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Edge fixed monophonic number of a graph

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Abstract

For an edge xy in a connected graph G of order $p \geq 3$, a set $S \subseteq V(G)$ is an xy-monophonic set of G if each vertex $v \in V(G)$ lies on an $x - u$ monophonic path or a $y - u$ monophonic path for some element u in S. The minimum cardinality of an xy-monophonic set of G is defined as the xy-monophonic number of G, denoted by $m_{xy}(G)$. An xy-monophonic set of cardinality $m_{xy}(G)$ is called a m_{xy}-set of G. We determine bounds for it and find the same for special classes of graphs. It is shown that for any three positive integers r, d and $n \geq 2$ with $2 \leq r \leq d$, there exists a connected graph G with monophonic radius r, monophonic diameter d and $m_{xy}(G) = n$ for some edge xy in G.

Key Words : *Monophonic path, vertex monophonic number, edge fixed monophonic number.*

Mathematics Subject Classification : *05C12.*

1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology we refer to [1, 2]. For vertices x and y in a connected graph G, the *distance* $d(x, y)$ is the length of a shortest $x - y$ path in G. An $x - y$ path of length $d(x, y)$ is called an $x - y$ *geodesic*. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. A vertex v is a *simplicial vertex* if the subgraph induced by its neighbors is complete. A *non-separable* graph is connected, non-trivial, and has no cut-vertices. A *block* of a graph is a maximal non-separable subgraph. A *connected block graph* is a connected graph in which each of its blocks is complete. A *caterpillar* is a tree for which the removal of all the end vertices gives a path.

A *chord* of a path P is an edge joining two non-adjacent vertices of P. A path P is called *monophonic* if it is a chordless path. The *closed intervel* $I_m[x, y]$ consists of all vertices lying on some $x - y$ monophonic of G. For any two vertices u and v in a connected graph G, the *monophonic distance* $d_m(u, v)$ from u to v is defined as the length of a longest $u - v$ monophonic path in G. The *monophonic eccentricity* $e_m(v)$ of a vertex v in G is $e_m(v) = max\{d_m(v, u) : u \in V(G)\}$. The *monophonic radius*, $rad_m(G)$ of G is $rad_m\{G\}$ $= min\{e_m(v) : v \in V(G)\}$ and the monophonic diameter, $diam_m\{G\}$ of G is $diam_m\{G\}$ $= max\{e_m(v) : v \in V(G)\}$. The monophonic distance was introduced in [3] and further studied in [4]. The concept of vertex monophonic number was introduced by Santhakumaran and Titus [5]. A set S of vertices of G is an *x-monophonic set* if each vertex v of G lies on an $x - y$ monophonic path in G for some element y in S. The minimum cardinality of an x-monophonic set of G is defined as the *x-monophonic number* of G and is denoted by $m_x(G)$ or simply m_x. An x-monophonic set of cardinality $m_x(G)$ is called a m_x*-set* of G. The following theorems will be used in the sequel.

Theorem 1.1. *[2] Let v be a vertex of a connected graph G. The following statements are equivalent:*

i) v is a cut-vertex of G.

ii) There exist vertices u and w distinct from v such that v is on every $u - w$ path.

iii) There exists a partition of the set of vertices $V - \{v\}$ into subsets U and W such that for any vertices $u \in U$ and $w \in W$, the vertex v is on every $u - w$ path.

Theorem 1.2. *[2] Every non-trivial connected graph has at least two vertices which are not cut-vertices.*

Theorem 1.3. *[2] Let G be a connected graph with at least three vertices. The following statements are equivalent:*

i) G is a block.

ii) Every two vertices of G lie on a common cycle.

Throughout this paper G denotes a connected graph with at least three vertices.

