Proyecciones Journal of Mathematics
Vol. 29, N^o 3, pp. 201-207, December 2010.
Universidad Católica del Norte
Antofagasta - Chile
DOI: 10.4067/S0716-09172010000300004

POLYNOMIAL SETS GENERATED BY $e^t\phi(xt)\psi(yt)$

MUMTAZ AHMAD KHAN
ALIGARH MUSLIM UNIVERSITY, INDIA
and
BAHMAN ALIDAD
GOLESTAN UNIVERSITY, IRAN
Received : August 2009. Accepted : October 2010

Abstract

The present paper deals with two variables polynomial sets generated by functions of the form $e^t\phi(xt)\psi(yt)$. *Its special case analogous to Laguerre polynomials have been discussed.*

1. INTRODUCTION

Laguerre polynomials $L_n^{(\alpha)}(x)$ possess the generating relation (see Rainville [8], pp. 130)

$$e^t \ {}_0F_1(-;\ 1+\alpha;\ -xt) = \sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(x) t^n}{(1+\alpha)_n} \tag{1.1}$$

By studying generating relation

$$e^t \ \psi(xt) = \sum_{n=0}^{\infty} \sigma_n(x) \ t^n \tag{1.2}$$

One arrives at properties held by $L_n^{(\alpha)}(x)$ (see Rainville [8], p. 132-133)

Motivated by (1.2) an attempt has been made to study two variable polynomials similar to one given in (1.2) and generated by functions of the form $e^t \phi(xt)\psi(yt)$.

2. TWO VARIABLE POLYNOMIAL SETS ANALOGOUS TO (1.2)

Let us consider the generating relation of the type

$$e^t \ \phi(xt) \ \psi(yt) = \sum_{n=0}^{\infty} \sigma_n(x,y) \ t^n \tag{2.1}$$

Let

$$F = e^t \ \phi(xt) \ \psi(yt) \tag{2.2}$$

Then

$$\frac{\partial F}{\partial x} = t \ e^t \ \phi' \ \psi \tag{2.3}$$

$$\frac{\partial F}{\partial y} = t \ e^t \ \phi \ \psi' \tag{2.4}$$

$$\frac{\partial F}{\partial t} = e^t \ \phi \ \psi + x \ e^t \ \phi' \ \psi + y \ e^t \ \phi \ \psi' \tag{2.5}$$

Eliminating ϕ, ϕ', ψ and ψ' from the four equations (2.2), (2.3), (2.4) and (2.5), we obtain

$$\left(x\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) F - t \ \frac{\partial F}{\partial t} = -t \ F \tag{2.6}$$

Since

$$F = e^t\ \phi(xt)\ \psi(yt) = \sum_{n=0}^{\infty} \sigma_n(x,y)\ t^n$$

Equation (2.6) yields

$$\sum_{n=0}^{\infty}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)\sigma_n(x,y)t^n - \sum_{n=0}^{\infty} n\ \sigma_n(x,y)\ t^n = -\sum_{n=0}^{\infty}\sigma_n(x,y)t^{n+1}$$

$$= -\sum_{n=1}^{\infty}\sigma_{n-1}(x,y)t^n$$

from which the next theorem follows.

Theorem 1 :
From

$$e^t\ \phi(xt)\ \psi(yt) = \sum_{n=0}^{\infty}\sigma_n(x,y)\ t^n$$

it follows that $\frac{\partial}{\partial x}\sigma_0(x,y) = \frac{\partial}{\partial y}\sigma_0(x,y) = 0$ and for $n\ \geq\ 1$,

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)\sigma_n(x,y) - n\ \sigma_n(x,y) = -\sigma_{n-1}(x,y) \tag{2.7}$$

Next, let us assume that the functions ϕ and ψ in (2.1) have the formal power - series expansion

$$\phi(u) = \sum_{n=0}^{\infty}\gamma_n\ u^n \tag{2.8}$$

and

$$\psi(v) = \sum_{n=0}^{\infty}\delta_n\ v^n \tag{2.9}$$

Then (2.1) yields

$$\sum_{n=0}^{\infty}\sigma_n(x,y)\ t^n = \left(\sum_{n=0}^{\infty}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\gamma_n\ x^n\ t^n\right)\left(\sum_{n=0}^{\infty}\delta_n\ y^n\ t^n\right)$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{\gamma_r \ \delta_s \ x^r \ y^s \ t^n}{(n-r-s)!}$$

so that

$$\sigma_n(x,y) = \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{\gamma_r \ \delta_s \ x^r \ y^s}{(n-r-s)!} \tag{2.10}$$

