DOI: 10.4067/S0716-09172017000300449

Proyecciones Journal of Mathematics
Vol. 36, N^o 3, pp. 449-460, September 2017.
Universidad Católica del Norte
Antofagasta - Chile

Existence of positive periodic solutions for delay dynamic equations

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Received : September 2016. Accepted : August 2017

Abstract

In this article we study the existence of positive periodic solutions for a dynamic equations on time scales. The main tool employed here is the Schauder's fixed point theorem. The results obtained here extend the work of Olach [12]. Two examples are also given to illustrate this work.

Subjclass[2000] : *Primary 34K13, 34A34; Secondary 34K30, 34L30.*

Keywords : *Positive periodic solutions, Schauder's fixed point theorem, dynamic equations, time scales.*

1. Introduction

In 1988, Stephan Hilger [8] introduced the theory of time scales (measure chains) as a means of unifying discrete and continuum calculi. Since Hilger's initial work there has been significant growth in the theory of dynamic equations on time scales, covering a variety of different problems; see [6, 7, 11] and references therein.

Let $\mathbf{T}$ be a periodic time scale such that $t_0 \in \mathbf{T}$. In this article, we are interested in the analysis of qualitative theory of positive periodic solutions of delay dynamic equations. Motivated by the papers [1]-[5], [9], [10], [12] and the references therein, we consider the following delay dynamic equation

$$x^{\triangle}(t) + p(t)\, x^{\sigma}(t) + q(t)\, x(\tau(t)) = 0,\ t \geq t_0, \tag{1.1}$$

Throughout this paper we assume that $p, q : [t_0, \infty) \cap \mathbf{T} \rightarrow \mathbf{R}$ are rd-continuous, $\tau : \mathbf{T} \rightarrow \mathbf{T}$ is increasing so that the function $x(\tau(t))$ is well defined over $\mathbf{T}$. We also assume that $\tau : [t_0, \infty) \cap \mathbf{T} \rightarrow [0, \infty) \cap \mathbf{T}$ is rd-continuous, $\tau(t) < t$ and $\lim_{t \to \infty} \tau(t) = \infty$. To reach our desired end we have to transform (1.1) into an integral equation and then use Schauder's fixed point theorem to show the existence of positive periodic solutions.

The organization of this paper is as follows. In Section 2, we introduce some notations and definitions, and state some preliminary material needed in later sections. We will state some facts about the exponential function on a time scale as well as the Schauder's fixed point theorem. For details on Schauder theorem we refer the reader to [13]. In Section 3, we establish our main results for positive periodic solutions by applying the Schauder's fixed point theorem. In Section 4, we present two examples to illustrate our results. The results presented in this paper extend the main results in [12].

2. Preliminaries

A time scale is an arbitrary nonempty closed subset of real numbers. The study of dynamic equations on time scales is a fairly new subject, and research in this area is rapidly growing (see [1]-[5], [9], [10], [12] and papers therein). The theory of dynamic equations unifies the theories of differential equations and difference equations. We suppose that the reader is familiar with the basic concepts concerning the calculus on time scales for dynamic equations. Otherwise one can find in Bohner and Peterson books [6, 7, 11] most of the material needed to read this paper. We start by giving some definitions necessary for our work. The notion of periodic time scales is introduced in Kaufmann and Raffoul [9]. The following two definitions are borrowed from [9].

Definition 2.1. *We say that a time scale* $\mathbf{T}$ *is periodic if there exist a* $p > 0$ *such that if* $t \in \mathbf{T}$ *then* $t \pm p \in \mathbf{T}$*. For* $\mathbf{T} \neq \mathbf{R}$*, the smallest positive* p *is called the period of the time scale.*

Example 2.2. *The following time scales are periodic.*

1. $\mathbf{T} = \bigcup_{i=-\infty}^{\infty}[2(i-1)h, 2ih],\ h > 0$ *has period* $p = 2h$.
2. $\mathbf{T} = hZ$ *has period* $p = h$.
3. $\mathbf{T} = \mathbf{R}$.
4. $\mathbf{T} = \{t = k - q^m : k \in Z,\ m \in N_0\}$ *where,* $0 < q < 1$ *has period* $p = 1$.