2. Edge fixed monophonic number

Definition 2.1. *Let $e = xy$ be any edge of a connected graph G of order at least three. A set S of vertices of G is an xy-monophonic set if every vertex of G lies on either an $x - u$ monophonic path or a $y - u$ monophonic path in G for some element u in S. The minimum cardinality of an xy-monophonic set of G is defined as the xy-monophonic number of G and is denoted by $m_{xy}(G)$ or $m_e(G)$. An xy-monophonic set of cardinality $m_{xy}(G)$ is called a m_{xy}-set or m_e-set of G.*

Example 2.2. *For the graph G given in Figure 2.1, the minimum edge fixed monophonic sets and the edge fixed monophonic numbers are given in Table 2.1.*

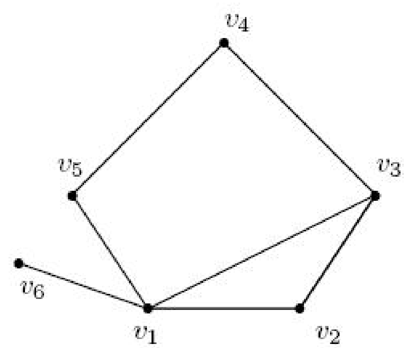

Figure 2.1 : G

Theorem 2.3. *For any edge xy in a connected graph G of order at least three, the vertices x and y do not belong to any minimum xy-monophonic set of G.*

Proof. Suppose that x belongs to a minimum xy-monophonic set, say S, of G. Since G is a connected graph with at least three vertices and xy in an edge, it follows from the definition of an xy-monophonic set that S contains a vertex v different from x and y. Since the vertex x lies on every $x - v$ monophonic path in G, it follows that $T = S - \{x\}$ is an xy-monophonic set of G, which is a contradiction to S a minimum xy-monophonic set of G. Similarly, y does not belong to any minimum xy-monophonic set of G. □

Table 2.1: The Edge Fixed Monophonic Number of a Graph

Edge e	minimum e-monophonic sets	e-monophonic number
v_1v_2	$\{v_4, v_6\}, \{v_5, v_6\}$	2
v_2v_3	$\{v_4, v_6\}, \{v_5, v_6\}$	2
v_3v_4	$\{v_2, v_6\}$	2
v_4v_5	$\{v_2, v_6\}$	2
v_5v_1	$\{v_2, v_6\}$	2
v_1v_6	$\{v_2, v_4\}$	2
v_1v_3	$\{v_2, v_6, v_4\}, \{v_2, v_6, v_5\}$	3

Theorem 2.4. *Let xy be any edge of a connected graph G of order at least three. Then*

i) every simplicial vertex of G other than the vertices x and y (whether x or y is simplicial or not) belongs to every m_{xy}-set.

ii) no cut-vertex of G belongs to any m_{xy}-set.

Proof. (i) By Theorem 2.3, the vertices x and y do not belong to any m_{xy}-set. So, let $u \neq x, y$ be a simplicial vertex of G. Let S be a m_{xy}-set of G such that $u \notin S$. Then u is an internal vertex of either an $x - v$ monophonic path or a $y - v$ monophonic path for some element v in S. Without loss of generality, let P be an $x - v$ monophonic path with u is an internal vertex. Then both the neighbors of u on P are not adjacent and hence u is not a simplicial vertex, which is a contradiction.
(ii) Let v be a cut-vertex of G. Then by Theorem 1.1, there exists a partition of the set of vertices $V - \{v\}$ into subsets U and W such that for any vertex $u \in U$ and $w \in W$, the vertex v lies on every $u - w$ path. Let S be a m_{xy}-set of G. We consider three cases.

Case (i): Both x and y belong to U. Suppose that $S \cap W = \emptyset$. Let $w_1 \in W$. Since S is an xy-monophonic set, there exists an element z in S such that w_1 lies on either an $x - z$ monophonic path or a $y - z$ monophonic path in G. Suppose that w_1 lies on an $x - z$ monophonic path $P : x = z_0, z_1, \ldots, w_1, \ldots, z_n = z$ in G. Then the $x - w_1$ subpath of P and $w_1 - z$ subpath of P both contain v so that P is not a path in G, which is a contradiction. Hence $S \cap W \neq \emptyset$. Let $w_2 \in S \cap W$. Then v is an internal vertex of any $x - w_2$ monophonic path and v is also an internal vertex of any $y - w_2$ monophonic path. If $v \in S$, then let $S' = S - \{v\}$. It is clear that every vertex that lies on an $x - v$ monophonic path also lies on an $x - w_2$ monophonic path. Hence it follows that S' is an xy-monophonic set of G, which is a contradiction to S a minimum xy-monophonic set of G. Thus v does not belong to any minimum xy-monophonic set of G.

Case (ii): Both x and y belong to W. It is simillar to Case (i).

