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GENERALIZED ULAM-HYERS STABILITIES OF QUARTIC DERIVATIONS ON BANACH ALGEBRAS

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Abstract

Let $\mathcal{A}$,$\mathcal{B}$ be two rings. A mapping $\delta : \mathcal{A} \to \mathcal{B}$ is called quartic derivation, if δ is a quartic function satisfies $\delta(ab) = a^4\delta(b) + \delta(a)b^4$ for all $a, b \in \mathcal{A}$. The main purpose of this paper to prove the generalized Hyers–Ulam–Rassias stability of the quartic derivations on Banach algebras.

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1. Introduction

The study of stability problems as just mentioned originated from a famous talk given by S.M. Ulam [65] in 1940: *Under what condition does there exists a homomorphism near an approximate homomorphism?* In 1941, D. H. Hyers [28] gave the first affirmative answer to the question of Ulam for Banach spaces. Let $f : E \longrightarrow E'$ be a mapping between Banach spaces such that

$$\|f(x+y) - f(x) - f(y)\| \leq \delta$$

for all $x, y \in E$, and for some $\delta > 0$. Then there exists a unique additive mapping $T : E \longrightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \delta$$

for all $x \in E$. Moreover if $f(tx)$ is continuous in $t \in \mathbf{R}$ for each fixed $x \in E$, then T is linear. Finally in 1978, Th. M. Rassias [60] proved the following theorem.

Theorem 1.1. *Let $f : E \longrightarrow E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $T : E \longrightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.2}$$

for all $x \in E$. If $p < 0$ then inequality (1.1) holds for all $x, y \neq 0$, and (1.2) for $x \neq 0$. Also, if the function $t \mapsto f(tx)$ from $\mathbf{R}$ into E' is continuous in real t for each fixed $x \in E$, then T is linear.

In 1991, Z. Gajda [20] answered the question for the case $p > 1$, which was raised by Rassias. This new concept is known as Hyers–Ulam–Rassias stability of functional equations.

In 1982–1994, J.M. Rassias (see [46]–[53]) solved the Ulam problem for different mappings and for many Euler–Lagrange type quadratic mappings, by involving a product of different powers of norms. In addition, J.M. Rassias considered the mixed product–sum of powers of norms control function

[61]. This concept is known as Ulam–Gavrua–Rassias stability of functional equations. For more details about the results concerning such problems and mixed product-sum stability (JMRassias Stability) the reader is referred to [1, 5, 6, 7, 8, 17, 19, 22, 24, 25, 26, 27, 30, 32, 34, 36, 37, 43, 45, 54, 55] and [56].

In 1994, a generalization of the Rassias, theorem was obtained by Găvruta as follows [21] (see also [23], [31]).
Suppose (G,+) is an abelian group, E is a Banach space, and that the so-called admissible control function $\varphi : G \times G \to \mathbf{R}$ satisfies

$$\tilde{\varphi}(x,y) := 2^{-1} \sum_{n=0}^{\infty} 2^{-n} \varphi(2^n x, 2^n y) < \infty$$

for all $x, y \in G$. If $f : G \to E$ is a mapping with

$$\|f(x+y) - f(x) - f(y)\| \leq \varphi(x,y)$$

for all $x, y \in G$, then there exists a unique mapping $T : G \to E$ such that $T(x+y) = T(x) + T(y)$ and $\|f(x) - T(x)\| \leq \tilde{\varphi}(x,x)$ for all $x, y \in G$.
In [40], Won-Gil Park and Jea Hyeong Bae, considered the following functional equation:

$$f(2x+y) + f(2x-y) = 4(f(x+y) + f(x-y)) + 24f(x) - 6f(y). \tag{1.3}$$

In fact they proved that a function f between real vector spaces X and Y is a solution of (1.3) if and only if there exists a unique symmetric multi-additive function $B : X \times X \times X \times X \longrightarrow Y$ such that $f(x) = B(x,x,x,x)$ for all $x \in X$. It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (1.3), which is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic function (see also [2]).
Let $\mathcal{A}$ be an algebra over the real or complex field $\mathbf{F}$ and $\mathcal{X}$ a left $\mathcal{A}$-module (respectively $\mathcal{A}$-bimodule). An additive map $\delta : \mathcal{A} \to \mathcal{X}$ said to be a module left derivation (respectively module derivation) if $\delta(xy) = x.\delta(y) + y.\delta(x)$ (respectively $\delta(xy) = x.\delta(y) + \delta(x).y$) holds for all $x, y \in \mathcal{A}$ where . denotes the module multiplication on $\mathcal{X}$. Since $\mathcal{A}$ is a left $\mathcal{A}$-module (respectively $\mathcal{A}$-bimodule) with the product of $\mathcal{A}$ giving the module multiplication (respectively two module multiplications), the module left derivation (respectively module derivation) $\delta : \mathcal{A} \to \mathcal{A}$ is said to be a ring left derivation (respectively ring derivation) on $\mathcal{A}$. Furthermore, if the identity $\delta(kx) = k\delta(x)$ holds for all $k \in \mathbf{F}$ and all $x \in \mathcal{A}$, then δ is a linear left derivation (respectively linear derivation).

Let us introduce the background of our investigation.
Recently, T. Miura et al. [35] considered the stability of ring derivations on Banach algebras: Under suitable conditions, every approximate ring derivation f on a Banach algebra $\mathcal{A}$ is an exact ring derivation. In particular, if $\mathcal{A}$ is a commutative semisimple Banach algebra with the maximal ideal space without isolated points, then f is identically zero. The first stability result concerning derivations between operator algebras was obtained by P. Šemrl [62] (see also [4]–[18] and [38]–[44]).
In this paper, we investigate the generalized Hyers–Ulam–Rassias stability of quartic derivations from a Banach algebra into its Banach modules.

2. Main result

In this section, we assume that $\mathcal{A}$ is a commutative Banach algebra and $\mathcal{X}$ a Banach $\mathcal{A}$-module.

Definition 2.1. *A mapping $\delta : \mathcal{A} \to \mathcal{X}$ is called a quartic derivation if δ is a quartic function satisfies $\delta(ab) = \delta(a)b^4 + a^4\delta(b)$ for all $a, b \in \mathcal{A}$.*

Example 2.2. *We take*

$$\mathcal{T} = \begin{bmatrix} 0 & \mathcal{A} & \mathcal{A} & \mathcal{A} & \mathcal{A} \\ 0 & 0 & \mathcal{A} & \mathcal{A} & \mathcal{A} \\ 0 & 0 & 0 & \mathcal{A} & \mathcal{A} \\ 0 & 0 & 0 & 0 & \mathcal{A} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then $\mathcal{T}$ is a Banach algebra equipped with the usual matrix-like operations and the following norm:

$$\left\| \begin{bmatrix} 0 & a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & a_5 & a_6 & a_7 \\ 0 & 0 & 0 & a_8 & a_9 \\ 0 & 0 & 0 & 0 & a_{10} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\| = \sum_{i=1}^{10} \|a_i\| \qquad (a_i \in \mathcal{A}).$$

It is known that

$$\mathcal{T}^* = \begin{bmatrix} 0 & \mathcal{A}^* & \mathcal{A}^* & \mathcal{A}^* & \mathcal{A}^* \\ 0 & 0 & \mathcal{A}^* & \mathcal{A}^* & \mathcal{A}^* \\ 0 & 0 & 0 & \mathcal{A}^* & \mathcal{A}^* \\ 0 & 0 & 0 & 0 & \mathcal{A}^* \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is the dual of $\mathcal{T}$ under the following norm

$$\left\| \begin{bmatrix} 0 & f_1 & f_2 & f_3 & f_4 \\ 0 & 0 & f_5 & f_6 & f_7 \\ 0 & 0 & 0 & f_8 & f_9 \\ 0 & 0 & 0 & 0 & f_{10} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\| = \max\{\|f_i\|; \quad f_i \in \mathcal{T}^*(i = 1, 2, ..., 10)\}.$$