Remark 2.3 ([9]). *All periodic time scales are unbounded above and below.*

Definition 2.4. *Let* $\mathbf{T} \neq \mathbf{R}$ *be a periodic time scale with period* p*. We say that the function* $f : \mathbf{T} \to \mathbf{R}$ *is periodic with period* ω *if there exists a natural number* n *such that* $\omega = np$*,* $f(t \pm \omega) = f(t)$ *for all* $t \in \mathbf{T}$ *and* ω *is the smallest number such that* $f(t \pm \omega) = f(t)$*.*

If $\mathbf{T} = \mathbf{R}$*, we say that* f *is periodic with period* $\omega > 0$ *if* ω *is the smallest positive number such that* $f(t \pm \omega) = f(t)$ *for all* $t \in \mathbf{T}$.

Remark 2.5 ([9]). *If* $\mathbf{T}$ *is a periodic time scale with period* p*, then* $\sigma(t \pm np) = \sigma(t) \pm np$*. Consequently, the graininess function* μ *satisfies* $\mu(t \pm np) = \sigma(t \pm np) - (t \pm np) = \sigma(t) - t = \mu(t)$ *and so, is a periodic function with period* p*.*

Definition 2.6 ([6]). *A function* $f : \mathbf{T} \to \mathbf{R}$ *is called rd-continuous provided it is continuous at every right-dense point* $t \in \mathbf{T}$ *and its left-sided limits exist, and is finite at every left-dense point* $t \in \mathbf{T}$*. The set of rd-continuous functions* $f : \mathbf{T} \to \mathbf{R}$ *will be denoted by*

$$C_{rd} = C_{rd}(\mathbf{T}) = C_{rd}(\mathbf{T}, \mathbf{R}).$$

Definition 2.7 ([6]). *For $f : \mathbf{T} \to \mathbf{R}$, we define $f^{\Delta}(t)$ to be the number (if it exists) with the property that for any given $\varepsilon > 0$, there exists a neighborhood U of t such that*

$$\left|(f(\sigma(t)) - f(s)) - f^{\Delta}(t)\,(\sigma(t) - s)\right| < \varepsilon\,|\sigma(t) - s| \text{ for all } s \in U.$$

The function $f^{\Delta} : \mathbf{T}^k \to \mathbf{R}$ is called the delta (or Hilger) derivative of f on $\mathbf{T}^k$.

Definition 2.8 ([6]). *A function $p : \mathbf{T} \to \mathbf{R}$ is called regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbf{T}$. The set of all regressive and rd-continuous functions $p : \mathbf{T} \to \mathbf{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbf{T}, \mathbf{R})$. We define the set $\mathcal{R}^+$ of all positively regressive elements of $\mathcal{R}$ by*

$$\mathcal{R}^+ = \mathcal{R}^+(\mathbf{T}, \mathbf{R}) = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0,\ \forall t \in \mathbf{T}\}.$$

Definition 2.9 ([6]). *Let $p \in \mathcal{R}$, then the generalized exponential function e_p is defined as the unique solution of the initial value problem*

$$x^{\Delta}(t) = p(t)x(t),\ x(s) = 1,\ \text{where } s \in \mathbf{T}.$$

An explicit formula for $e_p(t, s)$ is given by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(v)}(p(v))\Delta v\right),\ \text{for all } s, t \in \mathbf{T},$$

with

$$\xi_h(v) = \begin{cases} \frac{\log(1+hv)}{h} & \text{if } h \neq 0, \\ v & \text{if } h = 0, \end{cases}$$

where log *is the principal logarithm function.*

Lemma 2.10 ([6]). *Let $p, q \in \mathcal{R}$. Then*

(i) $e_0(t, s) \equiv 1$ and $e_p(t, t) \equiv 1$,

(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$,

(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t, s)$ where, $\ominus p(t) = -\frac{p(t)}{1+\mu(t)p(t)}$,

(iv) $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$,

(v) $e_p(t, s)e_p(s, r) = e_p(t, r)$,

(vi) $\left(\frac{1}{e_p(\cdot,s)}\right)^{\Delta} = -\frac{p(t)}{e_p^{\sigma}(\cdot,s)}$.