Case (iii): Either $x = v$ or $y = v$. By Theorem 2.3, v does not belong to any m_{xy}-set. □

Corollary 2.5. *Let T be a tree with k end vertices. Then $m_{xy}(T) = k - 1$ or k according as xy is an end edge or cut-edge.*

Proof. This follows from Theorem 2.4. □

Corollary 2.6. *Let $K_{1,n}(n \geq 2)$ be a star. Then $m_{xy}(K_{1,n}) = n - 1$ for any edge xy in $K_{1,n}$.*

Corollary 2.7. *Let G be a complete graph $K_p(p \geq 3)$. Then $m_{xy}(G) = p - 2$ for any edge xy in G.*

Theorem 2.8. *For any edge xy in the cube $Q_n(n \geq 3)$, $m_{xy}(Q_n) = 1$.*

Proof. Let $e = xy$ be an edge in Q_n and let $x = (a_1, a_2, \ldots, a_n)$, where $a_i \in \{0, 1\}$. Let $x' = (a'_1, a'_2, \ldots, a'_n)$ be another vertex of Q_n such that a'_i is the compliment of a_i. Let u be any vertex in Q_n. For convenience, let $u = (a_1, a'_2, a_3, \ldots, a_n)$. Then u lies on an $x - x'$ monophonic path $P : x = (a_1, a_2, \ldots, a_n), (a_1, a'_2, a_3, \ldots, a_n), \ldots, (a'_1, a'_2, \ldots, a'_{n-1}, a_n), (a'_1, a'_2, \ldots, a'_n) = x'$. Hence $\{x'\}$ is an xy-monophonic set of Q_n and so $m_{xy}(Q_n) = 1$. □

Theorem 2.9. *i) For any edge xy in the wheel $W_n = K_1 + C_{n-1}(n \geq 5)$, $m_{xy}(W_n) = 1$.*

ii) For any edge xy in the complete bipartite graph $K_{m,n}(1 \leq m \leq n)$,

$$m_{xy}(K_{m,n}) = \begin{cases} n-1 & \text{if } m = 1 \\ 1 & \text{if } m = 2 \\ 2 & \text{if } m \geq 3. \end{cases}$$

Proof. (i) Let xy be an edge in W_n. Then either x or y is a vertex of C_{n-1}. Let $x \in V(C_{n-1})$ and let z be a non-adjacent vertex of x in C_{n-1}. It is clear that every vertex of W_n lies on an $x - z$ monophonic path. Hence $\{z\}$ is a m_{xy}-set of W_n and so $m_{xy}(W_n) = 1$.

(ii) Let $U = \{u_1, u_2, \ldots, u_m\}$ and $W = \{w_1, w_2, \ldots, w_n\}$ be the vertex subsets of the bipartition of the vertices of $K_{m,n}$. If $m = 1$, then by Corollary 2.6, $m_{xy}(K_{1,n}) = n - 1$ for any edge xy in $K_{1,n}$. If $m = 2$, let e be an edge in $K_{m,n}$, say $e = u_1w_1$. It is clear that every vertex of $K_{m,n}$ lies on an $u_1 - u_2$ monophonic path. Hence $\{u_2\}$ is an e-monophonic set of $K_{m,n}$ and so $m_e(K_{m,n}) = 1$. If $m \geq 3$, then it is clear that no singleton subset of V is an e-monophonic set of $K_{m,n}$ and so $m_e(K_{m,n}) \geq 2$. Without loss of generality, take $e = u_1w_1$. Let $S = \{u_2, w_2\}$. Then every vertex of U lies on a $w_1 - w_2$ monophonic path and every vertex of W lies on a $u_1 - u_2$ monophonic path. Hence S is an e-monophonic set of $K_{m,n}$ and so $m_e(K_{m,n}) = 2$. □

Theorem 2.10. *For any edge xy in a connected graph G of order $p \geq 3$, $1 \leq m_{xy}(G) \leq p - 2$.*

Proof. It is clear from the definition of m_{xy}-set that $m_{xy}(G) \geq 1$. Also, since the vertices x and y do not belong to any m_{xy}-set, it follows that $m_{xy}(G) \leq p - 2$. □

Remark 2.11. *The bounds for $m_{xy}(G)$ in Theorem 2.10 are sharp. If C is any cycle, then $m_{xy}(C) = 1$ for any edge xy in C. For any edge xy in a complete graph K_p $(p \geq 3)$, $m_{xy}(K_p) = p - 2$.*

Now we proceed to characterize graphs for which the upper bound in Theorem 2.10 is attained.