Let the left module action of $\mathcal{T}$ on $\mathcal{T}^*$ be trivial and let the right module action of $\mathcal{T}$ on $\mathcal{T}^*$ is defined as follows:

$$\left\langle \begin{bmatrix} 0 & f_1 & f_2 & f_3 & f_4 \\ 0 & 0 & f_5 & f_6 & f_7 \\ 0 & 0 & 0 & f_8 & f_9 \\ 0 & 0 & 0 & 0 & f_{10} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & a_5 & a_6 & a_7 \\ 0 & 0 & 0 & a_8 & a_9 \\ 0 & 0 & 0 & 0 & a_{10} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & b_1 & b_2 & b_3 & b_4 \\ 0 & 0 & b_5 & b_6 & b_7 \\ 0 & 0 & 0 & b_8 & b_9 \\ 0 & 0 & 0 & 0 & b_{10} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\rangle$$

$$= \sum_{i=1}^{10} f_i(a_i b_i)$$

for all $f_i \in \mathcal{A}^*$, $a_i, b_i \in \mathcal{A}(i = 1, ..., 10)$. Then $\mathcal{T}^*$ is a Banach $\mathcal{T}$-module. Let

$$\begin{bmatrix} 0 & f_1 & f_2 & f_3 & f_4 \\ 0 & 0 & f_5 & f_6 & f_7 \\ 0 & 0 & 0 & f_8 & f_9 \\ 0 & 0 & 0 & 0 & f_{10} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathcal{T}^*.$$

We define $\delta : \mathcal{T} \to \mathcal{T}^*$ by

$$\delta\left(\begin{bmatrix} 0 & a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & a_5 & a_6 & a_7 \\ 0 & 0 & 0 & a_8 & a_9 \\ 0 & 0 & 0 & 0 & a_{10} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right) = \left[\begin{bmatrix} 0 & f_1 & f_2 & f_3 & f_4 \\ 0 & 0 & f_5 & f_6 & f_7 \\ 0 & 0 & 0 & f_8 & f_9 \\ 0 & 0 & 0 & 0 & f_{10} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & a_1a_2 & a_3a_4 & a_5a_6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right].$$

Then we can see that δ is a quartic derivation from $\mathcal{T}$ into $\mathcal{T}^*$.

Now, we investigate the generalized Hyers–Ulam–Rassias stability of quartic derivations from $\mathcal{A}$ into $\mathcal{X}$. For convenience, we use the following abbreviation for a given mapping $f : \mathcal{A} \to \mathcal{X}$;

$$D_f(x, y) = f(2x + y) + f(2x - y) - 4[f(x + y) + f(x - y)] - 24f(x) + 6f(y)$$

for all $x, y \in \mathcal{A}$.

Theorem 2.3. *Let $f : \mathcal{A} \to \mathcal{X}$ with $f(0) = 0$ be a mapping for which there exists function $\varphi : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to [0, \infty)$ such that*

$$\|D_f(x, y) + f(zt) - z^4 f(t) - f(z)t^4\| \leq \varphi(x, y, z, t), \tag{2.1}$$

$$\tilde{\varphi}(x) := \sum_{i=0}^{\infty} \frac{1}{16^i} \varphi(2^i x, 0, 0, 0) < \infty, \tag{2.2}$$

$$\lim_{i\to\infty} \frac{1}{16^i} \varphi(2^i x, 2^i y, 2^i z, 2^i t) = 0 \tag{2.3}$$

for all $x, y, z, t \in \mathcal{A}$. Then there exists a unique quartic derivation $\delta : \mathcal{A} \to \mathcal{X}$ such that

$$\|\delta(x) - f(x)\| \leq \frac{1}{32} \tilde{\varphi}(x) \tag{2.4}$$

for all $x \in \mathcal{A}$.

Proof. Letting $z = t = y = 0$ in (2.1), we get

$$\|\frac{1}{16} f(2x) - f(x)\| \leq \frac{1}{32} \varphi(x, 0, 0, 0) \tag{2.5}$$

for all $x \in \mathcal{A}$. By induction, we have

$$\|\frac{1}{16^n} f(2^n x) - f(x)\| \leq \frac{1}{32} \sum_{i=0}^{n-1} \frac{1}{16^i} \varphi(2^i x, 0, 0, 0) \tag{2.6}$$

for all $x \in \mathcal{A}$. In order to show that functions $\delta_n(x) = \frac{1}{16^n} f(2^n x)$ form a Convergent sequence, we used Cauchy convergence criterion. In deed, replace x by $2^m x$ in (2.6) and result divide by 16^m, where m is an arbitrary positive integer, we find that

$$\|\frac{1}{16^{n+m}} f(2^{n+m} x) - \frac{1}{16^m} f(2^m x)\| \leq \frac{1}{32} \sum_{i=m}^{m+n-1} \frac{1}{16^i} \varphi(2^i x, 0, 0, 0) \tag{2.7}$$

for all $x \in \mathcal{A}$. By (2.2) and since $\mathcal{X}$ is complete then by $n \to \infty$, $\lim_{n\to\infty} \delta_n(x)$ exists for all $x \in \mathcal{A}$.

