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ON THE DISTRIBUTIONS OF THE DENSITIES INVOLVING NON-ZERO ZEROS OF BESSEL AND LEGENDRE FUNCTIONS AND THEIR INFINITE DIVISIBILITY

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Abstract

In the present paper, we introduce the probability density functions involving non-zero zeros of the Bessel and Legendre functions. Then, we evaluate the distributions of the characteristic functions defined by these probability density functions and again obtain their related functions and polynomials. Finally, we prove the infinite divisibility of these probability density functions.

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1. Introduction

Hochstadt [4] has derived the infinite product representation of the Bessel function of order real v, such that

$$(1.1)\quad J_v(z) = \frac{\left(\frac{z}{2}\right)^v}{\Gamma(v+1)} \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{z_k^2}\right),\ z \in C,\ z_k \neq 0,\ \forall\ k = 1, 2, ...$$

as considering the sequence of integrals

$$(1.2)\qquad I_n^{(1)} = \frac{1}{2\pi i} \int_{C_n} \frac{F_1(\xi)}{\xi(\xi - z)} d\xi,\ i = \sqrt{(-1)},\ n = 1, 2, ...$$

where, the sequence of circles $\{C_n\}$ about the origin such that C_n includes all non-zero real zeros, $\pm z_1, ..., \pm z_n$, of Bessel function $J_v(z)$, the point z such that for fixed $z \neq z_n$ and not passing through any zeros, also, $\mathrm{F}_1(z) = \frac{G'(z)}{G(z)}, G(z) = z^{-v}\ J_v(z), \forall\ z \in \mathbf{C}$ (the set of complex numbers).

Again, Kumar and Srivastava [6] have obtained the infinite product representation of the Legendre function such that

$$P_\mu(z) = z^{\mu-2} \prod_{k=1}^{\infty} \left(1 - \frac{1 - z^2}{1 - \varsigma_k^2}\right), z \neq 0, z \in \mathbf{C}$$

and $\varsigma_k \neq 0$, and $-1 < \varsigma_k < 1$, $\forall k = 1, 2...$
(1.3)

by considering the sequence of integrals

$$(1.4)\qquad I_n^{(2)} = \frac{1}{2\pi i} \int_{C_n} \frac{F_2(t)}{(t - z)} dt,\ i = \sqrt{(-1)},\ n = 1, 2, ...$$

where, the sequence of circles C_n about the origin such that C_n includes all non-zero and real zeros, $\pm\varsigma_1, ..., \pm\varsigma_(n,)$ lie between -1 to $+1$ (i.e. $-1 < \varsigma_k < 1,\ \forall\ k = 1, 2, ..., n$) of Legendre polynomial of degree μ, the point z twice such that for fixed $z \neq \varsigma_n$ and not passes through any zeros. Also, it has $F_2(z) = \frac{H^(z)}{H(z)}, H(z) = z^{-\mu} P_\mu(z), z \neq 0$ and $z \in \mathbf{C}$.

The concept of infinite divisibility and the decomposition of distributions arise in probability and statistics in relation to seeking families of probability distributions that might be a natural choice in certain applications, in the same way that the normal distribution is. The term infinitely divisible characteristic function is used for the characteristic function of any infinitely divisible distribution.These distributions play a very important role in probability theory in the context of limit theorems. Takano [10] has conjectured that following probability density in normed conjugate product of Gamma functions

$$\frac{2}{\pi}\Gamma\left(1-ix\right)\Gamma\left(1+ix\right)=\frac{2ix}{\sin\pi ix}=\frac{2}{\pi}\frac{1}{\prod_{k=1}^{\infty}\left(1+\frac{x^{2}}{k^{2}}\right)},\ -\infty<x<\infty$$

(1.5)

is infinite divisible.