Lemma 2.11 ([1]). *If* $p \in \mathcal{R}^+$, *then*

$$0 < e_p(t,s) \le \exp\left(\int_s^t p(v)\Delta v\right), \ \forall t \in \mathbf{T}.$$

The proof of the main results in the next section is based upon an application of the following Schauder's fixed point theorem.

Theorem 2.12 (Schauder's fixed point theorem [13]). *Let* Ω *be a closed, convex and nonempty subset of a Banach space* X. *Let* $S : \Omega \to \Omega$ *be a continuous mapping such that* $S\Omega$ *is a relatively compact subset of* X. *Then* S *has at least one fixed point in* Ω. *That is there exists an* $x \in \Omega$ *such that* $Sx = x$.

3. The existence of periodic solutions

In this the section we will study existence of positive ω-periodic solution of Eq. (1.1). In the next lemma and theorem we choose $T \in \mathbf{T}$ sufficiently large that $\tau(t) \ge t_0$ for $t \ge T$.

Lemma 3.1. *Suppose that there exists a rd-continuous function* $k : [T,\infty)\cap \mathbf{T} \to (0,\infty)$ *such that*

$$p + qk \in \mathcal{R}^+, \int_t^{t+\omega} \xi_{\mu(s)}\left[\ominus\left(p(s)+q(s)k(s)\right)\right]\Delta s = 0, \ t \ge T. \tag{3.1}$$

Then the function

$$f(t) = \exp\left(\int_T^t \xi_{\mu(s)}\left[\ominus(p(s)+q(s)k(s))\right]\Delta s\right), \ t \ge T,$$

is ω*-periodic.*

Proof. For $t \ge T$ we obtain f(t+ω)
$= \exp\left(\int_T^{t+\omega} \xi_{\mu(s)}\left[\ominus(p(s)+q(s)k(s))\right]\Delta s\right)$
$= \exp\left(\int_T^t \xi_{\mu(s)}\left[\ominus(p(s)+q(s)k(s)\right]\Delta s\right)\exp\left(\int_t^{t+\omega} \xi_{\mu(s)}\left[\ominus(p(s)+q(s)k(s))\right]\Delta s\right)$
$= f(t)$.

Thus the function f is ω-periodic. □

Theorem 3.2. *Suppose that there exists a rd-continuous function $k : [T,\infty)\cap\mathbf{T}\to(0,\infty)$ such that (3.1) holds and*

$$\int_{\sigma(t)}^{\tau(t)} \xi_{\mu(s)}\left[\ominus(p(s)+q(s)k(s))\right]\Delta s=\log(k(t)),\ \tau(t)\ge T. \tag{3.2}$$

Then Eq. (1.1) has a positive ω-periodic solution.

Proof. Let $X=C_{rd}([t_0,\infty)\cap\mathbf{T},\mathbf{R})$ be the Banach space with the norm $\|x\|=\sup_{t\ge t_0}|x(t)|$. With regard to Lemma 3.1 we define

$$\begin{aligned} M&=\max_{t\in[T,\infty)\cap\mathbf{T}}\left\{\exp\left(\int_T^t \xi_{\mu(s)}\left[\ominus(p(s)+q(s)k(s))\right]\Delta s\right)\right\},\\ m&=\min_{t\in[T,\infty)\cap\mathbf{T}}\left\{\exp\left(\int_T^t \xi_{\mu(s)}\left[\ominus(p(s)+q(s)k(s))\right]\Delta s\right)\right\}.\end{aligned} \tag{3.3}$$

We now define a closed, bounded and convex subset Ω of X as follows

$$\begin{aligned}\Omega=\{x\in X: {}&x(t+\omega)=x(t),\ t\ge T,\\ &m\le x(t)\le M,\ t\ge T,\\ &k(t)x^\sigma(t)=x(\tau(t)),\ t\ge T,\\ &x(t)=1,\ t_0\le t\le T\}.\end{aligned}$$