Let $m = 0$ and $n \to \infty$ in (2.7), we have

$$\|\delta(x) - f(x)\| \leq \frac{1}{32} \sum_{i=0}^{\infty} \frac{1}{16^i} \varphi(2^i x, 0, 0, 0) = \frac{1}{32} \tilde{\varphi}(x)$$

such that δ is defined $\delta : \mathcal{A} \to \mathcal{X}$ by $\delta(x) = \lim_{n\to\infty} \frac{1}{16^n} f(2^n x)$ for all $x \in \mathcal{A}$. Letting $z = t = 0$ and replacing x, y by $2^n x, 2^n y$, respectively, in the inequality (2.1), we get

$$\|D_f(2^n x, 2^n y)\| \leq \varphi(2^n x, 2^n y, 0, 0)$$

for all $x, y \in \mathcal{A}$, that is,

$$\|\frac{1}{16^n} D_f(2^n x, 2^n y)\| \leq \frac{1}{16^n} \varphi(2^n x, 2^n y, 0, 0)$$

for all $x, y \in \mathcal{A}$. Passing the limit $n \to \infty$, we have

$$D_\delta(x, y) = 0$$

for all $x, y \in \mathcal{A}$. Hence δ is a quartic functional equation. On the other hand, letting $x = y = 0$ and replacing z, t by $2^n z, 2^n t$, respectively, in (2.1), we obtain

$$\|f(2^{2n} zt) - 16^n z f(2^n t) - f(2^n z) 16^n t\| \leq \varphi(0, 0, 2^n z, 2^n t)$$

for all $z, t \in \mathcal{A}$. Hence

$$\|\frac{1}{16^{2n}} f(2^{2n} zt) - \frac{1}{16^n} z f(2^n t) - f(2^n z) \frac{1}{16^n} t\| \leq \frac{1}{16^{2n}} \varphi(0, 0, 2^n z, 2^n t)$$

for all $z, t \in \mathcal{A}$. Passing the limit $n \to \infty$, we obtain

$$\delta(zt) = z^4 \delta(t) + \delta(z) t^4$$

for all $z, t \in \mathcal{A}$.

Now, suppose there exists a function $\delta' : \mathcal{A} \to \mathcal{X}$ with

$$D_{\delta'}(x, y) = 0$$

for all $x, y \in \mathcal{A}$ and

$$\|\delta'(x) - f(x)\| \leq \frac{1}{32} \tilde{\varphi}(x)$$

for all $x \in \mathcal{A}$.

We have $\|\delta(x)-\delta'(x)\| = \frac{1}{16^n}\|\delta(2^nx)-\delta'(2^nx)\| = \frac{1}{16^n}(\|\delta(2^nx)-f(2^nx)\|+\|\delta'(2^nx)-f(2^nx)\|) \leq \frac{1}{16}\sum_{i=n}^{\infty}\frac{1}{16^i}\varphi(2^ix,0)$ for all $x \in \mathcal{A}$. Passing the limit $n \to \infty$, we obtain $\delta(x) = \delta'(x)$ for all $x \in \mathcal{A}$. □

Now, we establish the Ulam–Gavruta–Rassias stability of quadratic derivations as follows:

Corollary 2.4. *Let $p > 0, q_j > 0, (j = 1,2,3,4)$ and θ be positive real numbers with*

$$Max\{p, \sum_{j=1}^{4} q_j\} < 4.$$

If $f : \mathcal{A} \to \mathcal{X}$ with $f(0) = 0$ is a mapping such that

$$\|D_f(x,y)+f(zt)-z^4f(t)-f(z)t^4\|$$

$$\leq \theta(\|x\|^p+\|y\|^p+\|z\|^p+\|t\|^p+\|x\|^{q_1}\|y\|^{q_2}\|z\|^{q_3}\|t\|^{q_4})$$

for all $x,y,z,t \in \mathcal{A}$, then there is a unique quartic derivation $\delta : \mathcal{A} \to \mathcal{X}$ such that

$$\|\delta(x)-f(x)\| \leq \frac{\theta}{32-2^{p+1}}\|x\|^p$$

for all $x \in \mathcal{A}$.