Several authors studied infinite divisibility of many functions and their related topics (See,, Bondesson [2], Goovaerts, D Hooge and Pril [3], Kelkar [5], Steutel [8], Takano [9, 10] and Thorin [11] etc.). Motivated by the above work and from (1.1), (1.3) and (1.5), we introduce following equalities in the form of probability density functions:

$$\frac{(ix)^{v}}{2^{v}\Gamma\left(v+1\right)J_{v}\left(ix\right)}=\frac{1}{\prod_{k=1}^{\infty}\left(1+\frac{x^{2}}{z_{k}^{2}}\right)},\ -\infty<x<\infty \tag{1.6}$$

and

$$\frac{(ix)^{\mu}}{x^{2}P_{\mu}\left(ix\right)}=\frac{1}{\prod_{k=1}^{\infty}\left(\frac{x^{2}+\varsigma_{k}^{2}}{1-\varsigma_{k}^{2}}\right)},\ -\infty<x<\infty \tag{1.7}$$

respectively, and where, $z_k \neq 0$ and $\varsigma_k \neq 0-4$, and $-1<\varsigma_k<1$, $\forall\ k=1,2...,n$, are zeros of Bessel and Legendre functions, respectively. Here, we conjecture that above densities (1.6) and (1.7) are infinitely divisible. Therefore, for that we define following probability density functions:

$$f_{1}\left(x\right)=\lambda_{1}\frac{\prod_{k=1}^{n}\left(z_{k}^{2}\right)}{\prod_{k=1}^{n}\left(x^{2}+z_{k}^{2}\right)},-\infty<x<\infty \tag{1.8}$$

and

$$f_{2}\left(x\right)=\lambda_{2}\frac{\prod_{k=1}^{n}\left(1-\varsigma_{k}^{2}\right)}{\prod_{k=1}^{n}\left(x^{2}+\varsigma_{k}^{2}\right)},-\infty<x<\infty \tag{1.9}$$

where, $z_k \neq 0$ and $\varsigma_k \neq 0$, and $-1 < \varsigma_k < 1, \forall\ k = 1, 2, ..., n$, are zeros of Bessel and Legendre functions, respectively and λ_1 and λ_2 are arbitrary constants.

2. The Distributions of the Characteristic Function Involving Non-zero Zeros of Bessel Function and its Related Polynomials

Theorem-1 : If in the complex upper half plane for the probability density function (1.8) there exists a characteristic function

$$\Phi_1(t) = \int_{-\infty}^{\infty} e^{itx} \frac{\lambda_1 \prod_{k=1}^{n}(z_k^2)}{\prod_{k=1}^{n}(x^2+z_k^2)} dx,\ -\infty < t < \infty \tag{2.1}$$

then, there holds the distribution formula of non-zero real zeros, z_k, (i.e. $z_k \neq 0$), $\forall k = 1, 2, ..., n$, of Bessel function on the real curve of t such that

$$\Phi_1(t) = \pi\lambda_1 \sum_{k=1}^{n} z_k \frac{\prod_{l=1,l\neq k}^{n}(z_l^2)\ \exp[-z_k|t|]}{\prod_{l=1,l\neq k}^{n}(-z_k^2+z_l^2)},\ -\infty < t < \infty \tag{2.2}$$

Proof : In the complex plane, consider the contour integral

$$\int_C e^{itz} \frac{\lambda_1 \prod_{k=1}^{n}(z_k^2)}{\prod_{k=1}^{n}(z^2+z_k^2)}\ dz,\ -\ -\infty < t < \infty \tag{2.3}$$

where, C is the closed contour consisting of γ, the upper half of the large circle $|z| = R$, the real axis from $-R$ to R. The poles of the integrand $e^{itz}\frac{\lambda_1\prod_{k=1}^{n}(z_k^2)}{\prod_{k=1}^{n}(z^2+z_k^2)}$ are at $z = \pm iz_k, \forall k = 1, 2..., n$ and $-\infty < t < \infty$. Also, inside C it has the simple poles at $z = iz_k, = 1, 2..., n$ in the upper half plane and $\lim_{|z|\to\infty}\frac{\lambda_1\prod_{k=1}^{n}(z_k^2)}{\prod_{k=1}^{n}(z^2+z_k^2)} = 0$ Hence by Cauchys residue theorem and due to Jordan lemma, we find

