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GRAPHIC AND REPRESENTABLE FUZZIFYING MATROIDS*

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Abstract

In this paper, a fuzzifying matroid is induced respectively from a fuzzy graph and a fuzzy vector subspace. The concepts of graphic fuzzifying matroid and representable fuzzifying matroid are presented and some properties of them are discussed. In general, a graphic fuzzifying matriod can not be representable over any field. But when a fuzzifying matroid is isomorphic to a fuzzifying cycle matroid which is induced by a fuzzy tree, it is a representable over any field.

Keywords : *fuzzifying matroid, fuzzifying cycle matroid, fuzzifying vector matroid, graphic fuzzifying matroid, representable fuzzifying matroid.*

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1. Introduction

In 1935, Whitney presented the definition of matroid and used graphic matroids and vector matroids as the two fundamental examples of matroids. Graphic matroids and representable matroids form respectively a fundamental class of matroids, there have been numerous basic problems associated with these classes, many scholars researched them (see [9, 10], etc). Moreover, numerous operations and results for graphs and matrices provide the inspiration or motivation for corresponding operations and results for matroids.

Recently, Shi [7] introduced a new approach to fuzzification of matroids, namely fuzzifying matroids. His approach to the fuzzification of matroids preserves many basic properties of crisp matroids, and a fuzzifying matroid and its fuzzy rank function are one-to-one corresponding. Further Shi [8] presented the concept of (L, M)-fuzzy matroid which is a wider generalization of M-fuzzifying matroids. A $(2, [0, 1])$-fuzzy matroid is precise a fuzzifying matroid.

In this paper, we induce a fuzzifying matroid from a fuzzy graph and fuzzy vector subspace, respectively. The concepts of graphic fuzzifying matroid and representable fuzzifying matroid are introduced and some properties of them are discussed. In general, a graphic fuzzifying matriod can not be representable over any field. But when a fuzzifying matroid is isomorphic to a fuzzifying cycle matroid which is induced by a fuzzy tree, it is representable over any field.

2. Preliminaries

Throughout this paper, let D be a finite set, we denote the power set of D by 2^D. For any $X \subseteq D$, we denote by $|X|$ the cardinality of X. Let X be a non-empty subset of D, fuzzy sets on X are all the mappings from X to $[0, 1]$, denote by $[0, 1]^X$. Let $A \subseteq [0, 1]^X$, we shall use the following notations

$$\begin{aligned}&\mathrm{Im}A = \{A(x) : x \in X\},\\ &A_{[a]} = \{x \in X : A(x) \geq a\}, \quad \forall a \in (0, 1],\\ &A_{(a)} = \{x \in X : A(x) > a\}, \quad \forall a \in [0, 1).\end{aligned}$$

Definition 2.1 ([2]). *Let V be a vector space over field F. Define a fuzzy set $\lambda : V \to [0, 1]$, if for any $v_1, v_2 \in V$ and $a, b \in F$, $\lambda(av_1 + bv_2) \geq \lambda(v_1) \wedge \lambda(v_2)$ holds, then the pair (V, λ) is called a fuzzy vector subspace.*

For a fuzzy vector subspace $\widetilde{V} = (V, \lambda)$, $\lambda_{[a]}$ and $\lambda_{(a)}$ are vector subspaces of V [2].

Definition 2.2 ([5]). *Let D be a finite set and $\mathcal{I} \subseteq 2^D$. If $\mathcal{I}$ satisfies the following statements:*

(**I1**) *$\mathcal{I}$ is non-empty;*

(**I2**) *For any $A, B \in 2^D$, $A \subseteq B$, if $B \in \mathcal{I}$, then $A \in \mathcal{I}$;*

(**I3**) *If $A, B \in \mathcal{I}$ and $|B| > |A|$, then there is an element $e \in B - A$ such that $A \cup e \in \mathcal{I}$,*

then the pair $(D, \mathcal{I})$ is called a matroid. $\mathcal{I}$ is called the independent family on D and its members are called independent sets.

Shi [8] generated the concept of matroids as follows.

Definition 2.3 ([8]). *Let $\mathcal{I} : 2^D \to [0, 1]$ be a mapping. If it satisfies the following statements:*

(**FI1**) *$\mathcal{I}(\emptyset) = 1$;*

(**FI2**) *For any $A, B \in 2^D$, if $A \subseteq B$, then $\mathcal{I}(A) \geq \mathcal{I}(B)$;*

(**FI3**) *If $A, B \subseteq 2^D$ and $|A| < |B|$, then there is $e \in B - A$ such that $\mathcal{I}(A \cup e) \geq \mathcal{I}(A) \wedge \mathcal{I}(B)$,*

then the pair $(D, \mathcal{I})$ is called a fuzzifying matroid. For each $A \in 2^D$, $\mathcal{I}(A)$ can be regarded as the degree which A is an independent set.