Define the operator $S:\Omega\longrightarrow X$ as follows

$$(Sx)(t)=\begin{cases}\exp\left(\int_T^t \xi_{\mu(s)}\left[\ominus\left(p(s)+q(s)\dfrac{x(\tau(s))}{x^\sigma(s)}\right)\right]\Delta s\right), & t\ge T,\\ 1, & t_0\le t\le T.\end{cases}$$

We will show that for any $x\in\Omega$ we have $Sx\in\Omega$. For every $x\in\Omega$ and $t\ge T$ we get

$$\begin{aligned}(Sx)(t)&=\exp\left(\int_T^t \xi_{\mu(s)}\left[\ominus\left(p(s)+q(s)\tfrac{x(\tau(s))}{x^\sigma(s)}\right)\right]\Delta s\right)\\ &=\exp\left(\int_T^t \xi_{\mu(s)}\left[\ominus(p(s)+q(s)k(s))\right]\Delta s\right)\le M.\end{aligned}$$

Furthermore for $x\in\Omega$ and $t\ge T$ we obtain

$$(Sx)(t)=\exp\left(\int_T^t \xi_{\mu(s)}\left[\ominus(p(s)+q(s)k(s))\right]\Delta s\right)\ge m.$$

For $t \in [t_0, T] \cap \mathbf{T}$ we have $(Sx)(t) = 1$, that is $(Sx)(t) \in \Omega$.

Further for every $x \in \Omega$ and $\tau(t) \geq T$ we get

$$(Sx)(\tau(t)) = \exp\left(\int_T^{\tau(t)} \xi_{\mu(s)}\left[\ominus\left(p(s) + q(s)\frac{x(\tau(s))}{x^\sigma(s)}\right)\right]\Delta s\right)$$

$$= (Sx)^\sigma(t)\exp\left(\int_{\sigma(t)}^{\tau(t)} \xi_{\mu(s)}\left[\ominus\left(p(s) + q(s)\frac{x(\tau(s))}{x^\sigma(s)}\right)\right]\Delta s\right).$$

With regard to (3.2) and (3) for $\tau(t) \geq T$ it follows that

$$(Sx)(\tau(t)) = (Sx)^\sigma(t)\exp\left(\int_{\sigma(t)}^{\tau(t)} \xi_{\mu(s)}\left[\ominus(p(s) + q(s)k(s))\right]\Delta s\right) = k(t)(Sx)^\sigma(t).$$

Finally we will show that for $x \in \Omega$, $t \geq T$ the function Sx is ω-periodic. For $x \in \Omega$, $t \geq T$ and according to (3.1) we have

$$\begin{aligned}
&(Sx)(t+\omega)\\
&= \exp\left(\int_T^{t+\omega} \xi_{\mu(s)}\left[\ominus(p(s) + q(s)k(s))\right]\Delta s\right)\\
&= \exp\left(\int_T^{t} \xi_{\mu(s)}\left[\ominus(p(s) + q(s)k(s))\right]\Delta s\right)\exp\left(\int_t^{t+\omega} \xi_{\mu(s)}\left[\ominus(p(s) + q(s)k(s))\right]\Delta s\right)\\
&= (Sx)(t).
\end{aligned}$$

So Sx is ω-periodic on $[T, \infty) \cap \mathbf{T}$. Thus we have proved that $Sx \in \Omega$ for any $x \in \Omega$.

We now show that S is completely continuous. Let $x_i \in \Omega$ be such that $x_i \longrightarrow x \in \Omega$ as $i \longrightarrow \infty$. For $t \geq T$, we have

$$|(Sx_i)(t) - (Sx)(t)| = \left|\exp\left(\int_T^t \xi_{\mu(s)}\left[\ominus\left(p(s) + q(s)\frac{x_i(\tau(s))}{x_i^\sigma(s)}\right)\right]\Delta s\right)\right.$$

$$\left. - \exp\left(\int_T^t \xi_{\mu(s)}\left[\ominus\left(p(s) + q(s)\frac{x(\tau(s))}{x^\sigma(s)}\right)\right]\Delta s\right)\right|.$$

By applying the Lebesgue dominated convergence theorem we obtain that

$$\lim_{i \longrightarrow \infty} \|Sx_i - Sx\| = 0.$$

For $t \in [t_0, T] \cap \mathbf{T}$ the relation above is also valid. This means that S is continuous.