$$\int_{-\infty}^{\infty} e^{itx} \frac{\lambda_1 \prod_{k=1}^{n}(z_k^2)}{\prod_{k=1}^{n}(x^2+z_k^2)} dx = \pi\lambda_1 \sum_{k=1}^{n} z_k \frac{\prod_{l=1,l\neq k}^{n}(z_l^2)\ \exp[-z_k|t|]}{\prod_{l=1,l\neq k}^{n}(-z_k^2+z_l^2)},$$

when $t > 0$.

(2.4)

In the similar way we have

$$\int_{-\infty}^{\infty} e^{itx}\,\frac{\lambda_1 \prod_{k=1}^{n}\left(z_k^2\right)}{\prod_{k=1}^{n}\left(x^2+z_k^2\right)}\,dx = \pi\lambda_1 \sum_{k=1}^{n} z_k\,\frac{\prod_{l=1,l\neq k}^{n}\left(z_l^2\right)\,\exp\left[z_k t\right]}{\prod_{l=1,l\neq k}^{n}\left(-z_k^2+z_l^2\right)},$$

when $t<0$.

(2.5)

$$\int_{-\infty}^{\infty} e^{itx}\,\frac{\lambda_1 \prod_{k=1}^{n}\left(z_k^2\right)}{\prod_{k=1}^{n}\left(x^2+z_k^2\right)}\,dx = \pi\lambda_1 \sum_{k=1}^{n} z_k\,\frac{\prod_{l=1,l\neq k}^{n}\left(z_l^2\right)}{\prod_{l=1,l\neq k}^{n}\left(-z_k^2+z_l^2\right)},$$

when $t=0$.

(2.6)

Therefore, on combining the equations (2.4a), (2.4b) and (2.4c) we evaluate

$$\int_{-\infty}^{\infty} e^{itx}\,\frac{\lambda_1 \prod_{k=1}^{n}\left(z_k^2\right)}{\prod_{k=1}^{n}\left(x^2+z_k^2\right)}\,dx = \pi\lambda_1 \sum_{k=1}^{n} z_k\,\frac{\prod_{l=1,l\neq k}^{n}\left(z_l^2\right)\,\exp\left[z_k\left|t\right|\right]}{\prod_{l=1,l\neq k}^{n}\left(-z_k^2+z_l^2\right)},$$

$-\infty<t<\infty$.

(2.7)

Hence, the theorem has been proved.

2.1. Related Functions and Polynomials:

Set $u=e^{-|t|}$ (i.e. $|t|=logu^{-1}$), $0<u<1$ in (2.2), we find a function

$$\Phi_1\left(u\right) = \pi\lambda_1 u \sum_{k=1}^{n} z_k\,\frac{\prod_{l=1,l\neq k}^{n}\left(z_l^2\right)\,\left(u\right)^{z}k^{-1}}{\prod_{l=1,l\neq k}^{n}\left(-z_k^2+z_l^2\right)}. \tag{2.8}$$

Again, set $\pi\lambda_1 u = \frac{(-1)^{n-1} z_n \prod_{l=1}^{n-1}\left(-z_n^2+z_l^2\right)}{\prod_{l=1}^{n} z_l}$ in (2.6), we find another function

$$Q_1\left(u\right) = \frac{\left(-1\right)^{n-1} z_n\,\prod_{l=1}^{n-1}\left(-z_n^2+z_l^2\right)}{\prod_{l=1}^{n} z_l}\,\sum_{k=1}^{n} z_k\,\frac{\prod_{l=1,l\neq k}^{n}\left(z_l^2\right)\left(u\right)^{z_k^{-1}}}{\prod_{l=1,l\neq k}^{n}\left(-z_k^2+z_l^2\right)}$$

