$g(x) = \frac{e^{\lambda(x+z_0)} - e^{-\lambda(x-z_0)}}{2}, \quad x \in \mathbf{R},$$

for some non-zero complex numbers α, λ.

(d)

$$\begin{aligned} f(x) &= \alpha e^{i\frac{(2n+1)\pi}{2z_0}x} + \beta e^{-i\frac{(2n+1)\pi}{2z_0}x} \\ &= (\alpha+\beta)\cos\left(\frac{(2n+1)\pi}{2z_0}x\right) + i(\alpha-\beta)\sin\left(\frac{(2n+1)\pi}{2z_0}x\right), \end{aligned}$$

and

$$g(x) = (-1)^{n+1}\sin\left(\frac{(2n+1)\pi}{2z_0}x\right), \quad x \in \mathbf{R},$$

for some $\alpha, \beta \in \mathbf{C}\backslash\{0\}$ *and* $n \in \mathbf{Z}$.

Now we seek the solutions $f, g \in C(\mathbf{R})\backslash\{0\}$ *of the functional equation (1.5) which is here*

$$f(x+y+z_0) - f(x-y+z_0) = 2g(x)f(x), \quad x, y \in \mathbf{R}.$$

According to Theorem 4.2, we see that there exists a constant $\lambda \in \mathbf{C}$ *such that*

$$g(x) = \frac{e^{i\lambda(x+z_0)} + e^{-i\lambda(x+z_0)}}{2} = \cos\left(\lambda(x+z_0)\right), \quad x \in \mathbf{R}.$$

Furthermore, we have the following possibilities:

(a) If $\lambda \neq 0$, *then there exists* $\alpha \in \mathbf{C}\backslash\{0\}$ *such that*

$$f(x) = \alpha\frac{e^{i\lambda x} - e^{-i\lambda x}}{2i} = \alpha\sin\left(\lambda x\right), \quad x \in \mathbf{R}.$$

(b) If $\lambda = 0$, *then* $g = 1$ *and there exists* $\alpha \in \mathbf{C}\backslash\{0\}$ *such that*

$$f(x) = \alpha x, \quad x \in \mathbf{R}.$$

Finally, by help of Corollary 3.5 we see that the solutions $f \in C(\mathbf{R})\backslash\{0\}$ *of the functional equation (1.6), which is here Van Vleck's equation (1.1), are the sine functions*

$$f(x) = (-1)^n\sin\left(\frac{(2n+1)\pi}{2z_0}x\right), \quad x \in \mathbf{R}, \quad n \in \mathbf{Z}.$$

This result can be found e.g. in [7].

Example 5.2. *For an application of our results on a non-abelian group, consider the 3-dimensional Heisenberg group $G = H_3$ described in [6, Example A.17(a)], and take as the involutive automorphism*

$$\sigma\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b & c \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -b & -c+ab \\ 0 & 1 & -a \\ 0 & 0 & 1 \end{pmatrix} \quad \text{for } a,b,c \in \mathbf{R}.$$

Let $Z_0 = \begin{pmatrix} 1 & a_0 & c_0 \\ 0 & 1 & b_0 \\ 0 & 0 & 1 \end{pmatrix}$ *be a fixed element of H_3 and let* $\mu = \frac{1}{2}\delta_{Z_0}$. *We indicate here the corresponding continuous solutions of Eqs. (1.4), (1.5) and (1.6).*

The continuous characters on H_3 are parametrized by $(\alpha, \beta) \in \mathbf{C}^2$ as follows (see, e.g., [1, Example 5.2]).

$$\chi_{\alpha,\beta}\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = e^{\alpha a+\beta b} \quad \text{for } a,b,c \in \mathbf{R}.$$

We compute that $\chi_{\alpha,\beta} \circ \sigma = \chi_{-\beta,-\alpha}$, so $\chi_{\alpha,\beta} \circ \sigma = \chi_{\alpha,\beta}$, if and only if $\beta = -\alpha$, and in that case

$$\chi_{\alpha,-\alpha}\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = e^{\alpha(a-b)} \quad \text{for } a,b,c \in \mathbf{R}.$$

In view of [1, Example 5.2], the continuous odd additive functions on H_3 are parametrized by $\gamma \in \mathbf{C}$ as follows

$$A_\gamma\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \gamma(a+b) \quad \text{for } a,b,c \in \mathbf{R}.$$

Now we are in the position to describe the solutions $f, g \in C(H_3)\backslash\{0\}$ of the functional equation (1.5), which is here

$$f(XYZ_0) - f(\sigma(Y)XZ_0) = 2g(X)f(Y), \quad X, Y \in H_3.$$

By help of Theorem 4.2 we see that there exist constants $\alpha, \beta \in \mathbf{C}$ such that

$$g\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \frac{e^{\alpha(a+a_0)+\beta(b+b_0)} + e^{-\beta(a+a_0)-\alpha(b+b_0)}}{2}, \quad a,b,c \in \mathbf{R}.$$

