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Bipartite theory of irredundant set

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Abstract

The bipartite version of irredundant set, edge-vertex irredundant set and vertex-edge irredundant set are introduced. Using the bipartite theory of graph, $IR_{ve}(G) + \gamma(G) \leq |V|$ *and* $\gamma_{ve}(G) + IR(G) \leq |V|$ *are proved.*

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1. Introduction

All graphs considered here are simple and undirected. [4,5] suggests that given any problem, say P, on an arbitrary graph G, there is very likely a corresponding problem Q on a bipartite graph G^1, such that a solution for Q provides a solution for P. The bipartite theory of graphs was introduced in [4] and a parameter called $X-$domination number of a bipartite graph was defined. Let $G = (X, Y, E)$ be a bipartite graph with $|X| = p$ and $|Y| = q$. Two vertices u and v in X are $X-$adjacent if they have a common adjacent vertex $y \in Y$. Let $y \in X$ and $\Delta_Y = max\{|N_Y(u)| : y \in X\}$ where the $X-$neighbor set $N_Y(u)$ is defined as $N_Y(u) = \{v \in X : u \ and \ v \ are \ X-adjacent\}$.

A subset $X \subseteq X$ is an $X-$dominating set [4] if every $x \in X - D$ is $X-$adjacent to some vertex in D. The minimum cardinality of a $X-$dominating set is called $X-$domination number and is denoted by $\gamma_X(G)$.

We say a vertex $x \in X$ hyper $Y-$dominates $y \in Y$ if $y \in N(x)$ or $y \in N(N_Y(x))$. A subset $S \subseteq X$ is a hyper $Y-$dominating set [6] if every $y \in Y$ is hyper $Y-$dominated by a vertex of S. The minimum cardinality of a hyper Y-dominating set is called hyper $Y-$domination number and is denoted by $\gamma_{hY}(G)$.

Given an arbitrary graph $G = (V, E)$, a vertex $u \in V(G)$ ve-dominates an edge $vw \in E(G)$ if (a) $u = v$ or $u = w$ (u incident to vw) or (b) uv or uw is an edge in G. A subset $S \subseteq V(G)$ is a vertex-edge dominating set [3] if for all edges $e \in E(G)$, there exists a vertex $v \in S$ such that v dominates e. The minimum cardinality of a ve-dominating set of G is called the vertex-edge domination number and is denoted as $\gamma_{ve}(G)$.

An edge $e = uv \in E(G)$ ev-dominates a vertex $w \in V(G)$ if (i) $u = w$ or $v = w$ (w is incident to e) or (ii) uw or vw is an edge in G. (w is adjacent to u or v). A set $S \subseteq E(G)$ is an edge-vertex dominating set [3] if for all vertices $v \in V(G)$, there exists an edge $e \in S$ such that e dominates v. The minimum cardinality of a ev-dominating set of G is called the edge-vertex domination number and is denoted as $\gamma_{ev}(G)$.

Observation: 1. *Let G be an arbitrary graph. A vertex $u \in V(G)$ ve-dominates the edge $e \in E(G)$ if and only if the edge e ev-dominates the vertex $u \in V(G)$.*

2. Bipartite Construction

The bipartite graph $VE(G)$ constructed from an arbitrary graph $G = (V, E)$ is defined as in [4]. $VE(G) = (V, E, F)$ is defined by the edges $F = \{(u, e) : e = (u, v) \in E\}$. $VE(G) \cong S(G)$, where $S(G)$ denotes the subdivision graph of G.

The bipartite graph $EV(G)$ [4] constructed from an arbitrary graph $G = (V, E)$ is defined as $EV(G) = (E, V, J)$ where $J = \{(e, u)(e, v) : e = (u, v) \in E\}$.

A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called a dominating set [2] if every $v \in V$ is either an element of S or is adjacent to an element of S. The minimum cardinality of a dominating set of a graph G is called the domination number and is denoted by $\gamma(G)$.

A set $F \subseteq E(G)$ of edges in a graph $G = (V, E)$ is called an edge dominating set [2] if every $e \in E(G)$ is either an element of F or is adjacent to an element of $E - F$. The minimum cardinality of an edge dominating set of a graph G is called the edge domination number and is denoted by $\gamma_1(G)$.

Theorem:2.1 [4] For any graph G,
(a) $\gamma_X(VE(G)) = \gamma(G)$
(b) $\gamma_X(EV(G)) = \gamma_1(G)$.

Theorem:2.2 [6] For any graph G,
(a) $\gamma_{hY}(VE(G)) = \gamma_{ve}(G)$
(b) $\gamma_{hY}(EV(G)) = \gamma_{ev}(G)$.