(2.9)

Now, set $z_l = l, l = 1, 2, ..., n$, in (2.7), we find a polynomial such that

$$Q_1^{(n-1)}(u) = \sum_{k=1}^{n} \frac{(n-1)!\ (2n)!}{(n-k)!\ (n+k)!}\, k\,(-u)^{k-1}. \tag{2.10}$$

For the probability density (1.5) the Takanos polynomial [10] is given by

$$P_{n-1}(u) = \sum_{k=1}^{n} \frac{(2n)!}{(n-k)!\ (n+k)!}\ \frac{k}{n}\ (-u)^{k-1},\ n \geq 1. \tag{2.11}$$

Then, make an appeal to the eqns. (2.8) and (2.9) to get a relation

$$Q_1^{(n-1)}(u) = n! P_{(n-1)}(u), n \geq 1. \tag{2.12}$$

Thus due to the relation (2.10) and the analysis given in [10, section-2, eqns. (2.3)-(2.8)] we may guess the curve of the trigonometric sum which has non- zero zeros (in form of natural numbers) of Bessel function.

Further, set $\pi\lambda_1 u = \frac{(-1)^{n-1}\ z_n\ \prod_{l=1}^{n-1}(-z_n^2+z_l^2)}{\prod_{l=1}^{n} z_l^2}$ in (2.6) and then put $z_l = \frac{1}{l}, \forall l = 1, 2, ..., n$, to find a function

$$R_1^{(n-1)}(u) = \sum_{k=1}^{n} \frac{(k)^{2n-1}}{(n)^{2n-1}}\ \frac{(2n)!}{(n-k)!\ (n+k)!}\ (-u_k)^{k-1}, \text{where,}$$

$u_k = u^{(-1/k)}, 0 < u < 1, n \geq 1.$
(2.13)

The proof of (2.11) follows readily from the formula (2.6) by setting
$\pi\lambda_1 u = \frac{(-1)^{n-1}\ z_n\ \prod_{l=1}^{n-1}(-z_n^2+z_l^2)}{\prod_{l=1}^{n} z_l^2}$
and then putting $z_l = \frac{1}{l}, \forall l = 1, 2, ..., n$, to get

$$R_1^{(n-1)}(u) = (-1)^{n-1} \prod_{l=1}^{n-1} \left(\frac{n^2-l^2}{n^2}\right) \sum_{k=1}^{n} \binom{n}{k} \prod_{l=1, l\neq k}^{n} \left(\frac{k^2}{k^2-l^2}\right) (u)^{1/k^{-1}}$$

(2.14)

Then from the equation (2.12), we easily obtain (2.11).

If we suppose that

$$S_1^{(n-1)}(u) = uR_1^{((n-1)}(u), S_1^{((n-1)}(u) \text{ is many times differentiable,}$$

n ≥ 1,
(2.15)

and

$$H_{n-1}(u) = \sum_{k=1}^{n} \frac{(2n)!\ (-1)^{k-1}}{(n-k)!\ (n+k)!} \left(\frac{k}{n}\right)(u)^{\frac{1}{k}},\ n \geq 1$$

Then with the aid of (2.11), (2.13) and (2.14), we find that

$$\left(u\frac{d}{du}\right)^{2n-2} S_1^{(n-1)}(u) = \frac{1}{(n)^{2n-2}} H_{n-1}(u),\ n \geq 1. \tag{2.18}$$

Thus due to the relation (2.15) and the analysis given in [10, section-2, eqns. (2.3)-(2.8)] we may guess the curve of the trigonometric sum, this curve has non-zero zeros (in form of rational numbers) of Bessel function.