Theorem 2.4 ([8]). *Let $\mathcal{I} : 2^D \to [0, 1]$ be a mapping. Then the following statements are equivalent:*

(1) *$(D, \mathcal{I})$ is a fuzzifying matroid.*

(2) *For each $a \in (0, 1]$, $(D, \mathcal{I}_{[a]})$ is a matroid.*

(3) *For each $a \in [0, 1)$, $(D, \mathcal{I}_{(a)})$ is a matroid.*

Definition 2.5 ([5]). *Let D_1 and D_2 be two finite sets. Suppose that $H_1 = (D_1, \mathcal{I}_1)$ and $H_2 = (D_2, \mathcal{I}_2)$ are two matroids. H_1 and H_2 are isomorphic if there exists a mapping $\psi : D_1 \to D_2$ such that ψ satisfies the following statements:*

(**I**) *ψ is a one-to-one correspondence,*

(**II**) *For each $X \subseteq D_1$, $X \in \mathcal{I}_1$ if and only if $\psi(X) \in \mathcal{I}_2$,*

denoted by $H_1 \cong H_2$. The mapping ψ is called an isomorphic mapping from H_1 to H_2.

In crisp matroid theory, let $G = (E, V)$ be a graph, we can induce a cycle matroid $M(G)$. A set X of edges is independent in $M(G)$ if and only if X does not contain any cycles. A matroid $H = (D, \mathcal{I})$ is graphic if there exists a graph G such that $H \cong M(G)$, H is called a graphic matroid. Similarly, let B be a subset of a vector space V over field F, we can obtain a vector matroid $M[B]$. A set X of vectors is independent in $M[B]$ if and only if X is linear independence in V. $H = (D, \mathcal{I})$ is representable if there exists a subset B of some vector space such that $H \cong M[B]$, H is called a representable matroid.

Theorem 2.6 ([5]). *Let $H = (D, \mathcal{I})$ be a matroid. If H is graphic, then it is representable over any field.*

3. Isomorphism of fuzzifying matroids

In this section, we define the isomorphism of fuzzifying matroids and discuss its properties.

Definition 3.1. *Let $H_1 = (D_1, \mathcal{I}_1)$ and $H_2 = (D_2, \mathcal{I}_2)$ be two fuzzifying matroids. H_1 and H_2 are fuzzifying isomorphic if there exists a mapping $\psi : D_1 \to D_2$ such that ψ satisfies the following statements:*

(**FI**) *ψ is a one-to-one correspondence,*

(**FII**) *For each subset $X \subseteq D_1$, $\mathcal{I}_1(X) = \mathcal{I}_2(\psi(X))$,*

denoted by $H_1 \cong H_2$, ψ is called an isomorphic mapping from H_1 to H_2.

For a fuzzifying matroid $(D, \mathcal{I})$, $(D, \mathcal{I}_{[a]})$ is a matroid for each $a \in (0, 1]$. By the finiteness of D, then $\mathrm{Im}\mathcal{I}$ is a finite subsets of $[0, 1]$. It is easily to obtain the following.

Lemma 3.2. *Let $(E, \mathcal{I})$ be a fuzzifying matroid. Then there exists a finite sequence $1 \geq a_1 > a_2 > \cdots > a_r > 0$ such that*

(i) *If $a, b \in (a_{i+1}, a_i](1 \leq i \leq r-1)$, then $\mathcal{I}_{[a]} = \mathcal{I}_{[b]} = \mathcal{I}_{[a_i]}$;*

(ii) *If $a \in (a_{i+1}, a_i]$ and $b \in (a_i, a_{i-1}](2 \leq i \leq r-1)$, then $\mathcal{I}_{[b]} \subset \mathcal{I}_{[a]}$.*

Theorem 3.3. *Let $H_1 = (D_1, \mathcal{I}_1)$ and $H_2 = (D_2, \mathcal{I}_2)$ be two fuzzifying matroids. Then the following statements are equivalent:*

(1) *$H_1 \cong H_2$.*

(2) *For each $a \in (0, 1]$, $(D_1, (\mathcal{I}_1)_{[a]}) \cong (D_2, (\mathcal{I}_2)_{[a]})$.*

(3) *For each $a \in [0, 1)$, $(D_1, (\mathcal{I}_1)_{(a)}) \cong (D_2, (\mathcal{I}_2)_{(a)})$.*

Proof. (1) $\Rightarrow$ (2) Since $H_1 \cong H_2$, then there exists an isomorphic mapping $\psi : D_1 \to D_2$ such that $\mathcal{I}_1(X) = \mathcal{I}_2(\psi(X))$ for any $X \subseteq D_1$.

For each $a \in (0,1]$, $\forall X \subseteq D_1$, $X \in (\mathcal{I}_1)_{[a]}$ if and only if $\mathcal{I}_1(X) \geq a$, i.e. $\mathcal{I}_2(\psi(X)) \geq a$ if and only if $\psi(X) \in (\mathcal{I}_2)_{[a]}$. Thus $(D_1, (\mathcal{I}_1)_{[a]}) \cong (D_2, (\mathcal{I}_2)_{[a]})$.