Now consider the sum

$$\sum_{n=0}^{\infty} (c)_n \ \sigma_n(x,y) \ t^n = \sum_{n=0}^{\infty} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(c)_n \ \gamma_r \ \delta_s \ x^r \ y^s \ t^n}{(n-r-s)!}$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(c)_{n+r+s} \ \gamma_r \ \delta_s \ x^r \ y^s \ t^{n+r+s}}{n!}$$

$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (c)_{r+s} \ \gamma_r \ \delta_s \ (xt)^r \ (yt)^s \sum_{n=0}^{\infty} \frac{(c+r+s)_n \ t^n}{n!}$$

$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(c)_{r+s} \ \gamma_r \ \delta_s \ (xt)^r \ (yt)^s}{(1+t)^{c+r+s}} \tag{2.10}$$

We thus arrive at the following theorem:

Theorem 2 :
From
$e^t \ \phi(xt) \ \psi(yt) = \sum_{n=0}^{\infty} \sigma_n(x,y) \ t^n$, $\phi(u) = \sum_{n=0}^{\infty} \gamma_n \ u^n$, $\psi(v) = \sum_{n=0}^{\infty} \delta_n \ v^n$ it follows that for arbitrary c

$$(1-t)^{-c} \ F\left(\frac{xt}{1-t}, \frac{yt}{1-t}\right) = \sum_{n=0}^{\infty} (c)_n \ \sigma_n(x,y) \ t^n \tag{2.11}$$

in which

$$F(u,v) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (c)_{n+k} \ \gamma_n \ \delta_k \ u^n \ v^k \tag{2.12}$$

The role of Theorem 2 is as follows: If a set $\sigma_n(x,y)$ has a generating function of the form $e^t \ \phi(xt) \ \psi(yt)$, Theorem 2 yields for $\sigma_n(x,y)$ another generating function of the form exhibited in (2.11). For instance, if $\phi(u)$ and $\psi(v)$ are specified ${}_pF_q$, the theorem gives for $\sigma_n(x,y)$ a class (c arbitrary) of generating functions involving two variables hypergeometric functions.

Let us now apply Theorems 1 and 2 to Laguerre polynomials of two variables $L_n^{(\alpha,\beta)}(x,y)$ due to S. F. Ragab [7] defined by

$$L_n^{(\alpha,\beta)}(x,y) = \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!}\sum_{r=0}^{n}\frac{(-y)^r\ L_{n-r}^{\alpha}(x)}{r!\ \Gamma(\alpha+n-r+1)\Gamma(\beta+r+1)} \tag{2.13}$$

Where $L_n^{(\alpha)}(x)$ is the well - known Laguerre polynomials of one variable.

The definition (2.13) is equivalent to the following explicit representation of $L_n^{(\alpha,\beta)}(x,y)$, given by Ragab:

$$L_n^{(\alpha,\beta)}(x,y) = \frac{(\alpha+1)_n(\beta+1)_n}{(n!)^2}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}x^s\ y^r}{(\alpha+1)_s\ (\beta+1)_r\ r!\ s!} \tag{2.14}$$

Later, the same year Chatterjea [1] gave the following generating function for $L_n^{(\alpha,\beta)}(x,y)$:

$$e^t\ {}_0F_1(-;\ \alpha+1;\ -xt)\ {}_0F_1(-;\ \beta+1;\ -yt) = \sum_{n=0}^{\infty}\frac{n!\ L_n^{(\alpha,\beta)}(x,y)\ t^n}{(\alpha+1)_n(\beta+1)_n} \tag{2.15}$$

We use Theorem 1 to conclude that $L_0^{(\alpha,\beta)}(x,y)$ is a constant and, and for $n \geq 1$.

