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A generalization of variant of Wilson′s type Hilbert space valued functional equations

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Abstract

In the present paper we characterize, in terms of characters, multiplicative functions, the continuous solutions of some functional equations for mappings defined on a monoid and taking their values in a complex Hilbert space with the Hadamard product. In addition, we investigate a superstability result for these equations.

Keywords : *D'Alembert's functional equation, Hilbert space, Hadamard product, superstability.*

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1. Introduction

Let M be a monoid i.e., is a semigroup with an identify element that we denote by e and $\sigma, \tau : M \to M$ are two involutive automorphisms. That is $\sigma(xy) = \sigma(x)\sigma(y)$, $\tau(xy) = \tau(x)\tau(y)$ and $\sigma(\sigma(x)) = x$, $\tau(\tau(x)) = x$ for all $x, y \in M$. By a variant of Wilson's functional equation on M we mean the functional equation

$$f(x\sigma(y)) + f(\tau(y)x) = 2f(x)g(y), \quad x, y \in M, \tag{1.1}$$

where $f, g : M \to \mathbf{C}$ are the unknown functions. A special case of Wilson's functional equation is d'Alembert's functional equation:

$$f(x\sigma(y)) + f(\tau(y)x) = 2f(x)f(y), \quad x, y \in M, \tag{1.2}$$

The solutions of equation (1.2) are known [2]. Further contextual and historical discussion on the functional equation (1.1) and (1.2) can be found, e.g., in [6.2].

The present paper studies an extension to a situation where the unknown functions f, g map a possibly non-abelian group or monoid into a complex Hilbert space H with the Hadamard product. Our considerations refer mainly to results by Rezaei [4], Zeglami [11]. It has been proved [3] that the functional equation (1.2) with $\sigma = id$ is superstable in the class of functions $f : G \to \mathbf{C}$, if every such function satisfies the inequality

$$|f(xy) + f(\tau(y)x) - 2f(x)f(y)| \leq \epsilon \ for \ all \ x, y \in G,$$

where ϵ is a fixed positive real number. Then either f is a bounded function or

$$f(xy) + f(\tau(y)x) = 2f(x)f(y), \quad x, y \in G.$$

Let H be a separable Hilbert space with a orthonormal basis $\{e_n, n \in \mathbf{N}\}$. For two vectors $x, y \in H$, the Hadamard product, also known as the entrywise product on the Hilbert space H is defined by

$$x * y = \sum_{n=0}^{\infty} \langle x, e_n\rangle \langle y, e_n\rangle e_n, \quad x, y \in H. \tag{1.3}$$

The Cauchy-Schwarz inequality together with the Parseval identity ensure that the Hadamard multiplication is well defined. In fact,

$$\|x * y\| \leq (\sum_{n=0}^{\infty} |\langle x, e_n\rangle|^2)^{\frac{1}{2}} (\sum_{n=0}^{\infty} |\langle y, e_n\rangle|^2)^{\frac{1}{2}} = ||x||||y||. \tag{1.4}$$

The purpose of this work is first to give a characterization, in terms of multiplicative functions, the solutions of the Hilbert space valued functional equation by Hadamard product:

$$f(x\sigma(y)) + f(\tau(y)x) = 2g(x) * f(y), \quad x, y \in M. \tag{1.5}$$

When f we determine the solutions of the functional equation

$$f(x\sigma(y)) + f(\tau(y)x) = 2f(x) * g(y), \quad x, y \in M, \tag{1.6}$$

where $f, g : M \to H$ are the unknown functions. Second, we determine a characterization of the following d'Alembert-Hilbert-valued functional equation:

$$f(x\sigma(y)) + f(\tau(y)x) = 2f(x) * f(y), \quad x, y \in M. \tag{1.7}$$

Throughout the paper, $\mathbf{N}$, $\mathbf{R}$ and $\mathbf{C}$ stand for the sets of positive integers, real numbers and complex numbers, respectively. We let G denote a group and S denote a semigroup i.e., a set with an associative composition rule.

A function $A : M \to \mathbf{C}$ is called additive, if it satisfies $A(xy) = A(x) + A(y)$ for all $x, y \in M$.

A multiplicative function on M is a map $\chi : M \to \mathbf{C}$ such that $\chi(xy) = \chi(x)\chi(y)$ for all $x, y \in M$.

A monoid M is generated by its squares if for every $x \in I_\chi$, $x = x_1^2 x_2^2 \cdots x_n^2$ for some $x_1, x_2, \cdots, x_n \in M$.