3. Irredundant sets

3.1. Vertex-edge irredundant set

A vertex $v \in S \subseteq V(G)$ has a private edge $e = uw \in E(G)$ (with respect to a set S), if: 1. v is incident to e or v is adjacent to either u or w, and 2. for every vertices $x \in S - \{v\}$, x is not incident to e and x is not adjacent to either u or w.

A set S is a vertex-edge irredundant set [3] (simply a ve-irredundant set) if every vertex $v \in S$ has a private edge. The vertex-edge irredundance of a graph G is the cardinality of a maximal ve-irredundant set with minimum number of vertices and is denoted by $ir_{ve}(G)$. The upper vertex-edge irredundance number of a graph G is the cardinality of a maximum ve-irredundant set of vertices and is denoted by $IR_{ve}(G)$.

Theorem: 3.1.1 [3] Every minimal ve-dominating set is a maximal ve-irredundant set.

3.2. Edge-vertex irredundant set

An edge $e = uv \in F \subseteq E(G)$ has a private vertex $w \in V(G)$ (with respect to a set F), if: 1. e is incident to w, and 2. for all edges $f = xy \in F - \{e\}$, f is not incident to w and neither x nor y is adjacent to w.

A set F is an edge-vertex irredundant set [3](simply a ev-irredundant set) if every edge $e \in F$ has a private vertex. The edge-vertex irredundance of a graph G is the cardinality of a maximal ev-irredundant set with minimum number of vertices and is denoted by $ir_{ev}(G)$. The upper edge-vertex irredundance number of a graph G is the cardinality of a maximum ev-irredundant set of vertices and is denoted by $IR_{ev}(G)$.

Theorem 3.2.1:[3] Every minimal ev-dominating set of G is a maximal ev-irredundant set.

3.3. Hyper $Y-$ Irredundant set

Let $G = (X, Y, E)$ be a bipartite graph. Let $S \subseteq X$. A vertex $x \in S$ has a private hyper $Y-$neighbor $y \in Y$ if 1. x is adjacent to y or $y \in N(N_Y(x))$ and 2. for all vertices $x_1 \in S - \{x\}$, x_1 is not adjacent to y and $y \notin N(N_Y(x_1))$.

A set S is hyper $Y-$irredundant set if every $v \in S$ has a private hyper $Y-$neighbor. The hyper $Y-$irredundance number of a graph G is the minimum cardinality of a maximal hyper $Y-$irredundant set of vertices and is denoted by $ir_{hY}(G)$. The upper hyper $Y-$irredundance number of a graph G is the maximum cardinality of a maximal hyper $Y-$irredundant set of vertices and is denoted by $IR_{hY}(G)$.

Theorem: 3.3.1 A hyper $Y-$dominating set S is a minimal hyper $Y-$dominating set if and only if it is hyper $Y-$dominating set and hyper $Y-$irredundant set.

Proof: Let S be a hyper Y-dominating set. Then S is a minimal hyper Y-dominating set if and only if $\forall u \in S$, $\exists y \in Y$ which is not hyper Y-dominated by $S - \{u\}$. Equivalently, S is a minimal hyper Y-dominating set if and only if $\forall u \in S$, u has atleast one private hyper Y-neighbour. Thus S is minimal hyper Y-dominating set if and only if it is hyper Y-irredundant set.

Conversely, let S be both hyper Y-dominating and hyper Y-irredundant.

Claim: S is a minimal hyper Y-dominating set.
If S is not minimal hyper Y-dominating set, there exists $v \in S$ for which $S - \{v\}$ is hyper Y-dominating. Since S is hyper Y-irredundant, v has a private hyper Y-neighbor of u. By definition u is not hyper Y-adjacent to any vertex in $S - \{v\}$. That is, $S - \{v\}$ is not hyper Y-dominating set, a contradiction. Hence, S is a minimal hyper Y-dominating set.
Theorem: 3.3.2 Every minimal hyper $Y-$dominating set is a maximal hyper $Y-$irredundant set.

Proof: Every minimal hyper Y-dominating set S is hyper Y-irredundant set.

Claim: S is a maximal hyper Y-irredundant set.
Suppose S is not maximal hyper Y-irredundant set. Then there exists a vertex $u \in X - S$ for which $S \cup \{u\}$ is hyper Y-irredundant. There exists atleast one vertex $y \in Y$ which is a private hyper Y-neighbor of u with respect to $S \cup \{u\}$. That is no vertex in S is hyper Y-adjacent to y. Hence, S is not a hyper Y-dominating set, a contradiction. Hence, S is a maximal hyper Y-irredundant set.