Theorem 2.12. *Let G be a connected graph of order at least 3. Then G is either K_p or $K_{1,p-1}$ if and only if $m_{xy}(G) = p-2$ for every edge xy in G.*

Proof. If $G = K_p$, then by Corollary 2.7, $m_{xy}(G) = p-2$ for every edge xy in G. If $G = K_{1,p-1}$, then by Corollary 2.6, $m_{xy}(G) = p-2$ for any edge xy in G. Conversely, suppose that $m_{xy}(G) = p-2$ for every edge xy in G. By Theorem 1.2, G has at least two vertices which are not cut-vertices. Let xy be an edge of G with x is not a cut-vertex. If G has two or more cut-vertices, then by Theorem 2.4(ii), $m_{xy}(G) \leq p-3$, which is a contradiction. Thus the number of cut-vertices k of G is at most one.

Case (i) $k = 0$. Then the graph G is a block. Now we claim that G is complete. If G is not complete, then there exist two vertices x and y in G such that $d(x,y) \geq 2$. By Theorem 1.3, x and y lie on a common cycle and hence x and y lie on a smallest cycle $C : x, x_1, x_2, \ldots, y, \ldots, x_n, x$ of length at least 4. Then $(V(G) - V(C)) \cup \{y\}$ is an xx_1-monophonic set of G and so $m_{xx_1}(G) \leq p-3$, which is a contradiction. Hence G is the complete graph.

Case (ii) $k = 1$. Let x be the cut-vertex of G. If $p = 3$, then $G = P_3$, a star with three vertices. If $p \geq 4$, we claim that $G = K_{1,p-1}$. It is enough to prove that degree of every vertex other than x is one. Suppose that there exists a vertex, say y, with $deg\ y \geq 2$. Let $z \neq x$ be an adjacent vertex of y in G. Let $e = yz$. Since the vertices y and z do not lie on any minimum yz-monophonic set of G and by Theorem 2.4(ii), we have $m_{yz}(G) \leq p-3$, which is a contradiction. Thus every vertex of G other than x is of degree one. Hence G is a star. □

Theorem 2.13. *For any edge xy in a connected graph G, every x-monophonic set of G is an xy-monophonic set of G.*

Proof. Let S be an x-monophonic set of G. Then every vertex of G lies on an $x-z$ monophonic path for some z in S. It follows that S is an xy-monophonic set of G. □

Corollary 2.14. *For any edge xy in a connected graph G, $m_{xy}(G) \leq min\{m_x(G), m_y(G)\}$.*

Theorem 2.15. *For every pair a, b of integers with $1 \leq a \leq b$, there is a connected graph G with $m_{xy}(G) = a$ and $m_x(G) = b$ for some edge xy in G.*

Proof. Let $C_4 : x, y, z, u, x$ be a cycle of order 4. Add $b-1$ new vertices $v_1, v_2, \ldots, v_{a-1}, w_1, w_2, \ldots, w_{b-a}$ and joining each $v_i (1 \leq i \leq a-1)$ to x and joining each $w_j (1 \leq j \leq b-a)$ to the vertices y and u, thereby producing the graph G given in Figure 2.2. Let $S = \{v_1, v_2, \ldots, v_{a-1}\}$ be the set of all simplicial vertices of G. Since S is not an xy-monophonic set, it follows from Theorem 2.4(i) that $m_{xy}(G) \geq a$. On the other hand, $S_1 = S \cup \{u\}$

is an xy-monophonic set of G and so $m_{xy}(G) = |S_1| = a$. Clearly, $S_2 = \{v_1, v_2, \ldots, v_{a-1}, z, w_1, w_2, \ldots, w_{b-a}\}$ is the unique x-monophonic set of G and so $m_x(G) = |S_2| = b$.