A character on a group G is a homomorphism from G into the multiplicative of non-zero complex numbers. While a non-zero multiplicative function on a group can never take the value 0, it is possible for a multiplicative function on a monoid M to take the value 0 on a proper, non-empty subset of M. If $\chi : M \to \mathbf{C}$ is multiplicative and $\chi \neq 0$, then

$$I_\chi = \{x \in M / \chi(x) = 0\}$$

is either empty or a proper subset of M. The fact that χ is multiplicative establishes that I_χ is a two-sided ideal in M if not empty (for us an ideal is never the empty set). It follows also that $M \setminus I_\chi$ is a subsemigroup of M.

Let $C(M)$ denote the algebra of continuous functions from M into $\mathbf{C}$.

2. Solutions of (1.5) and (1.6)

In this section, we solve the functional equation (1.5) by expressing its solutions in terms of multiplicative functions.

Theorem 2.1. *Let M be a monoid, let $\sigma, \tau : M \to M$ be involutive automorphisms. Assume that the functions $f, g : M \to H$ satisfy (1.5). Then, there exists a positive integer N such that*

$$f(x) = \textstyle\sum_{n=1}^{N} \langle f(x), e_n\rangle e_n \text{ and } x \to \langle g(x), e_{N+k}\rangle \text{ is arbitrary}$$

for all $x \in M$ and $k > 0$. Furthermore, for every $k \in \{1, 2, ..., N\}$, we have the following possibilities:

$$\begin{cases} \langle g(x), e_k\rangle = \frac{\chi_k(x)+\chi_k\circ\sigma\circ\tau(x)}{2} \\ \langle f(x), e_k\rangle = \frac{\alpha_k(\chi_k(x)+\chi_k\circ\sigma\circ\tau(x))}{2} \end{cases} ; \begin{cases} \langle g(x), e_k\rangle \text{ is an arbitrary function,} \\ \langle f(x), e_k\rangle = 0 \end{cases}$$

for all $x \in M$, where χ_k is a non-zero multiplicative function of M such that $\chi_k \circ \sigma \circ \tau = \chi_k \circ \tau \circ \sigma$ and $\alpha_k \in \mathbf{C}\backslash\{0\}$. If M is a topological monoid and $f \in C(M)$, then χ_k, $\chi_k \circ \sigma \circ \tau \in C(M)$.

Proof. For every integer $k \geq 0$, consider the functions $f_k, g_k : M \to \mathbf{C}$ defined by

$$f_k(x) = \langle f(x), e_k\rangle \text{ and } g_k(x) = \langle g(x), e_k\rangle \text{ for all } x \in M.$$

Since (f, g) satisfies (1.5), for all $x, y \in M$, we have

$$\begin{aligned} \sum_{k=0}^{+\infty}\{\langle f(x\sigma(y)), e_k\rangle + \langle f(\tau(y)x), e_k\rangle\}e_k &= \sum_{k=0}^{+\infty}\langle\{f(x\sigma(y)) + f(\tau(y)x)\}, e_k\rangle e_k \\ &= f(x\sigma(y)) + f(\tau(y)x) \\ &= 2g(x) * f(y) \\ &= 2\sum_{k=0}^{+\infty}\langle g(x), e_k\rangle\langle f\rangle, \end{aligned}$$

This yields for all $k \in \mathbf{N}$,

$$f_k(x\sigma(y)) + f_k(\tau(y)x) = 2g_k(x)f_k(y) \quad \text{for all } x, y \in M. \tag{2.1}$$

If we put $y = e$ in (2.1), we find that $f_k(x) = f_k(e)g_k(x)$. So, if we take $\alpha_k = f_k(e)$, equation (2.1) can be written as follows:

$$\alpha_k g_k(x\sigma(y)) + \alpha_k g_k(\tau(y)x) = 2\alpha_k g_k(x)g_k(y) \ \ for\ all\ x, y \in M.$$

Then, either $\alpha_k = 0$ or g_k is a solution of equation (1.6). In view of [2, Theorem 3.2], one of the following statements holds:
(a) We have that

$$f_k = 0 \ and\ g_k\ is\ an\ arbitrary\ function.$$

(b) There exists a multiplicative function χ_k such that
$g_k(x) = \frac{\chi_k(x)+\chi_k\circ\sigma\circ\tau(x)}{2}$ and $f_k(x) = \frac{\alpha_k(\chi_k(x)+\chi_k\circ\sigma\circ\tau(x))}{2}$ *for* $x \in M$.