(2) $\Rightarrow$ (1) Suppose that $\text{Im}\mathcal{I}_1 = \{a_1, a_2, \cdots, a_r\}(a_1 > a_2 > \cdots > a_r)$ and $\text{Im}\mathcal{I}_2 = \{b_1, b_2, \cdots, b_s\}(b_1 > b_2 > \cdots > b_s)$. We show that $\text{Im}\mathcal{I}_1 = \text{Im}\mathcal{I}_2$ as follows. Suppose that there is an element $a_i \in \text{Im}\mathcal{I}_1 \setminus \text{Im}\mathcal{I}_2$, then there exists $1 \leq j \leq s$ such that $b_j < a_i < b_{j-1}$. We have

$$(\mathcal{I}_1)_{[b_j]} = (\mathcal{I}_1)_{[a_i]} \cong (\mathcal{I}_2)_{[a_i]} = (\mathcal{I}_2)_{[b_{j-1}]} \subset (\mathcal{I}_1)_{[b_j]}.$$

This is a constriction. Thus $\text{Im}\mathcal{I}_1 = \text{Im}\mathcal{I}_2 = \{a_1, a_2, \cdots, a_r\}$. By Lemma 3.2, we can obtain two nest sets of matroids as follows.

$$\{\emptyset\} \subseteq (\mathcal{I}_1)_{[a_1]} \subset (\mathcal{I}_1)_{[a_2]} \subset \cdots \subset (\mathcal{I}_1)_{[a_r]} \subseteq 2^{D_1},$$

$$\{\emptyset\} \subseteq (\mathcal{I}_2)_{[a_1]} \subseteq (\mathcal{I}_2)_{[a_2]} \subset \cdots \subset (\mathcal{I}_2)_{[a_r]} \subseteq 2^{D_2}.$$

Let $\psi_{a_1} : \mu_{[a_1]} \to \lambda_{[a_1]}$ be an isomorphic mapping from $(\mathcal{I}_1)_{[a_1]}$ to $(\mathcal{I}_2)_{[a_1]}$. We can obtain an isomorphic mapping ψ_{a_2} from $(\mathcal{I}_1)_{[a_2]}$ to $(\mathcal{I}_2)_{[a_2]}$ such that $\psi_{a_2}(A) = \psi_{a_1}(A)$ for each $A \in (\mathcal{I}_1)_{[a_1]}$. Further we can obtain an isomorphic mapping ψ_{a_3} from $(\mathcal{I}_1)_{[a_3]}$ to $(\mathcal{I}_2)_{[a_3]}$ such that $\psi_{a_3}(A) = \psi_{a_2}(A)$ for each $A \in (\mathcal{I}_1)_{[a_2]}$. Analogously we can obtain an isomorphic mapping ψ_{a_r} from $(\mathcal{I}_1)_{[a_r]}$ to $(\mathcal{I}_2)_{[a_r]}$ such that $\psi_{a_r}(A) = \psi_{a_{r-1}}(A)$ for each $A \in (\mathcal{I}_1)_{[a_{r-1}]}$. By decomposition theorem of fuzzy sets, we can obtain $\forall A \in 2^{D_1}$,

$$\begin{aligned}\mathcal{I}_1(A) &= \bigvee\{a \in (0,1] : A \in (\mathcal{I}_1)_{[a]}\} \\ &= \bigvee\{a \in (0,1] : \psi_a(A) \in (\mathcal{I}_2)_{[a]}\} \\ &= \mathcal{I}_2(\psi_a(A)) = \mathcal{I}_2(\psi_{a_r}(A)).\end{aligned}$$

Thus $\mathcal{I}_1 \cong \mathcal{I}_2$ and ψ_{a_r} is precise the corresponding isomorphic mapping. □

4. Graphic and representable fuzzifying matroids

In this section, the definitions of graphic fuzzifying matroid and representable fuzzifying matroid are introduced and some properties of them are discussed. The relation between graphic fuzzifying matroids and representable fuzzifying matroids are considered.

Let $G = (V, E)$ be a graph with finite node sets V and edge sets $E \subseteq V \times V$. δ and μ are fuzzy sets from V and $V \times V$ to $[0,1]$, respectively. Then $\widetilde{G} = (G, \delta, \mu)$ is called a fuzzy graph if

$$\mu(x,y) \leq \delta(x) \wedge \delta(y), \quad \text{for any } x, y \in V.$$

A fuzzy graph $\widetilde{G} = (G, \delta, \mu)$ is connected if the corresponding graph G is connected. When G is a tree, fuzzy graph (G, δ, μ) is also called a fuzzy tree.

In this paper, let G be a simple graph and $\delta(x) \equiv 1$, we denote a fuzzy graph by $\widetilde{G} = (G, \mu)$.

In the following, we can obtain a fuzzifying matroid from a fuzzy graph.

Theorem 4.1. *Let $\widetilde{G} = (G, \mu)$ be a fuzzy graph with $G = (V, E)$. Define a mapping $\mathcal{I}_{\widetilde{G}} : 2^E \to [0,1]$ such that for each $A \subseteq E$,*

$$\mathcal{I}_{\widetilde{G}}(A) = \bigvee \{a \in (0,1] : A \subseteq \mu_{[a]} \text{and} A \textit{ does not contain any cycles}\}.$$

Then $(E, \mathcal{I}_{\widetilde{G}})$ is a fuzzifying matroid.