We now show that $S\Omega$ is relatively compact. It is sufficient to show by the Arzela-Ascoli theorem that the family of function $\{Sx : x \in \Omega\}$ is uniformly bounded and equicontinuous on $[t_0,\infty)\cap\mathbf{T}$. The uniform boundedness follows from the definition of Ω. With regard to (3.3) for $t\geq T$, $x\in\Omega$ we get

$$\begin{aligned}
&\left|(Sx)^{\Delta}(t)\right| \\
&= \left|-\left(p(t)+q(t)\tfrac{x(\tau(t))}{x^{\sigma}(t)}\right)\right| \exp\left(\textstyle\int_T^{\sigma(t)} \xi_{\mu(s)}\left[\ominus\left(p(s)+q(s)\tfrac{x(\tau(s))}{x^{\sigma}(s)}\right)\right]\Delta s\right) \\
&= |p(t)+q(t)k(t)| \exp\left(\textstyle\int_T^{\sigma(t)} \xi_{\mu(s)}\left[\ominus(p(s)+q(s)k(s))\right]\Delta s\right) \\
&\leq M_1,
\end{aligned}$$

For $t\in[t_0,T]\cap\mathbf{T}$, $x\in\Omega$, we have

$$\left|(Sx)^{\Delta}(t)\right| = 0.$$

This shows the equicontinuity of the family $S\Omega$. Hence $S\Omega$ is relatively compact and therefore S is completely continuous. By Theorem 2.12 there is an $x_0\in\Omega$ such that $Sx_0=x_0$. We see that x_0 is a positive ω-periodic solution of Eq. (1.1). The proof is complete. □

4. Two examples

In this section, we give two examples to illustrate the applications of Theorem 3.2.

Example 4.1. *Consider the delay dynamic equation on* $\mathbf{T}$ *with* $\mu(t)\neq 0$,

$$x^{\triangle}(t) - \frac{1}{\mu(t)}x^{\sigma}(t) + \frac{e^{(\cos\sigma(t)-\cos(t))}}{\mu(t)}x(\sigma(t)-2\pi) = 0,\ t\geq 0. \tag{4.1}$$

We take $k(t)=1$. *Then for conditions (3.1), (3.2) and* $\omega=2\pi$ *we obtain*

$\int_t^{t+\omega} \xi_{\mu(s)} [\ominus (p(s) + q(s)k(s))] \Delta s$

$= \int_t^{t+2\pi} \frac{1}{\mu(s)} \log [\ominus (p(s) + q(s)k(s)) \mu(s) + 1] \Delta s$

$= \int_t^{t+2\pi} \frac{1}{\mu(s)} \log \left[\frac{-(p(s)+q(s)k(s))}{1+\mu(s)(p(s)+q(s)k(s))} \mu(s) + 1\right] \Delta s$

$= \int_t^{t+2\pi} \frac{1}{\mu(s)} \log \left[\frac{-(p(s)+q(s))}{1+\mu(s)(p(s)+q(s))} \mu(s) + 1\right] \Delta s$

$= \int_t^{t+2\pi} \frac{1}{\mu(s)} \log \left[\frac{1}{1+\mu(s)(p(s)+q(s))}\right] \Delta s$

$= \int_t^{t+2\pi} -\frac{1}{\mu(s)} \log [1 + \mu(t) (p(s) + q(s))] \Delta s$

$= \int_t^{t+2\pi} -\frac{(\cos\sigma(s)-\cos(s))}{\mu(s)} \Delta s$

$= -\cos s \mid_t^{t+2\pi}$

$= 0,$

and

$\int_{\sigma(t)}^{\tau(t)} \xi_{\mu(s)} [\ominus(p(s) + q(s)k(s))] \Delta s$

$= \int_{\sigma(t)}^{\sigma(t)-2\pi} \xi_{\mu(s)} [\ominus(p(s) + q(s)k(s))] \Delta s = 0, \ t \geq 0.$