Theorem:3.3.3 For any graph G,

(a) $ir_{hY}(VE(G)) = ir_{ve}(G)$

(b) $ir_{hY}(EV(G)) = ir_{ev}(G)$.

Proof: Let S be a $ir_{hY}-$set of $VE(G) = (X, Y, E)$. Every $x \in S$ has a private hyper $Y-$neighbor $y \in Y$. x is adjacent to y or $y \in N(N_Y(x))$ and for all vertices $x_1 \in S - \{x\}$, x_1 is not adjacent to y and $y \notin N(N_Y(x_1))$. In graph G, $x \in S \subseteq V$ is incident with $y \in E$ or x is adjacent to either u or v where $y = uv$ and for every $x_1 \in S - \{x\}$, $y \in E$ is not incident with

x_1 and x_1 is not adjacent to either u or v. S is a vertex edge irredundant set.

$ir_{ev}(G) \leq |S| = ir_{hY}(VE(G))$.

Let U be a $ir_{ve}-$set of G. Every vertex $v \in S$ has a private edge $e = uw$ with respect to U. Equivalently, v is incident with e or v is adjacent to either u or w and for every $x \in U - \{v\}$, x is not incident with e and x is not adjacent to either u or w. In $VE(G)$, every $v \in S$ has private hyper $Y-$neighbor e. Therefore, $U \subseteq X$ is a hyper $Y-$irredundant set of $VE(G)$. Hence, $ir_{hY}(VE(G)) \leq |U| = ir_{ve}(G)$.

Similarly (b) can be proved.

3.4. X-Irredundant set

Let $G = (X, Y, E)$ be a bipartite graph.Let $S \subseteq X$. Let $u \in S$. A vertex v is a private X-neighbor of u with respect to S if u is the only point of S, X-adjacent to v.

A set S is X-irredundant set if every $u \in S$ has a private X-neighbor. The X-irredundance number of a graph G is the cardinality of a maximal X-irredundant set of vertices with minimum cardinality and is denoted by $ir_X(G)$. The upper X-irredundance number of a graph G is the cardinality of a X-irredundant set of vertices with maximum cardinality and is denoted by $IR_X(G)$.

Theorem:3.4.1 A X-dominating set S is a minimal X-dominating set if and only if it is X-dominating and X-irredundant.

Proof: Let S be a X-dominating set. Then S is a minimal X-dominating set if and only if for every $u \in S$ there exists $v \in X - (S - \{u\})$ which is not X-dominated by $S - \{u\}$. Equivalently, S is a minimal X-dominating set if and only if $\forall u \in S$, u has atleast one private X-neighbor with respect to S. Thus S is minimal X-dominating set if and only if it is X-irredundant.

Conversely, Let S is both X-dominating and X-irredundant.

Claim: S is a minimal X-dominating set.

If S is not a minimal X-dominating set, then there exists $v \in S$ for which $S - \{v\}$ is X-dominating. Since S is X-irredundant, v has a private X-neighbor of with respect to S say u (u may be equal to v). By definition, u is not X-adjacent to any vertex in $S - \{v\}$. Therefore, $S - \{v\}$ is not a X-dominating set, a contradiction. Hence, S is a minimal X-dominating set.

Theorem:3.4.2 Every minimal X-dominating set is a maximal X-irredundant set.

Proof: Every minimal X-dominating set S is X-irredundant set.

Claim: S is a maximal X-irredundant set.

Suppose S is not a maximal X-irredundant set. Then there exists a vertex $u \in X - S$ for which $S \cup \{u\}$ is X-irredundant. Therefore, there exists atleast one vertex x which is a private X-neighbor of u with respect to $S \cup \{u\}$. Hence, no vertex in S is X-adjacent to x. Thus S is not X-dominating set, a contradiction. Hence, S is maximal X-irredundant set.

A vertex v is a private neighbor of a vertex u in a set $S \subseteq V(G)$ with respect to S if $N[v] \cap S = \{u\}$. The private neighbor set of $u \in S$ with respect to S is defined as $pn[u, S] = \{v : N[v] \cap S = \{u\}\}$. A set S is called irredundant set [2] if for every vertex $u \in S$, $pn[u, S] \neq \phi$. The irredundance number of agraph G is the cardinality of a maximal irredundant set with minimum number of vertices and is denoted by $ir(G)$. The upper irredundance number of a graph G is the cardinality of a maximum irredundant set of vertices and is denoted by $IR(G)$.