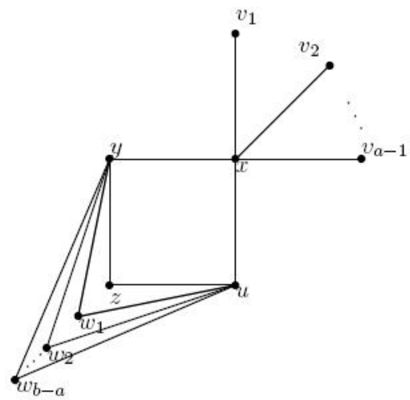

Figure 2.2 : G

We have seen that if G is a connected graph of order $p \geq 3$, then $1 \leq m_{xy}(G) \leq p-2$ for any edge xy in G. In the following theorem we give an improved upper bound for the edge fixed monophonic number of a tree in terms of its order and monophonic diameter.

Theorem 2.16. *If T is a tree of order p and monophonic diameter d_m, then $m_{xy}(T) \leq p - d_m + 1$ for any edge xy in T.*

Proof. Let $P : v_0, v_1, v_2, \ldots, v_{d_m}$ be a monophonic path of length d_m. Now, let $S = V(G) - \{v_1, v_2, \ldots, v_{d_m-1}\}$. If e is an internal edge of P, then clearly S is an e-monophonic set of T so that $m_e(T) \leq |S| = p - d_m + 1$. If e is an end edge of P, say $e = v_0v_1$, then $S_1 = S - \{v_0\}$ is an e-monophonic set of T so that $m_e(T) \leq |S_1| = p - d_m$. If $e = xy$ is an edge lies out side P, then $S_2 = S - \{x, y\}$ is an e-monophonic set of T so that $m_e(T) \leq |S_2| = p - d_m$. Hence for any edge xy in T, $m_{xy}(T) \leq p - d_m + 1$. □

Remark 2.17. *The bound in Theorem 2.16 is not true for any graph. For example, consider the graph G given in Figure 2.3. Here $p = 7$, $d_m(G) = 4$, $m_e(G) = 5$ and $p - d_m + 1 = 4$. Hence $m_e(G) > p - d_m + 1$.*

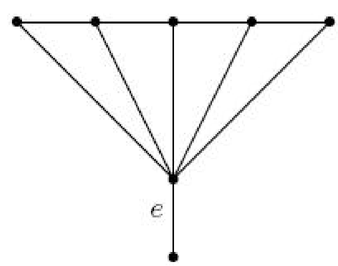

Figure 2.3 : G

Theorem 2.18. *For any edge xy in a non-trivial tree T of order p and monophonic diameter d_m, $m_{xy}(T) = p - d_m$ or $p - d_m + 1$ if and only if T is a caterpillar.*

Proof. Let T be any non-trivial tree. Let $P : v_0, v_1, \ldots, v_d$ be a monophonic path of length d_m. Let k be the number of end vertices of T and let l be the number of internal vertices of T other than $v_1, v_2, \ldots, v_{d-1}$. Then $d_m - 1 + l + k = p$. By Corollary 2.5, $m_{xy}(T) = k$ or $k-1$ for any edge xy in T and so $m_{xy}(T) = p - d_m - l + 1$ or $p - d_m - l$ for any edge xy in T. Hence $m_{xy}(T) = p - d_m + 1$ or $p - d_m$ for any edge xy in T if and only if $l = 0$, if and only if all the internal vertices of T lie on the monophonic path P, if and only if T is a caterpillar.

For any connected graph G, $rad_m(G) \leq diam_m(G)$. It is shown in [3] that every two positive integers a and b with $a \leq b$ are realizable as the monophonic radius and monophonic diameter, respectively, of some connected graph. This result can be extended so that the edge fixed monophonic number can be prescribed. □

Theorem 2.19. *For positive integers r, d and $n \geq 2$ with $2 \leq r \leq d$, there exists a connected graph G with $rad_m(G) = r$, $diam_m(G) = d$ and $m_{xy}(G) = n$ for some edge xy in G.*

Proof. **Case (i)** $2 \leq r = d$. Let $C_{r+2} : v_1, v_2, \ldots, v_{r+2}, v_1$ be the cycle of order $r+2$. Let G be the graph obtained from C_{r+2} by adding n vertices $u_1, u_2, \ldots, u_n$ and joining each vertex u_i $(1 \leq i \leq n)$ to both v_2 and v_{r+2}, and also adding the edge v_1u_1. The graph G is shown in Figure 2.4. It is easily verified that the monophonic eccentricity of each vertex of G is r and so $rad_m(G) = diam_m(G) = r$. Also, for the edge v_1u_1, it is clear that $S = \{v_{r+1}, u_2, \ldots, u_n\}$ is a minimum xy-monophonic set of G and so $m_{xy}(G) = n$.