Proof. The proof follows from Theorem 2.1 taking

$$\varphi(x,y,z,t) := \theta(\|x\|^p+\|y\|^p+\|z\|^p+\|t\|^p+\|x\|^{q_1}\|y\|^{q_2}\|z\|^{q_3}\|t\|^{q_4})$$

for all $x,y,z,t \in \mathcal{A}$. □

Moreover, we investigate the superstability of quartic derivations as follows:

Corollary 2.5. *Let $q_j > 0, (j = 1,2,3,4)$ with $\sum_{j=1}^{4} q_j < 4$, and θ be positive real numbers. If $f : \mathcal{A} \to \mathcal{X}$ with $f(0) = 0$ is a mapping*

$$\|D_f(x,y)+f(zt)-z^4f(t)-f(z)t^4\| \leq \theta(\|x\|^{q_1}\|y\|^{q_2}\|z\|^{q_3}\|t\|^{q_4})$$

for all $x,y,z,t \in \mathcal{A}$, then f is a quartic derivation.

Proof. It follows from Theorem 2.1 by putting

$$\varphi(x,y,z,t) := \theta(\|x\|^{q_1}\|y\|^{q_2}\|z\|^{q_3}\|t\|^{q_4})$$

for all $x,y,z,t \in \mathcal{A}$.

□

Theorem 2.6. *Let $p_1+p_2 < 4, q_1+q_2 < 8$ and θ be positive real numbers. If $f : \mathcal{A} \to \mathcal{X}$ is a mapping*

$$\|D_f(x,y) + f(zt) - z^4 f(t) - f(z)t^4\| \leq \theta(\|x\|^{p_1}\|y\|^{p_2} + \|z\|^{q_1}\|t\|^{q_2}) \quad (2.8)$$

for all $x,y,z,t \in \mathcal{A}$, then there is a unique quartic derivation $\delta : \mathcal{A} \to \mathcal{X}$ such that

$$\|\delta(x) - f(x)\| \leq \frac{\theta}{3^4 - 3^{p_1+p_2}}\|x\|^{p_1+p_2} \quad (2.9)$$

for all $x \in \mathcal{A}$.

Proof. In the inequality (2.8), let $y = x = z = t = 0$, then $23\|f(0)\| \leq 0$. Hence $f(0) = 0$. Letting $y = z = t = 0$ in (2.8), we see that $2f(x) = 2^4 f(x)$ for all $x \in \mathcal{A}$. In the inequality (2.8), put $z = t = 0$ and replace y with x. Then we obtain

$$\|f(3x) - 81f(x)\| \leq \theta\|x\|^{p_1+p_2} \quad (2.10)$$

for all $x \in \mathcal{A}$. Hence

$$\|\frac{f(3x)}{81} - f(x)\| \leq \frac{\theta}{81}\|x\|^{p_1+p_2} \quad (2.11)$$

for all $x \in \mathcal{A}$. By using the induction, we can get that

$$\|\frac{f(3^n x)}{81^n} - f(x)\| \leq \frac{\theta\|x\|^{p_1+p_2}}{81}\sum_{i=0}^{n-1}\frac{3^{i(p_1+p_2)}}{81^i} \quad (2.12)$$

for all $x \in \mathcal{A}$. It follows from $p_1 + p_2 < 4$ that the sequence $\{\frac{1}{81^n}f(3^n x)\}$ is Cauchy sequence and so it is convergent since $\mathcal{X}$ is complete. Thus we can define a function $\delta : \mathcal{A} \to \mathcal{X}$ given by

$$\delta(x) := \lim_{n\to\infty}\frac{1}{81^n}f(3^n x) \quad (2.13)$$

for all $x \in \mathcal{A}$. In (2.12), passing the limit $n \to \infty$, we obtain the inequality (2.9). The proof of the uniqueness of δ, is similar to the proof of Theorem 2.1. □

Theorem 2.7. *Let $p_1+p_2>4, q_1+q_2>8$ and θ be positive real numbers. If $f:\mathcal{A}\to\mathcal{X}$ is a mapping satisfying* (2.8), *then there is a unique quartic derivation $\delta:\mathcal{A}\to\mathcal{X}$ such that*

$$\|f(x)-\delta(x)\|\leq\frac{\theta 3^{-(p_1+p_2)}}{1-3^{4-(p_1+p_2)}}\|x\|^{p_1+p_2} \tag{2.14}$$

for all $x\in\mathcal{A}$.