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)L_n^{(\alpha,\beta)}(x,y) - n\ L_n^{(\alpha,\beta)}(x,y) = -\frac{(\alpha+n)(\beta+n)}{n}L_{n-1}^{(\alpha,\beta)}(x,y) \tag{2.16}$$

In applying theorem 2 to Laguerre polynomials of two variables $L_n^{(\alpha,\beta)}(x,y)$, note that $\sigma_n(x,y) = \frac{n!\ L_n^{(\alpha,\beta)}(x,y)}{(\alpha+1)_n(\beta+1)_n}$ and that

$$\phi(u) = {}_0F_1(-;\ 1+\alpha;\ -u) = \sum_{n=0}^{\infty}\frac{(-1)^n\ u^n}{n!\ (1+\alpha)_n}$$

$$\psi(v) = {}_0F_1(-;\ 1+\beta;\ -v) = \sum_{n=0}^{\infty}\frac{(-1)^n\ v^n}{n!\ (1+\beta)_n}$$

Then $\gamma_n = \frac{(-1)^n}{n!\ (1+\alpha)_n}$, $\delta_n = \frac{(-1)^n}{n!\ (1+\beta)_n}$ and

$$F(u,v) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(c)_{n+k}\ \gamma_n\ \delta_k\ u^n\ v^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(c)_{n+k}\ (-1)^{n+k}\ u^n\ v^k}{n!\ k!\ (1+\alpha)_n\ (1+\beta)_k}$$

$$= \psi_2[c; 1+\alpha, 1+\beta; -u, -v]$$

Therefore Theorem 2, yields

$$(1-t)^{-c}\psi_2\left[c; 1+\alpha, 1+\beta; -\frac{xt}{1-t}, -\frac{yt}{1-t}\right] = \sum_{n=0}^{\infty}\frac{n!\ (c)_n\ L_n^{(\alpha,\beta)}(x,y)t^n}{(1+\alpha)_n\ (1+\beta)_n} \tag{2.17}$$

a class of generating relations for $L_n^{(\alpha,\beta)}(x,y)$ due to M.A. Khan and A.K. Shukla [2].

Concluding Remark

Application of the theorems given in this paper have already been shown in case of Laguerre polynomials of two variables. Thus, this class of product may be used whenever Laguerre polynomials of two variables occur.

References

[1] CHATTERJEA, S. K. *A note on Laguerre polynomials of two variables*, *Bull. Cal. Math. Soc.*, Vol. 83, pp. 263, (1991).

[2] KHAN, M. A. AND SHUKLA, A. K. *On Laguerre polynomials of several variables*, *Bull. Cal. Math. Soc.*, Vol. 89, pp. 155-164, (1997).

[3] KHAN, M. A. AND SHUKLA, A. K. *A note on Laguerre polynomials of m variables*, *Bulletin of the Greek Mathematical Society.*, Vol. 40, pp. 113-117, (1998).

[4] KHAN, M. A. AND ABUKHAMMASH, G. S. *On Hermite polynomials of two variables suggested by S. F. Ragab's Laguerre polynomials of two variables*, *Bull. Cal. Math. Soc.*, Vol. 90, pp. 61-76, (1998).

[5] KHAN, M. A. AND AHMAD, K. *On a general class of polynomials* $L_n^{(\alpha,\beta;\gamma,\delta)}(x,y)$ *of two variables suggested by the polynomials* $L_n^{(\alpha,\beta)}(x,y)$ *of Ragab and* $L_n^{(\alpha,\beta)}(x)$ *of Prabhakar and Rekha*, *Pro Mathematica, Vol. xix/Nos. 37-38*, pp. 21-38, (2005).

[6] KHAN, M. A. AND ALIDAD, B. *Polynomial sets generated by functions of the form* $G(2xt-t^2)K(2yt-t^2)$, *Communicated for publication.*

[7] RAGAB, S. F. *On Laguerse polynomials of two variables* $L_n^{(\alpha,\beta)}(x,y)$, *Bull. Cal. Math. Soc.*, Vol. 83, pp. 253, (1991).

[8] RAINVILLE, E. D. *Special Functions*, *Macmillan, New York; Reprinted by Chelsea Publ. Co., Bronx., New York*, (1971).

[9] SRIVASTAVA, H. M. AND MANOCHA, H. L. *A Treatise on Generating Functions*, *Ellis Horwood Limited Publishers*, Chichester, Halsted Press, a division of John Wiley and Sons, New York, (1984).

[10] SRIVASTAVA, H. M. AND KARLSSON, P. W. *Multiple Gaussian Hypergeometric Series*, *John Wiley & Sons (Halsted Press), New York; Ellis Horwood, Chichester*, (1985).

MUMTAZ AHMAD KHAN
Department of Applied Mathematics,
Faculty of Engineering and Technology,
Aligarh Muslim University,
Aligarh - 202002, U. P.,
India
e-mail : mumtaz_ahmad_khan_2008@yahoo.com

AND

BAHMAN ALIDAD
Department of Mathematics
Faculty of Science,
Golestan University,
Gorgan,
Iran
e-mail: bahman_alidad@yahoo.com