Proof. By the definition of $\mathcal{I}_{\widetilde{G}}$, it is easy to prove it satisfies (**FI1**) and (**FI2**), we show that (**FI3**) holds as follows. If $A, B \in 2^E$ and $|B| > |A|$, in order to prove that $\bigvee_{e \in B-A} \mathcal{I}_{\widetilde{G}}(A \cup e) \geq \mathcal{I}_{\widetilde{G}}(A) \wedge \mathcal{I}_{\widetilde{G}}(B)$, we suppose that $\mathcal{I}_{\widetilde{G}}(A) \wedge \mathcal{I}_{\widetilde{G}}(B) \neq 0$. Take any $b \in (0,1]$ and $b \leq \mathcal{I}_{\widetilde{G}}(A) \wedge \mathcal{I}_{\widetilde{G}}(B)$, then $\mathcal{I}_{\widetilde{G}}(A) \geq b$ and $\mathcal{I}_{\widetilde{G}}(B) \geq b$. By the definition of $\mathcal{I}_{\widetilde{G}}$ and the finity of $\mu_{[a]}(a \in (0,1])$, there exist $a_1 \geq b$ and $a_2 \geq b$ such that $A \subseteq \mu_{[a_1]} \subseteq \mu_{[b]}$, $B \subseteq \mu_{[a_2]} \subseteq \mu_{[b]}$ and A, B do not contain any cycles. Then in subgraph $\mu_{[b]}$, there exists an edge $e \in B - A$ such that $A \cup e$ does not contain any cycles. Thus $\mathcal{I}_{\widetilde{G}}(A \cup e) \geq b$. Therefore $\bigvee_{e \in B-A} \mathcal{I}_{\widetilde{G}}(A \cup e) \geq b$. By the arbitrariness of b, we have $\bigvee_{e \in B-A} \mathcal{I}_{\widetilde{G}}(A \cup e) \geq \mathcal{I}_{\widetilde{G}}(A) \wedge \mathcal{I}_{\widetilde{G}}(B)$. □

In above Theorem, the pair $(\widetilde{G}, \mathcal{I}_{\widetilde{G}})$ is called the fuzzifying cycle matroid of $\widetilde{G}$, denoted $M_F(\widetilde{G})$.

The next example shows that distinct fuzzy graphs may induce same fuzzifying cycle matroid and their corresponding graphs may not be isomorphic.

Example 4.2. *Let $\widetilde{G_1} = (G_1, \mu)$, $\widetilde{G_2} = (G_2, \mu)$ be fuzzy graphs with corresponding graphs of figure 1. We can easily obtain $M_F(\widetilde{G_1}) = M_F(\widetilde{G_2})$. But obviously G_1 is not isomorphic to G_2.* □

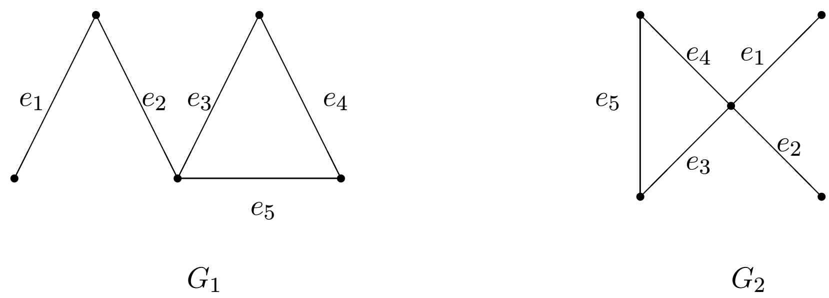

figure 1

Similarly, we can induce a fuzzifying matroid by a fuzzy vector subspace.

Theorem 4.3. *Let $\widetilde{V} = (V, \lambda)$ be a fuzzy vector subspace and U be a subset of V. We define a mapping $\mathcal{I}_U : 2^U \to [0, 1]$ such that for each $A \subseteq U$,*

$$\mathcal{I}_U(A) = \bigvee\{a \in (0,1] : A \subseteq \lambda_{[a]} \text{ and } A \text{ is linearly independent }\}$$

Then $(U, \mathcal{I}_U)$ is a fuzzifying matroid induced by U.

The proof is similar to that of Theorem 4.1.

The pair $(\widetilde{V}, \mathcal{I}_U)$ is called the fuzzifying vector matroid induced by U, denoted by $M_F[U]$.