Proof. It follows from (2.10) that

$$\|f(x)-81f(\frac{x}{3})\|\leq\frac{\theta}{3^{p_1+p_2}}\|x\|^{p_1+p_2} \tag{2.15}$$

for all $x\in\mathcal{X}$. By using the induction, we can get that

$$\|f(x)-81^n f(\frac{x}{3^n})\|\leq\frac{\theta\|x\|^{p_1+p_2}}{81}\sum_{i=1}^{n}\frac{81^i}{3^{i(p_1+p_2)}} \tag{2.16}$$

for all $x\in\mathcal{A}$. It follows from $p_1+p_2>4$ that the sequence $81^n f(\frac{x}{3^n})\}$ is Cauchy sequence and so it is convergent since $\mathcal{X}$ is complete. Thus we can define a function $\delta:\mathcal{A}\to\mathcal{X}$ given by

$$\delta(x):=\lim_{n\to\infty}81^n f(\frac{x}{3^n})$$

for all $x\in\mathcal{A}$. The rest of the proof is similar to the proof of Theorem 2.3. $\square$

Theorem 2.8. *Let $f:\mathcal{A}\to\mathcal{X}$ with $f(0)=0$ be a mapping for which there exists function $\varphi:\mathcal{A}\times\mathcal{A}\times\mathcal{A}\times\mathcal{A}\to[0,\infty)$ such that*

$$\|D_f(x,y)+f(zt)-z^4f(t)-f(z)t^4\|\leq\varphi(x,y,z,t), \tag{2.17}$$

$$\tilde{\varphi}(x):=\sum_{i=1}^{\infty}16^i\varphi(2^{-i}x,0,0,0)<\infty, \tag{2.18}$$

$$\lim_{i\to\infty}16^{2i}\varphi(2^{-i}x,2^{-i}y,2^{-i}z,2^{-i}t)=0 \tag{2.19}$$

for all $x,y,z,t\in\mathcal{A}$. Then there exists a unique quartic derivation $\delta:\mathcal{A}\to\mathcal{X}$ such that

$$\|f(x)-\delta(x)\|\leq\frac{1}{32}\tilde{\varphi}(x) \tag{2.20}$$

for all $x\in\mathcal{A}$.

Proof. It follows from (2.5) that

$$\|f(x) - 16f(2^{-1}x)\| \leq 2^{-1}\varphi(2^{-1}x, 0, 0, 0) \tag{2.21}$$

for all $x \in \mathcal{A}$. In (2.21), multiply the both sides by 16 and replace x with $2^{-1}x$, we have

$$\|16f(2^{-1}x) - 16^2 f(2^{-2}x)\| \leq 2^{-1}16\varphi(2^{-2}x, 0, 0, 0) \tag{2.22}$$

for all $x \in \mathcal{A}$. From two inequalities (2.21) and (2.22), we get

$$\|f(x) - 16^2 f(2^{-2}x)\| \leq 2^{-1}\varphi(2^{-1}x, 0, 0, 0) + 2^{-1}16\varphi(2^{-2}x, 0, 0, 0) \tag{2.23}$$

for all $x \in \mathcal{A}$. Continuing this way, we get

$$\|f(x) - 16^n f(2^{-n}x)\| \leq \frac{1}{32}\sum_{i=1}^{n} 16^i \varphi(2^{-i}x, 0, 0, 0) \tag{2.24}$$

for all $x \in \mathcal{A}$. For any positive integer m, multiply the both sides by 16^m and replace x by $2^{-m}x$ in (2.24), then we have

$$\|16^m f(2^{-m}x) - 16^{n+m} f(2^{-(n+m)}x)\| \leq \frac{1}{32}\sum_{i=1}^{n} 16^{i+m} \varphi(2^{-(i+m)}x, 0, 0, 0) \tag{2.25}$$

for all $x \in \mathcal{A}$. Passing the limit $m \to \infty$, the sequence $\{16^n f(2^{-n}x)\}$ is a Cauchy sequence in $\mathcal{X}$. By the completeness of $\mathcal{X}$, the sequence $\{16^n f(2^{-n}x)\}$ converges and so we can define a function $\delta : \mathcal{A} \to \mathcal{X}$ given by