Furthermore, we have the following possibilities:

(a) *If $\beta \neq -\alpha$, then there exists $\gamma \in \mathbf{C}\backslash\{0\}$ such that*

$$f\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \gamma\left[e^{\alpha a+\beta b} - e^{-\beta a-\alpha b}\right], \quad a,b,c \in \mathbf{R}.$$

(b) *If $\beta = -\alpha$, then $g\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = e^{\alpha(a-b+a_0-b_0)}$ and there exists $\gamma \in \mathbf{C}\backslash\{0\}$ such that*

$$f\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \gamma(a+b)e^{\alpha(a-b)}, \quad a,b,c \in \mathbf{R}.$$

Now we seek the solutions $f,g \in C(H_3)\backslash\{0\}$ of the functional equation (1.4) which is here

$$f(XYZ_0) - f(\sigma(Y)XZ_0) = 2f(X)g(Y), \quad X,Y \in H_3. \tag{5.1}$$

The condition $\mu(\chi_{\alpha,\beta} \circ \sigma) = -\mu(\chi_{\alpha,\beta})$, i.e. $\chi_{-\beta,-\alpha}(Z_0) = -\chi_{\alpha,\beta}(Z_0)$, of Theorem 3.3 (d) becomes $e^{(a_0+b_0)(\alpha+\beta)} = -1$, which breaks the job into two cases: $b_0 = -a_0$ or $b_0 \neq -a_0$.

Case 1: *Suppose that $b_0 = -a_0$. A small computation shows that $\chi_{-\beta,-\alpha}(Z_0) \neq -\chi_{\alpha,\beta}(Z_0)$, for all $\alpha, \beta \in \mathbf{C}$. Then, using Theorem 3.3, the solutions $f, g \in C(H_3)\backslash\{0\}$ of Eq. (5.1) are the functions of the forms:*

$$\begin{aligned} f\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} &= \gamma e^{\alpha a+\beta b}, \\ g\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} &= \frac{e^{\alpha(a+a_0)+\beta(b+b_0)} - e^{-\beta(a-b_0)-\alpha(b-a_0)}}{2}, \end{aligned}$$

for all $a,b,c \in \mathbf{R}$, where α, β, γ are complex numbers such that $\beta \neq -\alpha$ and $\gamma \neq 0$.

Case 2: *Suppose that $b_0 \neq -a_0$. A small computation shows that $\chi_{-\beta,-\alpha}(Z_0) = -\chi_{\alpha,\beta}(Z_0)$ if and only if $\beta = i\frac{(2n+1)\pi}{a_0+b_0} - \alpha$, where $n \in \mathbf{Z}$. Then, using Theorem 3.3, the solutions $f, g \in C(H_3)\backslash\{0\}$ of Eq. (5.1) are the ones of the forms:*

(1)

$$f\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \gamma e^{\alpha a+\beta b},$$
$$g\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \frac{e^{\alpha(a+a_0)+\beta(b+b_0)} - e^{-\beta(a-b_0)-\alpha(b-a_0)}}{2},$$

for all $a, b, c \in \mathbf{R}$, where α, β, γ are complex numbers such that $\beta \neq -\alpha$ and $\gamma \neq 0$.

(2)

$$f\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = e^{\alpha(a-b)}\left[\gamma e^{i\frac{(2n+1)\pi}{a_0+b_0}b} + \delta e^{-i\frac{(2n+1)\pi}{a_0+b_0}a}\right],$$
$$g\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = e^{\alpha(a-b+a_0-b_0)+i\frac{(2n+1)\pi}{a_0+b_0}b_0}\frac{e^{i\frac{(2n+1)\pi}{a_0+b_0}b} - e^{-i\frac{(2n+1)\pi}{a_0+b_0}a}}{2},$$

for all $a, b, c \in \mathbf{R}$, where $\alpha \in \mathbf{C}$, $\gamma, \delta \in \mathbf{C}\backslash\{0\}$, and $n \in \mathbf{Z}$.

We complete this section by solving the functional equation (1.6), which is here

$$f(\sigma(Y)XZ_0) - f(XYZ_0) = 2f(X)f(Y), \quad X, Y \in H_3. \tag{5.2}$$

By help of Corollary 3.5 and the previous discussion we see that we have the following two possibilities:
If $b_0 = -a_0$, then the only solution of Eq. (5.2) is the function $f \equiv 0$.
If $b_0 \neq -a_0$, then the solutions $f \in C(H_3)$ of Eq. (5.2) are either $f \equiv 0$ or

$$f\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = -e^{\alpha(a-b+a_0-b_0)+i\frac{(2n+1)\pi}{a_0+b_0}b_0}\frac{e^{i\frac{(2n+1)\pi}{a_0+b_0}b} - e^{-i\frac{(2n+1)\pi}{a_0+b_0}a}}{2},$$

for all $a, b, c \in \mathbf{R}$, where $\alpha \in \mathbf{C}$ and $n \in \mathbf{Z}$.

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