Definition 4.4. *Let $H = (D, \mathcal{I})$ be a fuzzifying matroid. If there exists a fuzzy graph $\widetilde{G} = (G, \mu)$ such that $H \cong M_F(\widetilde{G})$, then H is called a graphic fuzzifying matroid, we also say that H is graphic.*

Definition 4.5. *Let $H = (D, \mathcal{I})$ be a fuzzifying matroid. If we can find a subset U of some fuzzy vector subspace $\widetilde{V} = (V, \lambda)$ such that $H \cong M_F[U]$, then H is called a representable fuzzifying matroid, we also say that H is representable.*

Theorem 4.6. *Let $\mathcal{I} : 2^E \to [0, 1]$ be a mapping. Then the following statements are equivalent:*

(1) $H = (D, \mathcal{I})$ is a graphic fuzzifying matroid;
(2) For each $a \in (0, 1]$, $(D_{[a]}, \mathcal{I}_{[a]})$ is a graphic matroid;
(3) For each $a \in [0, 1)$, $(D_{(a)}, \mathcal{I}_{(a)})$ is a graphic matroid.
where

$$D_{[a]} = \{e \in D : \exists A \in \mathcal{I}_{[a]} \text{ such that } e \in A\},$$
$$D_{(a)} = \{e \in D : \exists A \in \mathcal{I}_{(a)} \text{ such that } e \in A\}.$$

Proof. (1) $\Rightarrow$ (2) By Theorem 2.4, $(D, \mathcal{I}_{[a]})$ is a matroid for each $a \in (0,1]$. Then $(D_{[a]}, \mathcal{I}_{[a]})$ is a matroid for each $a \in (0,1]$. Since H is a graphic fuzzifying matroid, then there exists a fuzzy graph $\widetilde{G} = (G, \mu)$ such that $H \cong M_F(\widetilde{G})$. Suppose that $\psi : D \to E(G)$ is the one-to-one correspondence such that $\mathcal{I}(A) = \mathcal{I}_{\widetilde{G}}(\psi(A))$ for any $A \in 2^D$. For each $a \in (0,1]$, $\mu_{[a]}$ is a subgraph of G, we denote the cycle matroid $M(\mu_{[a]})$ by $(E(\mu_{[a]}), \mathcal{I}_a)$. Define a mapping $\varphi : D_{[a]} \to E(\mu_{[a]})$ by

$$\varphi(x) = \psi(x), \quad \forall x \in D_{[a]},$$

then φ is a one-to-one correspondence. In the following, we show that $(D_{[a]}, \mathcal{I}_{[a]}) \cong M(\mu_{[a]})$. We need to prove that $X \in \mathcal{I}_{[a]}$ if and only if $\varphi(X) \in \mathcal{I}_a$. Since $H \cong M_F[\widetilde{G}]$ and the finite numbers of subgraph of G, we have

$$\begin{aligned}
\forall X \in \mathcal{I}_{[a]} \quad &\Rightarrow \quad \mathcal{I}(X) \geq a \\
&\Rightarrow \quad \mathcal{I}_{\widetilde{G}}(\varphi(X)) \geq a \\
&\Rightarrow \quad \bigvee\{b \in (0,1] : \varphi(X) \subseteq \mu_{[b]} \text{and} \varphi(X) \text{does not contain} \\
&\qquad \text{any cycles} \geq a \\
&\Rightarrow \quad \exists b \geq a \text{such that} \varphi(X) \subseteq \mu_{[b]} \text{and} \varphi(X) \text{does not} \\
&\qquad \text{contain any cycles.} \\
&\Rightarrow \quad \varphi(X) \subseteq \mu_{[b]} \subseteq \mu_{[a]} \text{and} \varphi(X) \text{does not contain any cycles.} \\
&\Rightarrow \quad \varphi(X) \in \mathcal{I}_a.
\end{aligned}$$

Conversely, $\forall \varphi(X) \in \mathcal{I}_a$, it implies $\varphi(X) \subseteq \mu_{[a]}$ and $\varphi(X)$ does not contain any cycles, we have $\mathcal{I}_{\widetilde{G}}(\varphi(X)) \geq a$, i.e. $\mathcal{I}_{\widetilde{G}}(\psi(X)) \geq a$, $\mathcal{I}(X) \geq a$, it follows that $X \in \mathcal{I}_{[a]}$. Thus $(D_{[a]}, \mathcal{I}_{[a]})$ is graphic.

(2) $\Rightarrow$ (1) If $(D_{[a]}, \mathcal{I}_{[a]})$ is a graphic matroid for each $a \in (0,1]$, then H is a fuzzifying matroid. We show that H is graphic as follows.