If H is infinite-dimensional, then

$$\langle g(x), e_k\rangle = g_k(x) \to 0\ as\ k \to +\infty$$

for every $x \in M$. Since $g_k(e) = 1$, statement (b) is not possible for infinitely many positive integers k. Hence, there exists some positive integer N such that $f_k = 0$ for every $k > N$. Thus, g_k is an arbitrary function for any $k > N$, f can be represented as

$$f(x) = \sum_{n=1}^{N} \langle f(x), e_n\rangle e_n,$$

and the expressions of the component functions f_n and g_n, $1 \leq n \leq N$, of f and g come from statements (a) and (b) above. In the case where H is finite- dimensional, the proof is clear.

As a consequence of Theorem 2.1 we derive formulas for the solutions of d'Alembert's Hilbert space valued functional equation (1.7). □

Corollary 2.2. *Let M be a monoid, let $\sigma, \tau : M \to M$ be involutive automorphisms. Assume that the functions $g : M \to H$ satisfy (1.7). Then, there exists a positive integer N such that*

$$f(x) = \textstyle\sum_{n=1}^{N} \langle f(x), e_n\rangle e_n \text{ and } x \to \langle g(x), e_{N+k}\rangle \text{ is arbitrary}$$

for all $x \in M$ and $k > 0$. Furthermore, for every $k \in \{1, 2, ..., N\}$, such that

$$g(x) = \frac{1}{2}\sum_{k=1}^{N} \epsilon_k(\chi_k(x) + \chi_k \circ \sigma \circ \tau(x))e_k,\ x \in M,$$

where $\epsilon_k = 1$ or 0 for every $k \in \{1, 2, ..., N\}$. for all $x \in M$, where χ_k is a non-zero multiplicative function of M such that $\chi_k \circ \sigma \circ \tau = \chi_k \circ \tau \circ \sigma$

If M is a topological monoid and $f \in C(M)$, then χ_k, $\chi_k \circ \sigma \circ \tau \in C(M)$.

Proof. The proof follows by putting $f = g$ in Theorem 2.1. □

Corollary 2.3. *Let M be a monoid, let $\tau : M \to M$ be involutive automorphisms. Assume that the functions $f, g : M \to H$ satisfy*

$$f(xy) + f(\tau(y)x) = 2g(x) * f(y).$$

Then, there exists a positive integer N such that

$$f(x) = \textstyle\sum_{n=1}^{N} \langle f(x), e_n \rangle e_n \text{ and } x \to \langle g(x), e_{N+k} \rangle \text{ is arbitrary}$$

for all $x \in M$ and $k > 0$. Furthermore, for every $k \in \{1, 2, ..., N\}$, we have the following possibilities:

$$\begin{cases} \langle g(x), e_k \rangle = \frac{\chi_k(x) + \chi_k \circ \tau(x)}{2} \\ \langle f(x), e_k \rangle = \frac{\alpha_k(\chi_k(x) + \chi_k \circ \tau(x))}{2} \end{cases} ; \begin{cases} \langle g(x), e_k \rangle \text{ is an arbitrary function,} \\ \langle f(x), e_k \rangle = 0 \end{cases}$$

for all $x \in M$, where χ_k is a non-zero multiplicative function of M and $\alpha_k \in \mathbf{C} \backslash \{0\}$.

If M is a topological monoid and $f \in C(M)$, then χ_k, $\chi_k \circ \tau \in C(M)$.

Proof. The proof follows by putting $\sigma = id$ in Theorem 2.1. □

We complete this section with a result concerning Wilson Hilbert space valued functional equation (1.6).

Theorem 2.4. *Let M be a monoid which is generated by its squares, let $\sigma, \tau : M \to M$ be involutive automorphisms. Assume that the pair $f, g : M \to \mathbf{C}$, satisfy Wilson's Hilbert valued functional equation (1.6). Then, there exists a positive integer N such that*

$$f(x) = \sum_{n=1}^{N} \langle f(x), e_n \rangle e_n \text{ and } \langle g(x), e_{N+k} \rangle \text{ is arbitrary}$$

for all $x \in M$ and $k > 0$. Furthermore, for every $k \in \{1, 2, ..., N\}$, we have the following possibilities:

(i)

$$\begin{cases} \langle g(x), e_k\rangle \text{ is an arbitrary function}, \\ \langle f(x), e_k\rangle = 0 \end{cases} ; \begin{cases} \langle g(x), e_k\rangle = \frac{\chi_k(x)+\chi_k\circ\sigma\circ\tau(x)}{2}, \\ \langle f(x), e_k\rangle = \alpha_k\chi_k\circ\sigma(x) \end{cases}$$

where $\chi_k : M \to \mathbf{C}$ *is a non-zero multiplicative function with* $\chi_k\circ\sigma\circ\tau = \chi_k\circ\tau\circ\sigma$*. and for some* $\alpha_k \in \mathbf{C}\backslash\{0\}$*.*

(ii) There exists a non-zero multiplicative function $\chi_k : M \to \mathbf{C}$ *with* $\chi_k\circ\sigma\circ\tau = \chi_k\circ\tau\circ\sigma$ *such that*

$$g_k = \frac{\chi_k + \chi_k\circ\sigma\circ\tau}{2}.$$

Furthermore, we have

(1) If $\chi_k \neq \chi_k\circ\sigma\circ\tau$*, then*

$$f_k = \alpha_k\chi_k\circ\sigma + \beta_k\chi_k\circ\tau$$

for some $\alpha_k, \beta_k \in \mathbf{C}\backslash\{0\}$*.*

(2) If $\chi_k = \chi_k\circ\sigma\circ\tau$*, then there exists a non-zero additive function* $A_k : M \setminus I_{\chi_k\circ\sigma} \to \mathbf{C}$ *with* $A_k\circ\tau = -A_k\circ\sigma$ *such that*

$$f_k(x) = \begin{cases} (\alpha_k + A_k(x))\chi_k(\sigma(x)) \ for \ x \in M\setminus I_{\chi_k\circ\sigma} \\ 0 \ for \ x \in I_{\chi_k\circ\sigma} \end{cases}$$

for some $\alpha_k, \in \mathbf{C}$*.*

Conversely, if f *and* g *have the forms described above, then the pair* (f, g) *is a solution of equation (1.6). Moreover, if* M *is a topological monoid generated by its squares, and* $f, g \in C(M)$*, then* $\chi_k, \chi_k\circ\sigma, \chi_k\circ\tau, \chi_k\circ\sigma\circ\tau \in C(M)$*, while* $A_k \in C(M\setminus I_{\chi_k\circ\sigma})$*.*

Proof. We proceed as in the proof of Theorem 2.1. For every integer $k \geq 0$, we consider the functions $f_k, g_k : M \to \mathbf{C}$, defined by

$$f_k(x) = \langle f(x), e_k\rangle \ and \ g_k(x) = \langle g(x), e_k\rangle \ for \ x \in M.$$

Since the pair (f, g) satisfies (1.6), for all $k \in \mathbf{N}$ we have

$$f_k(x\sigma(y)) + f_k(\tau(y)x) = 2f_k(x)g_k(y) \ for \ all \ x, y \in M. \tag{2.2}$$

By [6,Theorem 3.4] we infer that there are only the following cases

(a)

$$f_k = 0 \ and \ g_k \ is \ an \ arbitrary \ function.$$

(b) There exists a non -zero multiplicative function $\chi_k : M \to \mathbf{C}$ such that

$$f_k = \alpha_k \chi_k \circ \sigma \; and \; g_k = \frac{\chi_k + \chi_k \circ \sigma \circ \tau}{2}$$

for some $\alpha_k \in \mathbf{C}\backslash\{0\}$.

(c) There exists a non-zero multiplicative function $\chi_k : M \to \mathbf{C}$ with $\chi_k \circ \sigma \circ \tau = \chi_k \circ \tau \circ \sigma$ such that

$$g_k = \frac{\chi_k + \chi_k \circ \sigma \circ \tau}{2}.$$

Furthermore, we have.

(i) If $\chi_k \neq \chi_k \circ \sigma \circ \tau$, then

$$f_k = \alpha_k \chi_k \circ \sigma + \beta_k \chi_k \circ \tau$$

for some $\alpha_k, \beta_k \in \mathbf{C}\backslash\{0\}$.