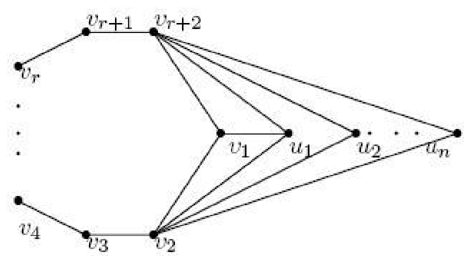

Figure 2.4 : G

Case (ii) $2 \leq r < d \leq 2r$. Let $C_{r+2} = v_1, v_2, \ldots, v_{r+2}, v_1$ be the cycle of order $r+2$ and let $P_{d-r+1} : u_0, u_1, \ldots, u_{d-r}$ be a path of order $d - r + 1$. Let H be the graph obtained from C_{r+2} and P_{d-r+1} by identifying v_1 in C_{r+2} and u_0 in P_{d-r+1}. Let G be the graph obtained from H by adding $n-1$ new vertices $w_1, w_2, \ldots, w_{n-1}$ and joining each $w_i (1 \leq i \leq n-1)$ with u_{d-r-1}. The graph G is shown in Figure 2.5. It is easily verified that $r \leq e_m(x) \leq d$ for any vertex x in G, $e_m(v_1) = r$ and $e_m(v_3) = d$. Thus $rad_m(G) = r$ and $diam_m(G) = d$. For the edge $e = u_{d-r-1}u_{d-r}$, $S = \{w_1, w_2, \ldots, w_{n-1}, v_3\}$ is a minimum e-monophonic set of G and so $m_e(G) = n$.

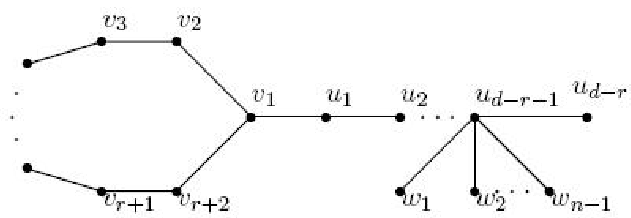

Figure 2.5 : G

Case (iii) $d > 2r$. Let $P_{2r-1} : v_1, v_2, \ldots, v_{2r-1}$ be a path of order $2r-1$. Let G be the graph obtained from the wheel $W = K_1 + C_{d+2}$ and the complete graph K_n by identifying the vertex v_1 of P_{2r-1} with the central vertex of W, and identifying the vertex v_{2r-1} of P_{2r-1} with a vertex of K_n. The graph G is shown in Figure 2.6. Since $d > 2r$, we have $e_m(x) = d$ for any vertex $x \in V(C_{d+2})$. Also, $e_m(x) = 2r$ for any vertex $x \in V(K_n) - v_{2r-1}$; $r \le e_m(x) \le 2r-1$ for any vertex $x \in V(P_{2r-1})$; and $e_m(x) = r$ for the central vertex x of P_{2r-1}. Thus $rad_m(G) = r$ and $diam_m(G) = d$.

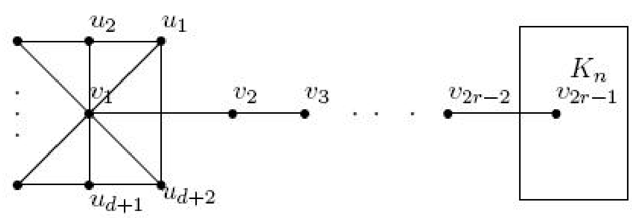

Figure 2.6 : G

Let $S = V(K_n) - \{v_{2r-1}\}$ be the set of all simplicial vertices of G. Then by Theorem 2.4(i), every m_e-set contains S for the edge $e = u_1u_2$. It is clear that S is not an e-monophonic set of G and so $m_e(G) > |S| = n-1$. Since $S' = S \cup \{u_{d+1}\}$ is an e-monophonic set of G, we have $m_e(G) = n$. □

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