3. The Distributions of the Characteristic Function Involving Non-zero Zeros of Legendre Function and its Related Polynomials

Theorem-2 : If in the complex upper half plane the characteristic function for the density function (1.9) exists by the formula

$$\Phi_2(t) = \int_{-\infty}^{\infty} e^{itx} \frac{\lambda_2 \prod_{k=1}^{n} (1-\varsigma_k^2)}{\prod_{k=1}^{n} (x^2+\varsigma_k^2)}\, dx,\ -\infty < t < \infty, \tag{3.1}$$

then, there holds the distribution formula of non-zero real zeros lying between -1 to $+1, \varsigma_k, (i.e. \varsigma_k \neq 0$, and $-1 < \varsigma_k < 1), \forall k = 1, 2..., n$, of Legendre function on the real curve of t such that

$$\Phi_2(t) = \pi\lambda_2 \sum_{k=1}^{n} \left(\varsigma_k^{-1} - \varsigma_k\right) \frac{\prod_{l=1, l\neq k}^{n} (1-\varsigma_k^2)\ \exp[-\varsigma_k |t|]}{\prod_{l=1, l\neq k}^{n} (-\varsigma_k^2+\varsigma_l^2)},\ -\infty < t < \infty,$$

(3.2)

Proof : In the similar manner, as analyzed to prove theorem-1, we prove theorem-2.

3.1. Related Functions and Polynomials:

Set $u = e^{-|t|}$ (i.e. $|t| = logu^{-1}, 0 < u < 1$ in (3.2), to find a function

$$\Phi_2(u) = \pi\lambda_2 u \sum_{k=1}^{n} \left(\varsigma_k^{-1} - \varsigma_k\right) \frac{\prod_{l=1,l\neq k}^{n} \left(1-\varsigma_l^2\right) (u)^{\varsigma_k - 1}}{\prod_{l=1,l\neq k}^{n} \left(-\varsigma_k^2 + \varsigma_l^2\right)},\ \varsigma_k \neq 0$$

and $-1 < \varsigma_k < 1,\ \forall\ k = 1, 2...$
(3.3)

Again, set $\pi\lambda_2 u = \frac{(-1)^{n-1}\prod_{l=1}^{n-1}\left(\varsigma_l^2 - \varsigma_n^2\right)}{\prod_{l=1}^{n}(1-\varsigma_l)}$ in (3.3), to find a function

$$Q_2(u) = (-1)^{n-1} \prod_{l=1}^{n-1} \left(\varsigma_l^2 - \varsigma_n^2\right) \sum_{k=1}^{n} \frac{1}{\varsigma_k} \frac{\prod_{l=1}^{n} (1+\varsigma_l)\ (u)^{\varsigma_k - 1}}{\prod_{l=1,l\neq k}^{n} \left(-\varsigma_k^2 + \varsigma_l^2\right)}, \varsigma \neq 0,$$

and $-1 < \varsigma_k < 1$
(3.4)

Now, set $\varsigma_l = l, l = 1, 2, ..., n$, in (3.4), we find a polynomial such that

$$Q_2^{(n-1)}(u) = \frac{(n+1)!}{n} \sum_{k=1}^{n} \frac{k}{n} \frac{(2n)!}{(n-k)!\ (n+k)!} (-u)^{k-1},\ n \geq 1,\ \forall k = 1, 2, ...$$

(3.5)

Now, with the aid of (2.9), and (3.5) we find that

$$Q_2^{(n-1)}(u) = \frac{(n+1)!}{n} P_{n-1},\ n \geq 1. \tag{3.6}$$

Thus due to the relation (3.6) and the analysis given in [10, section-2, eqns. (2.3)-(2.8)], we may guess the curve of the trigonometric sum which has no non- zero zeros (in form of natural numbers) of Legendre function.