Since H is a fuzzifying matroid, then there exists a finite sequence $1 \geq a_1 > a_2 > \cdots > a_r > 0$. By Lemma 3.2, we can obtain a nest set of matroids as follows

$$\{\emptyset\} \subseteq \mathcal{I}_{[a_1]} \subset \cdots \subset \mathcal{I}_{[a_r]} \subseteq 2^E.$$

Since $(D_{[a_1]}, \mathcal{I}_{[a_1]})$ is graphic, then there is a graph $G_{[a_1]}$ such that $(D_{[a_1]}, \mathcal{I}_{[a_1]}) \cong M(G_{[a_1]})$. We can obtain a graph $G_{[a_2]}$ such that $(D_{[a_2]}, \mathcal{I}_{[a_2]}) \cong M(G_{[a_2]})$ and $G_{[a_1]} \nsubseteq G_{[a_2]}$. Analogously we can obtain a graph $G_{[a_r]}$ such that $(D_{[a_r]}, \mathcal{I}_{[a_r]}) \cong M(G_{[a_r]})$ and $G_{[a_{r-1}]} \subset G_{[a_r]}$. Thus we obtain a sequence

$$G_{[a_1]} \subset G_{[a_2]} \subset \cdots \subset G_{[a_r]}.$$

Define a fuzzy set $\mu : G_{[a_r]} \to [0,1]$ by

$$\mu(e) = \bigvee\{a \in (0,1] : e \in G_{[a]}\}.$$

Thus we obtain a fuzzy graph $\widetilde{G} = (G_{[a_r]}, \mu)$. We prove that $H \cong M_F(G_{[a_r]})$ as follows. We denote the cycle matroid of $G_{[a_i]}(0 \leq i \leq r)$ by $M(G_{[a_i]}) = (E(G_{[a_i]}), \mathcal{I}_{a_i})$. Since $(D_{[a_1]}, \mathcal{I}_{[a_1]}) \cong M(G_{[a_1]})$, then there exists an isomorphic mapping $\psi_{a_1} : D_{[a_1]} \to E(G_{[a_1]})$. By $(D_{[a_2]}, \mathcal{I}_{[a_2]}) \cong M(G_{[a_2]})$ and $\mathcal{I}_{[a_1]} \nsubseteq \mathcal{I}_{[a_2]}$, we can obtain that an isomorphic mapping ψ_{a_2} such that $\psi_{a_2}(X) = \psi_{a_1}(X)$ for any $X \in \mathcal{I}_{[a_1]}$. Analogously we can obtain an isomorphic mapping ψ_{a_r} such that $\psi_{a_r}(X) = \psi_{a_{r-1}}(X)$ for any $X \in \mathcal{I}_{[a_r]}$.

Define a one-to-one mapping $\psi : D_{[a_r]} \to E(G_{[a_r]})$ such that $\psi(X) = \psi_{a_r}(X)$ for any $X \in \mathcal{I}_{[a_r]}$. For any $X \subseteq D_{[a_r]}$, we can obtain

$$\begin{aligned}
\mathcal{I}(X) &= \bigvee\{a \in (0,1] : X \in \mathcal{I}_{[a]}\} \\
&= \bigvee\{a \in (0,1] : \psi_a(X) \in \mathcal{I}_a\} \\
&= \bigvee\{a \in (0,1] : \psi_{a_r}(X) \in \mathcal{I}_a\} \\
&= \bigvee\{a \in (0,1] : \psi(X) \in \mathcal{I}_a\} \\
&= \bigvee\{a \in (0,1] : \psi(X) \subseteq G_{[a]} = \mu_{[a]} \text{and} \psi(X) \text{does not contains} \\
&\qquad \text{any cycles} \\
&= \mathcal{I}_{\widetilde{G}}(\psi(X))
\end{aligned}$$

Thus $(D_{[a_r]}, \mathcal{I})$ is a graphic fuzzifying matroid. Let $\mathcal{A} = \{A \subseteq D : \mathcal{I}(A) = 0\}$. Adding all members of $\mathcal{A}$ to $D_{[a_r]}$, we obtain the set D. In order to build the one-to-one correspondence between D and a fuzzy graph, we only need to add some edges such that their values of μ are zero. By above analysis, we obtain that H is graphic.

Similarly, we can prove that (1) $\Leftrightarrow$ (3). □

By Theorem 2.4 and Theorem 2.6, it is easy to obtain the following.

Corollary 4.7. *Let $H = (D, \mathcal{I})$ be a fuzzifying matroid. If H is graphic, then the following statements hold:*

(1) *For each $a \in (0,1]$, $(D_{[a]}, \mathcal{I}_{[a]})$ is representable over any field;*

(2) *For each $a \in [0,1)$, $(D_{(a)}, \mathcal{I}_{(a)})$ is representable over any field.*

Theorem 4.8. *Let $H = (D, \mathcal{I})$ is a representable fuzzifying matroid. Then the following statements hold:*

(1) *For each $a \in (0,1]$, $(D_{[a]}, \mathcal{I}_{[a]})$ is a representable matroid;*

(2) *For each $a \in [0,1)$, $(D_{(a)}, \mathcal{I}_{(a)})$ is a representable matroid.*

where

$$D_{[a]} = \{e \in D : \exists A \in \mathcal{I}_{[a]} \ \textit{such that} \ e \in A\},$$

$$D_{(a)} = \{e \in D : \exists A \in \mathcal{I}_{(a)} \ \textit{such that} \ e \in A\}.$$