(ii) If $\chi_k = \chi_k \circ \sigma \circ \tau$, then there exists a non-zero additive function $A_k : M \setminus I_{\chi_k \circ \sigma} \to \mathbf{C}$ with $A_k \circ \tau = -A_k \circ \sigma$ such that

$$f_k(x) = \begin{cases} (\alpha_k + A_k(x))\chi_k(\sigma(x)) \; for \; x \in M \setminus I_{\chi_k \circ \sigma} \\ 0 \; for \; x \in I_{\chi_k \circ \sigma} \end{cases}$$

for some $\alpha_k \in \mathbf{C}$. Conversely, the functions given with properties satisfy the functional equation (2.2). The continuation of the proof depends on the dimension of H. In fact, if H is infinite-dimensional, then

$$\langle g(x), e_k \rangle = g_k(x) \to 0 \; as \; k \to +\infty$$

for every $x \in M$. Statements (b) and (c) are not possible for infinitely positive integers n. Hence, there exists some positive integer N such that $f_k = 0$ for every $k > N$. Thus, f can be represented as

$$f(x) = \sum_{n=1}^{N} \langle f(x), e_n \rangle e_n,$$

g_k is an arbitrary function for any $k > N$, and expressions of the component functions f_n and g_n, $1 \leq n \leq N$ of f and g follow from the previous discussion. In the case where H is a finite-dimensional space, the proof is clear. □

Corollary 2.5. *Let M be a monoid which is generated by its squares, let $\tau : M \to M$ be an involutive automorphism, and let the pair $f, g : M \to H$ satisfy the functional equation*

$$f(xy) + f(\tau(y)x) = 2f(x) * g(y), \quad x, y \in M.$$

Then, there exists a positive integer N such that

$$f(x) = \sum_{n=1}^{N} \langle f(x), e_n \rangle e_n \text{ and } x \to \langle g(x), e_{N+k} \rangle \text{ is arbitrary}$$

for all $x \in M$ and $k > 0$. Furthermore, for every $k \in \{1, 2, ..., N\}$, we have the following possibilites:

(i) $\begin{cases} \langle g(x), e_k \rangle \ is\ an\ arbitrary\ function, \\ \langle f(x), e_k \rangle = 0 \end{cases}$

(ii) There exists a non-zero multiplicative function $\chi_k : M \to \mathbf{C}$ such that

$$g_k = \frac{\chi_k + \chi_k \circ \tau}{2}.$$

Furthermore, we have.

(1) If $\chi_k \neq \chi_k \circ \tau$, then

$$f_k = \alpha_k \chi_k + \beta_k \chi_k \circ \tau,$$

for some $\alpha_k, \beta_k \in \mathbf{C} \backslash \{0\}$.

(2) If $\chi_k = \chi_k \circ \tau$, then there exists an additive function $A_k : M \backslash I_{\chi_k} \to \mathbf{C}$ with $A_k \circ \tau = -A_k$ such that

$$f_k(x) = \begin{cases} (\alpha_k + A_k(x))\chi_k(x) \ for\ x \in M \setminus I_{\chi_k} \\ 0 \ for\ x \in I_{\chi_k} \end{cases}$$

for some $\alpha_k \in \mathbf{C}$.

Conversely, if f and g have the forms described above, then the pair (f, g) is a solution. Moreover, if M is a topological monoid generated by its squares, and $f, g \in C(M)$, then $\chi_k, \chi_k \circ \tau \in C(M)$, while $A_k \in C(M \setminus I_{\chi_k})$.

Proof. The proof follows by putting $\sigma = id$ in Theorem 2.4. $\square$

3. Superstability of Hilbert valued cosine type functional equations

The main result of this section is Theorem 3.3 that contains a superstability result for the functional equation (1.6). For the proof of our result we will begin by pointing out a superstability result for the equation

$$f(xy) + f(\sigma(y)x) = 2f(x)g(y) \tag{3.1}$$

where $f, g : G \to \mathbf{C}$ are the unknown functions.

Proposition 3.1. *Let $\delta > 0$ be given, let M be a monoid and let σ is an involutive morphism of M. Assume that the functions $f, g : M \to \mathbf{C}$ satisfies the inequality*

$$|f(xy) + f(\sigma(y)x) - 2f(x)g(y)| \leq \delta \text{ for all } x, y \in M,$$

and that g is unbounded. Then, the ordered pair (f, g) satisfies equation (3.1).

Proof. The proof is part of the proof of [3,Theorem 2.1 and Theorem 3.7] if we put $\chi = 1$ that deals with M being a group. □

Corollary 3.2. *Let $\delta > 0$ be given and let G be a monoid. Assume that the function $f : G \to \mathbf{C}$ satisfies the inequality*

$$|f(xy) + f(\sigma(y)x) - 2f(x)f(y)| \leq \delta \text{ for all } x, y \in G.$$

Then, either

$$|f(x)| \leq \frac{1 + \sqrt{1 + 2\delta}}{2} \text{ for all } x \in G,$$

or f has the form

$$f = \frac{\mu + \mu \circ \sigma}{2},$$

where μ is a multiplicative function.