Again, put $\varsigma_l = \frac{1}{l}, \forall l = 1, 2, ..., n$, in (3.4), we find that

$$R_2^{(n-1)}(u) = (-1)^{n-1} \prod_{l=1}^{n} -1 \left(\frac{n^2 - l^2}{n^2 l^2}\right) \sum_{k=1}^{n} \frac{k \prod_{l=1}^{n} \left(\frac{l+1}{l}\right) (u)^{1/k^{-1}}}{\prod_{l=1,l\neq k}^{n} \left(\frac{k^2 - l^2}{k^2 l^2}\right)}$$

(3.7)

Then on solving (3.7) we easily obtain

$$R_2^{(n-1)}(u) = \left(\frac{n+1}{n}\right)\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{2n-1}\frac{(2n)!}{(n-k)!\ (n+k)!}\ (-u)^{k-1},$$

where, $u_k = u^{-1/k},\ 0 < u < 1.$
(3.8)

Now we suppose that

$$S_2^{(n-1)}(u) = uR_2^{(n-1)}(u),\ S_2^{(n-1)}(u)$$

is many times differentiable, $n \geq 1$. (3.9)

Then we find the relation

$$\left(u\frac{d}{du}\right)^{2n-2} S_2^{(n-1)}(u) = \frac{(n+1)}{(n)^{2n-1}}\ H_{n-1}(u),\ n \geq 1. \tag{3.10}$$

The function $H_{n-1}(u)$ is defined in the equation (2.14).

From (3.10) we may guess the curve on which the distributions of non-zero real zeros lie between -1 to $+1$ of Legendre function.

4. On the Infinite Divisibility

Theorem-3 : The distributions, with the probability densities (1.8) and (1.9), are infinite divisible and thus (1.6) and (1.7) are also infinite divisible.

Proof : (A) : From (1.8), we write the equality (See Takano [10], p.3)

$$f_1(x) = \lambda_1 \frac{\prod_{k=1}^{n}\left(z_k^2\right)}{\prod_{k=1}^{n}\left(x^2+z_k^2\right)} = \lambda_1 \sum_{k=1}^{n}\frac{\prod_{l=1}^{n}\left(z_l^2\right)}{\prod_{l=1,l\neq k}^{n}\left(-z_k^2+z_l^2\right)\left(x^2+z_k^2\right)},$$
$$-\infty < x < \infty,\ z_k \neq 0,\ \forall\ k = 1,2,...,n.$$

(4.1)

Again, using the relation

$$\frac{1}{x^2+z_k^2} = \int_0^\infty \frac{1}{\sqrt{\pi}\,(v)^\alpha}\; e^{-x^2/v}\; \sqrt{\pi} e^{-z_k^2/v}\; (v)^{\alpha-2}\; dv,\; \alpha > 1$$

and $\alpha \neq 2, 3, ..., z_k \neq 0,$ in (4.1) we find that

$$f_1(x) = \int_0^\infty \frac{1}{\sqrt{\pi}\,(v)^\alpha}\; e^{-x^2/v} \sum_{k=1}^{n} \frac{\lambda_1\sqrt{\pi}\; \prod_{l==1}^{n}(z_l^2)}{\prod_{l=1,l\neq k}^{n}(-z_k^2+z_l^2)} e^{-z_k^2/v}\; (v)^{\alpha-2}\; dv,\; \alpha > 1$$

$\alpha > 1$ and $\alpha \neq 2, 3, ..., z_k \neq 0, \forall k = 1, 2, ..., n.$
(4.2)

Now, from (4.2) we set that

$$g_1(v) = \lambda_1\sqrt{\pi}\sum_{k=1}^{n} \frac{\prod_{l==1}^{n}(z_l^2)}{\prod_{l=1,l\neq k}^{n}(-z_k^2+z_l^2)} e^{-z_k^2/v}\; (v)^{\alpha-2}$$

(4.3)

$\alpha > 1$ and $\alpha \neq 2, 3, ..., z_k \neq 0, \forall k = 1, 2, ..., n, v > 0.$
When n = 1, in (4.3), we have

$$g_1(v) = \lambda_1\sqrt{\pi} z_1^2\; e^{-z_1^2/v}\; (v)^{\alpha-2},\; z_1 \neq 0,\; \alpha > 1$$

and $\alpha \neq 2, 3, ..., v > 0$
(4.4)