Proof. (1) By Corollary 2.4, we have $\mathcal{I}_{[a]}$ is a matroid for each $a \in (0,1]$. Since H is a representable fuzzifying matroid, there exists a fuzzy vector subspace $\widetilde{V} = (V, \lambda)$ and $U \subseteq V$ such that $H \cong M_F[\widetilde{U}]$. Suppose that $\psi : D \to U$ is the one-to-one correspondence such that $\mathcal{I}(A) = \mathcal{I}_{\widetilde{V}}(\psi(A))$ for any $A \in 2^D$. For each $a \in (0,1]$, $\lambda_{[a]}$ is a subspace of V, we denote the vector matroid $M[\lambda_{[a]}] = (\lambda_{[a]}, \mathcal{I}_a)$. Define a mapping $\varphi : D_{[a]} \to \lambda_{[a]}$ by

$$\varphi(x) = \psi(x), \quad \forall x \in D_{[a]},$$

then φ is a one-to-one correspondence. In the following, we show that $\mathcal{I}_{[a]} \cong M[\lambda_{[a]}]$. We need to prove that $X \in \mathcal{I}_{[a]}$ if and only if $\varphi(X) \in \mathcal{I}_a$. Since $H \cong M_F[U]$ and the finity of levels of λ, we have

$$\begin{aligned}
\forall X \in \mathcal{I}_{[a]} \quad &\Rightarrow \quad \mathcal{I}(X) \geq a \\
&\Rightarrow \quad \mathcal{I}_{\widetilde{G}}(\varphi(X)) \geq a \\
&\Rightarrow \quad \bigvee\{b \in (0,1] : \varphi(X) \subseteq \mu_{[b]} \text{and} \varphi(X) \text{is linear independent} \\
&\qquad \geq a \\
&\Rightarrow \quad \exists b \geq a \text{such that} \varphi(X) \subseteq \mu_{[b]} \text{and} \varphi(X) \text{is linear} \\
&\qquad \text{independent} \\
&\Rightarrow \quad \varphi(X) \subseteq \mu_{[b]} \subseteq \mu_{[a]} \text{and} \varphi(X) \text{is linear independent} \\
&\Rightarrow \quad \varphi(X) \in \mathcal{I}_a.
\end{aligned}$$

Conversely, $\forall \varphi(X) \in \mathcal{I}_a$, it implies $\varphi(X) \subseteq \lambda_{[a]}$ and $\varphi(X)$ does not contain any cycles, we have $\mathcal{I}_{\widetilde{G}}(\varphi(X)) \geq a$, i.e. $\mathcal{I}_{\widetilde{G}}(\psi(X)) \geq a$, then $\mathcal{I}(X) \geq a$, it means $X \in \mathcal{I}_{[a]}$. Thus $(D_{[a]}, \mathcal{I}_{[a]})$ is representable.

Similarly, we can prove (2) holds. □

Theorem 4.9. *Let $H = (D, \mathcal{I})$ be a graphic fuzzifying matroid. Then there is a connected fuzzy graph $\widetilde{G}$ such that $H \cong M_F(\widetilde{G})$.*

Proof. Since H is a graphic fuzzifying matroid, then there is a fuzzy graph $\widetilde{G'} = (G', \mu)$ such that $H \cong M_F(\widetilde{G'})$. If $\widetilde{G'}$ is not a connected fuzzy graph, it implies G' is not connected. We suppose that G' have k connected components, denote $G_1, G_2, \cdots, G_k$. We choose an vertex v_i from each connected component G_i, and identify $v_1, v_2, \cdots, v_k$ as a new vertex, then form a new graph G. Obviously, we have $E(G) = E(G')$ and G is connected. Thus $\widetilde{G} = (G, \mu)$ is a new fuzzy graph. $\forall X \subseteq E(G')$, X does not contain any cycles of G' if and only if X does not contain any cycles of G. By Theorem 4.1, we have $H \cong M_F(\widetilde{G})$. □

In crisp matroid theory, a graphic matriod is representable over any field. However, the next example shows that not all graphic fuzzifying matroids are representable over any field.

Example 4.10. *Let* $E = \{a, b, c\}$. *Define a mapping* $\mathcal{I} : 2^E \to [0, 1]$ *by*

$$\mathcal{I}(A) = \begin{cases} 1, & A = \{\emptyset\}, \\ \frac{1}{2}, & A \in \{\{a\}, \{b\}, \{a, b\}\}, \\ \frac{1}{3}, & A \in \{\{c\}, \{a, c\}, \{c, b\}\}, \\ 0, & A = \{a, b, c\}. \end{cases}$$

Then

$$\mathcal{I}_{[r]} = \begin{cases} \{\{\emptyset\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{c, b\}, \}, & r \in (0, \frac{1}{3}], \\ \{\{\emptyset\}, \{a\}, \{b\}, \{a, b\}\}, & r \in (\frac{1}{3}, \frac{1}{2}], \\ \{\emptyset\}, & r \in (\frac{1}{2}, 1]. \end{cases}$$

$(E, \mathcal{I}_{[r]})$ *is a matroid for each* $r \in (0, 1]$, *thus* $(E, \mathcal{I})$ *is a fuzzifying matroid.*