$$\delta(x) = \lim_{n\to\infty} 16^n f(2^{-n}x)$$

for all $x \in \mathcal{A}$. The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.9. *Let $p > 0, q_j > 0, (j = 1, 2, 3, 4)$ and θ be positive real numbers with*

$$Min\{p, \sum_{j=1}^{4} q_j\} > 4.$$

If $f : \mathcal{A} \to \mathcal{X}$ with $f(0) = 0$ is a mapping such that

$$\|D_f(x, y) + f(zt) - z^4 f(t) - f(z)t^4\|$$

$$\leq \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|t\|^p + \|x\|^{q_1}\|y\|^{q_2}\|z\|^{q_3}\|t\|^{q_4})$$

for all $x, y, z, t \in \mathcal{A}$, then there is a unique quartic derivation $\delta : \mathcal{A} \to \mathcal{X}$ such that

$$\|\delta(x) - f(x)\| \leq \frac{\theta}{32 - 2^{p+1}}\|x\|^p$$

for all $x \in \mathcal{A}$.

Proof. The proof follows from Theorem 2.5 taking

$$\varphi(x, y, z, t) := \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|t\|^p + \|x\|^{q_1}\|y\|^{q_2}\|z\|^{q_3}\|t\|^{q_4})$$

for all $x, y, z, t \in \mathcal{A}$. □

Also, we obtain a superstability result for quartic derivations as follows:

Corollary 2.10. *Let $q_j > 0, (j = 1, 2, 3, 4)$ with $\sum_{j=1}^{4} q_j > 4$, and θ be positive real numbers. If $f : \mathcal{A} \to \mathcal{X}$ with $f(0) = 0$ is a mapping*

$$\|D_f(x, y) + f(zt) - z^4 f(t) - f(z)t^4\| \leq \theta(\|x\|^{q_1}\|y\|^{q_2}\|z\|^{q_3}\|t\|^{q_4})$$

for all $x, y, z, t \in \mathcal{A}$, then f is a quartic derivation.

In the following example, we show that the superstability of quartic derivations does not hold in general case.

Example 2.11. *Let $x, y, z, t \in \mathcal{X}$ be fixed. We define $f : \mathcal{A} \to \mathcal{X}$ by $f(a) := a^4x - xa^4 + y$ for all $a \in \mathcal{A}$,*

$$\varphi(a, b, c, d) := \|D_f(x, y) - z^4 f(t) - f(z)t^4\| = \|y\|\|24 + z^4 + t^4\|.$$

Then we have

$$\sum_{i=0}^{\infty} \frac{\varphi(2^i a, 0, 0, 0)}{16^i} = \sum_{i=0}^{\infty} \frac{\|y\|\|24 + z^4 + t^4\|}{16^i} = \frac{16}{15}\|y\|\|24 + z^4 + t^4\|,$$

$$\lim_{n\to\infty} \frac{1}{16^n}\varphi(2^n a, 2^n b, 2^n c, 2^n d) = 0$$

for all $a, b, c, d \in \mathcal{A}$. Hence $\delta(a) = \lim_{n\to\infty} \frac{f(2^n a)}{16^n} = a^4x - xa^4$ for all $\in \mathcal{A}$. On the other hand we have

$$\delta(ab) = (ab)^4x - x(ab)^4 = a^4b^4x - xa^4b^4,$$

$$a^4\delta(b) + \delta(a)b^4 = a^4(b^4x - xb^4) + (a^4x - xa^4)b^4 = a^4b^4x - xa^4b^4.$$

Thus

$$\delta(ab) = a^4\delta(b) + \delta(a)b^4$$

for all $a, b \in \mathcal{A}$. *Furthermore,*

$$\delta(2a+b) + \delta(2a-b) = [(2a+b)^4x - x(2a+b)^4] + [(2a-b)^4x - x(2a-b)^4].$$

On the other hand we have

$$4[\delta(a+b) + \delta(a-b)] + 24\delta(a) - 6\delta(b)$$
$$= 4[((a+b)^4x - x(a+b)^4) + ((a-b)^4x - x(a-b)^4)]$$
$$+ 24[a^4x - xa^4] - 6[b^4x - xb^4].$$

Then δ *is quartic, that is,* $D_\delta(a, b) = 0$ *for all* $a, b \in \mathcal{A}$.

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