All conditions of Theorem 3.2 are satisfied. Thus Eq. (4.1) has a positive $\omega = 2\pi$-periodic solution

$$x(t) = \exp\left(\int_T^t -\frac{(\cos\sigma(s) - \cos(s))}{\mu(s)} \Delta s\right) = e^{\cos(T)-\cos(t)}, \ t \geq T.$$

Example 4.2. *Consider the delay differential equation on $\mathbf{T} = \mathbf{R}$,*

$$x'(t) - (\frac{1}{2}\sin t + e^{-t}) x(t) + e^{-t-\cos t} x(t-\pi) = 0, \ t \geq 0. \tag{4.2}$$

We choose $k(t) = e^{\cos t}$. Then for conditions (3.1), (3.2) and $\omega = 2\pi$ we have

$\int_t^{t+\omega} [p(s) + q(s)k(s)] \, ds = -\frac{1}{2} \int_t^{t+2\pi} \sin(s) ds = 0,$

$\int_{\tau(t)}^t [p(s) + q(s)k(s)] \, ds = -\frac{1}{2} \int_{t-\pi}^t \sin(s) ds = \cos t, \ t \geq 0.$

All conditions of Theorem 3.2 are satisfied. Thus Eq. (4.2) has a positive $\omega = 2\pi$*-periodic solution*

$$x(t) = \exp\left(\int_T^t \left(\frac{1}{2}\sin s\right) ds\right) = e^{\frac{1}{2}(\cos(T)-\cos(t))},$$

for $t \geq T$.

References

[1] M. Adivar and Y. N. Raffoul, Existence of periodic solutions in totally nonlinear delay dynamic equations, Electronic Journal of Qualitative Theory of Differential Equations, No. 1, pp. 1-20, (2009).

[2] A. Ardjouni and A. Djoudi, Existence of positive periodic solutions for nonlinear neutral dynamic equations with variable delay on a time scale, Malaya Journal of Matematik 2 (1), pp. 60-67, (2013).

[3] A. Ardjouni and A. Djoudi, Existence of periodic solutions for nonlinear neutral dynamic equations with functional delay on a time scale, Acta Univ. Palacki. Olomnc., Fac. rer. nat., Mathematica 52, 1, pp. 5-19, (2013).

[4] A. Ardjouni and A. Djoudi, Existence of periodic solutions for nonlinear neutral dynamic equations with variable delay on a time scale, Commun Nonlinear Sci Numer Simulat 17, pp. 3061–3069, (2012).

[5] A. Ardjouni and A. Djoudi, Periodic solutions in totally nonlinear dynamic equations with functional delay on a time scale, Rend. Sem. Mat. Univ. Politec. Torino Vol. 68, 4, pp. 349-359, (2010).

[6] M. Bohner, A. Peterson, Dynamic Equations on Time Scales, An Introduction with Applications, Birkhäuser, Boston, (2001).

[7] M. Bohner, A. Peterson, Advances in Dynamic Equations on Time Scales, Birkhäuser, Boston, (2003).

[8] S. Hilger, Ein Masskettenkalkül mit Anwendung auf Zentrumsmanningfaltigkeiten. PhD thesis, Universität Würzburg, (1988).

[9] E. R. Kaufmann, Y. N. Raffoul, Periodic solutions for a neutral nonlinear dynamical equation on a time scale, J. Math. Anal. Appl. 319, pp. 315-325, (2006).

[10] E. R. Kaufmann and Y. N. Raffoul, Periodicity and stability in neutral nonlinear dynamic equation with functional delay on a time scale, Electronic Journal of Differential Equations, No. 27, pp. 1-12, (2007).

[11] V. Lakshmikantham, S. Sivasundaram, B. Kaymarkcalan, Dynamic Systems on Measure Chains, Kluwer Academic Publishers, Dordrecht, (1996).

[12] R. Olach, Positive periodic solutions of delay differential equations, Applied Mathematics Letters 26, pp. 1141-1145, (2013).

[13] D. R. Smart, Fixed Points Theorems, Cambridge Univ. Press, Cambridge, UK, (1980).

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