Let $\widetilde{G} = (G, \mu)$ *be a fuzzy graph of figure 2.*

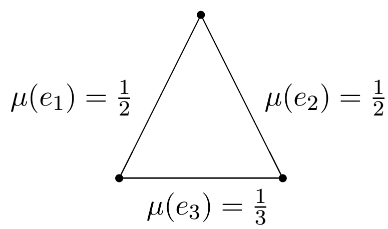

figure 2

We can check that $(E, \mathcal{I}) \cong M_F(\widetilde{G})$. *Thus* $(E, \mathcal{I})$ *is a graphic fuzzifying matriod.*

Suppose that there exists a fuzzy vector subspace $\widetilde{V} = (V, \lambda)$ *and* $U \subseteq V$ *such that* $M_F(\widetilde{G}) \cong M_F[U]$. *By Theorem 4.1 and Theorem 4.3, we obtain that* U *consists of three elements which are linear dependent, and any two elements of them are linear independent. Let* $U = \{v_1, v_2, v_3\}$, *by Definition 3.1, Theorem 4.1 and Theorem 4.3, then there always exists a vector such that its value in* λ *is precise* $\frac{1}{3}$. *Without loss of generality, we assume that* $\lambda(v_1) = \frac{1}{3}$. *Since* v_1 *is the linear combination of* v_2 *and* v_3, *it follows that*

$$\frac{1}{3} = \lambda(v_1) \geq \lambda(v_2) \wedge \lambda(v_3) = \frac{1}{2}.$$

This is a contradiction. Thus $(E, \mathcal{I})$ *is not representable.*

Next, we consider the conditions that a graphic fuzzifying matroid is representable over any field.

Lemma 4.11 ([3]). *Let V be a vector space with basis $B = \{\beta_\alpha\}_{\alpha\in\Gamma}$ (Γ is an index set) and $\mu_0 \in (0,1]$ is a constant. We suppose that $\{\mu_\alpha\}_{\alpha\in\Gamma} \subseteq (0,1]$ are any set of constants which satisfy $\mu_0 \geq \mu_\alpha$ for all $\alpha \in \Gamma$. We construct a function $\mu : V \to [0,1]$ in the following way. Any $z \neq 0, z \in V$ can be uniquely written as $z = \sum_{i=1}^{N} a_i\beta_{\alpha_i}$ with $a_i \neq 0$. We define*

$$\mu(z) = \wedge_{i=1}^{N}\mu(\beta_{\alpha_i}) = \wedge_{i=1}^{N}\mu_{\alpha_i} \quad \text{and} \quad \mu(0) = \mu_0.$$

Then $\widetilde{V} = (V, \mu)$ is a fuzzy vector subspace.

Theorem 4.12. *Let $H = (D, \mathcal{I})$ be a fuzzifying matroid. If $H \cong M_F(\widetilde{G})$, where $\widetilde{G} = (G, \mu)$ is a fuzzy tree, then H is representable over any field F.*

Proof. Let m and n be the edge-number and vertex-number of $\widetilde{G}$. Without loss of generality, we suppose that $n > 0$. Since F is a field, we suppose that 1 and 0 are its multiplicative and additive unit-element, respectively. Assume that $V(n, F)$ denotes the n dimension vector space on F. We suppose further that E_i is the following family:

$$E_i = \{e_i \in V(n, F) \mid \text{ only the } i\text{th value of } e_i \text{ is } 1 \text{ , the others are zero } \}.$$

We show that how to obtain a fuzzy vector subspace $\widetilde{V}$ such that the fuzzifying matroid induced by $\widetilde{V}$ is isomorphic to H.

Step 1 Suppose that the vertex set $V(G) = \{v_1, v_2, ..., v_n\}$.

Step 2 Suppose a mapping $\phi : E(G) \to V(n, F)$, such that for each edge $e = v_iv_j$(let $i \leq j$), $\phi(e) = e_j - e_i$.

Step 3 Let $B = \{\phi(e) : e \in E(G)\}$, $V = \langle B\rangle$ where $\langle B\rangle$ is the spanning subspace by B.

In crisp matroids, we know for any subset X of $E(G)$, X does not contain any cycles in $\widetilde{G}$ if and only if the vector family of $\phi(X)$ is independent on V.

Since $\widetilde{G}$ is a fuzzy tree, then the vectors of B are linear independent on $\langle B\rangle$. By Lemma 4.11, we can obtain a fuzzy vector subspace $\widetilde{V} = (\langle B\rangle, \lambda)$. By Theorem 4.1 and Theorem 4.3, we obtain that $M_F(\widetilde{G}) \cong M_F(\widetilde{V})$. Therefore H is representable. □

5. Conclusions

In this paper, we induce respectively a fuzzifying matroid from a fuzzy graph and a fuzzy vector subspace, introduce the concepts of graphic fuzzifying matroid and representable fuzzifying matroid, and obtain some properties of them. We give an example to show that not all graphic fuzzifying matriods can be representable over any field. But when a fuzzifying matroid is isomorphic to a fuzzifying cycle matroid which is induced by a fuzzy tree, we prove that it is a representable over any field.

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