Proof. The proof follows immediately from Propositon 3.1 and Theorem [1, Theorem 4]. □

Theorem 3.3. *Let $\delta > 0$ be given and let M be a monoid. Assume that the functions $f, g : M \to H$ satisfy the inequality*

$$||f(xy) + f(\sigma(y)x) - 2f(x) * g(y)|| \leq \delta \text{ for all } x, y \in M. \tag{3.2}$$

Then, either

(i) there exists $k \geq 1$ such that the function $x \mapsto \langle g(x), e_k\rangle$ is bounded, or

(ii) the pair (f, g) is a solution of the functional equation:

$$f(xy) + f(\sigma(y)x) = 2f(x) * g(y). \tag{3.3}$$

Proof. Suppose that the pair (f, g) satisfies (3.2). By applying the Parseval identity and the definition of Hadamard product with the inequality (3.2), we find that the scalar valued functions f_k, g_k defined by

$$f_k(x) = \langle f(x), e_k\rangle \text{ and } g_k(x) = \langle g(x), e_k\rangle \text{ for } x \in M,$$

satisfy the inequality

$$|f_k(xy) + f_k(\sigma(y)x) - 2f_k(x)g_k(y)| \leq \delta \text{ for all } x, y \in M.$$

According to Proposition 3.1, for all $k \in \mathbf{N}$, we have that either the function $x \mapsto \langle g(x), e_k\rangle$ is bounded or the pair (f_k, g_k) is a solution of (3.1). Then, we conclude that the pair (f, g) satisfies equation (3.3) if assertion (i) fails. □

In [4] it was proved that if $g : H \to H$ is surjective, then every component function $x \mapsto \langle g(x), e_n\rangle$ is unbounded. By applying Theorem (3.3), this leads to the following result.

Corollary 3.4. *Let $\delta > 0$ be given. Assume that functions $f, g : H \to H$, where g is surjective, satisfy the inequality*

$$||f(xy) + f(\sigma(y)x) - 2f(x) * g(y)|| \leq \delta \text{ for all } x, y \in H.$$

Then, the pair (f, g) satisfies the equation

$$f(xy) + f(\sigma(y)x) = 2f(x) * g(y) \text{ for all } x, y \in H.$$

Proof. Since g is surjective, then every component function $x \mapsto \langle g(x), e_n \rangle$ is unbounded. Thus, the proof follows immediately from Theorem 3.3. $\square$

Corollary 3.5. *Let $\delta > 0$ be given and let G be a topological group. Assume that the function $g : G \to H$ satisfies the inequality*

$$||g(xy) + g(\sigma(y)x) - 2g(x) * g(y)|| \leq \delta \; for \; all \; x, y \in G.$$

Then, either there exists $k \geq 1$ such that

$$|\langle g(x), e_k \rangle| \leq \frac{1 + \sqrt{1 + 2\delta}}{2} \; for \; all \; x \in G$$

or there exist a multiplicative function $\chi_k : M \to \mathbf{C} \backslash \{0\}$ and a positive integer N such that

$$g(x) = \frac{1}{2} \sum_{n=1}^{N} \epsilon_n (\chi_k(x) + \chi_k \circ \sigma(x)) e_n, \; for \; all \; x \in G,$$

where $\epsilon_n = 1$ or 0 for every $n \in \{1, 2, ..., N\}$.

Proof. If we put $f = g$ in Theorem 3.3, we immediately have that either there exists $k \geq 1$ such that the function $x \mapsto \langle g(x), e_k \rangle$ is bounded or g is a solution of the equation

$$g(xy) + g(\sigma(y)x) = 2g(x) * g(y), \quad x, y \in G.$$

The remainder of the proof follows if we put $\chi = 1$ from Corollary [3, Corollary 3.8] and Corollary 2.3. $\square$

Corollary 3.6. *Let $\delta > 0$ be given and let G be a group with identity element. Let $g : G \to H$ such that*

$$||g(xy) + g(yx) - 2g(x) * g(y)|| \leq \delta \; for \; all \; x, y \in G.$$

Then either g is bounded or g is multiplicative.

Proof. From Corollary 2.2 and Corollary 2.5 and then using [3, Corollary 3.9]. $\square$

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