Theorem:3.4.3 For any graph G,
(a) $ir_X(VE(G)) = ir(G)$
(b) $ir_X(EV(G)) = ir^1(G)$

Proof: Let S be a ir_X set of $VE(G) = (X, Y, E^1)$. Every v has a private X-neighbor u. Equivalently, v is X-adjacent to u and no other vertex in S is X-adjacent to u. In G, $v \in S$ is the only vertex adjacent to u and no other vertex in S is adjacent to u . Therefore, S is an irredundant set of G.

$ir(G) \leq |S| = ir_X(VE(G))$.

Let U be an $ir-$ set of G. For every vertex $v \in U$, $pn[v, U] \neq \phi$. Every vertex $v \in U$ has at least one private neighbor with respect to u. In $VE(G)$, that is every vertex $v \in U$ has at least one private X-neighbor. Therefore, U is an X-irredundant set. Hence, $ir_X(VE(G)) \leq |U| = ir(G)$. Hence, $ir_X(VE(G)) = ir(G)$.

(b) Let S be an ir_X set of $EV(G) = (X, Y, E^1)$. Every e has a private X-neighbor f. Equivalently, e is X-adjacent to f and no other vertex in S is X-adjacent to f. In G, $e \in S$ is the only edge adjacent to f and no other edge in S is adjacent to f. Therefore, S is an edge irredundant set of G. Hence, $ir^1(G) \leq |S| = ir_X(EV(G))$.

Let U be a ir^1- set of G. For every edge $e \in U$, $pn[e, U] \neq \phi$. Hence, every edge $e \in U$ has at least one private neighbor. That is, in $EV(G)$, every vertex $e \in U$ has at least one private X-neighbor. Therefore, U is an X-irredundant set in $EV(G)$. Thus, $ir_X(EV(G)) \leq |U| = ir^1(G)$. Hence, $ir_X(EV(G)) = ir^1(G)$.

4. Main Result

For any graph G, $IR_{ve}(G) + \gamma(G) \leq |V|$ and $\gamma_{ve}(G) + IR(G) \leq |V|$ are proved using bipartite theory of graphs, which are open problem in [3].

Theorem:4.1 Let $G = (X, Y, E)$ be a bipartite graph with $N_Y(x) \neq \phi$ for every $x \in X$. Then $IR_{hY}(G) + \gamma_X(G) \leq |X|$.

Proof: Let S be a IR_{hY} set of G. Then, S is a maximal hyper Y-irredundant set. Therefore, S is a hyper Y-irredundant set. That is every $x \in S$ has a private hyper Y-neighbor $y \in Y$. Then x is adjacent to y or $y \in N(N_Y(x))$ and for all vertices $x_1 \in S - \{x\}$, x_1 is not adjacent to y and $y \notin N(N_Y(x))$.

Case(i): x is adjacent with y.

Since $N_Y(v) \neq \phi$, x has X-neighbours. Let z be any X-neighbour of x. Suppose $z \in S$. Then z is not adjacent to y and $y \notin N(N_Y(z))$. But $y \in N(N_Y(x))$, since x is a X-neighbour of z, a contradiction. Therefore, any X-neighbour of x is in $X - S$.

Case(ii): $y \in N(N_Y(x))$.

Vertices in $N(y)$ are in $X - S$. Then $N(y) \subseteq X - S$. Other wise, we get a contradiction to $y \in Y$ is a private hyper Y-neighbor of $x \in S$. Hence, for every $x \in S$ there exists $x_1 \in X - S$ such that x and x_1 are X-adjacent. That is, $X - S$ is a X-dominating set. Therefore, $\gamma_X(G) \leq |X - S| = |X| - IR_{hY}(G)$. Hence, $IR_{hY}(G) + \gamma_X(G) \leq |X|$.

Corollary: 4.2 For any graph G,
(a)$IR_{ve}(G) + \gamma(G) \leq |V|$

(b) $IR_{ev}(G) + \gamma_1(G) \leq |E|$.

Theorem:4.3 Let $G = (X, Y, E)$ be a bipartite graph with $N_Y(x) \neq \phi$ for every $x \in X$ then $IR_X(G) + \gamma_{hY}(G) \leq |X|$.

Proof: Let S be a IR_X set of G. Every element $x \in S$ has a private X-neighbor. Consider the set $X - S$. Since $X - S$ is a X-dominating set elements of Y are either adjacent to $X - S$ or adjacent to vertices which are X-adjacent to elements of $X - S$. Therefore, $X - S$ is a hyper Y-dominating set. Therefore, $\gamma_{hY} \leq |X - S| = |X| - IR_X$. Hence, $IR_X + \gamma_{hY} \leq |X|$.

Corollary: 4.4 For any graph G ,

(a) $\gamma_{ve}(G) + IR(G) \leq |V|$

(b) $\gamma_{ev}(G) + IR^1(G) \leq |E|$.

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