When $n = 2$, we have

$$g_1(v) = \lambda_1\sqrt{\pi}\left[\frac{z_1^2}{(-z_1^2+z_2^2)}\; e^{-z_1^2/v} + \frac{z_2^2}{(-z_1^2+z_2^2)}\; e^{-z_2^2/v}\right](v)^{\alpha-2},\; z_1,\; z_2 \neq 0,\; \alpha > 1$$

and $\alpha \neq 2, 3, ..., v > 0$
(4.5)

In the above results (4.3)-(4.5) the distribution $g_1(v)$ is of type normal distribution and has the finite variables $(z_1, ..., z_n)$ and is continuous to the variable v so that it is continuous and partially discrete thus it is a mixing density function of normal distribution and it is positive for $\alpha > 1$ and $\alpha \neq 2, 3, ..., v > 0$. Again the distribution with the density $g_1(v)$ is infinitely divisible and hence for that following relations hold:

(4.6) $$-\chi_1'(s) = \chi_1(s) \int_0^\infty e^{-st}\, k_1(t)\, dt,$$

where

(4.7) $$\chi_1(s) \int_0^\infty e^{-sv}\, g_1(v)\, dv$$

The $k_1(t)$ is a non-negative bounded function given by

(4.8) $$k_1(t) = \lim_{R\to\infty} \frac{1}{2\pi i} \int_{\xi-iR}^{\xi+iR} e^{ts} \left(-\frac{\chi_1'(s)}{\chi_1(s)}\right) ds,\ \xi > 0,\ t > 0.$$

Then, make an appeal to the equations (4.4) and (4.7) we easily find that

$$\chi_1(s) = \lambda_1 \sqrt{\pi} z_1^2 \sum_{m=0}^{\infty} \frac{\Gamma(\alpha - m - 1)}{m!} \left(-z_1^2\right)^m (s)^{m-\alpha+1},\ \alpha > 1$$

and $\alpha \neq 2, 3, ...$
(4.9)

Then from (4.9) we have

$$\chi_1'(s) = -\lambda_1 \sqrt{\pi} z_1^2 \sum_{m=0}^{\infty} \frac{\Gamma(\alpha - m)}{m!} \left(-z_1^2\right)^m (s)^{m-\alpha},\ \alpha > 1$$

and $\alpha \neq 2, 3, ...$(4.10)

Thus from the equations (4.8), (4.9) and (4.10), we obtain

$$k_1(t) = \lim_{R\to\infty} \frac{1}{2\pi i} \int_{\xi-iR}^{\xi+iR} e^{ts} \left(\frac{\sum_{m=0}^{\infty} \frac{\Gamma(\alpha-m)}{m!} \left(-z_1^2\right)^m (s)^{m-\alpha}}{s \sum_{m=0}^{\infty} \frac{\Gamma(\alpha-m-1)}{m!} \left(-z_1^2\right)^m (s)^{m-\alpha}}\right) ds,$$

$\alpha > 1$ and $\alpha \neq 2, 3, ..., \xi > 0, t > 0$
(4.11)

Check the following inequalities for all $m = 0, 1, 2, ...,$ and $\alpha \neq 2, 3, ...$
$\alpha + m + 1 > \alpha + m \Rightarrow \Gamma(\alpha - m) > \Gamma(\alpha - m - 1)$, and

$$\alpha < \alpha + m + 1 \Rightarrow \Gamma(\alpha - m) < \alpha\Gamma(\alpha - m - 1) \tag{4.12}$$

Then, from the inequalities given in (4.12) we easily obtain

$$\frac{1}{s} < \left(\frac{\sum_{m=0}^{\infty} \frac{\Gamma(\alpha-m)}{m!} \left(-z_1^2\right)^m (s)^{m-\alpha}}{s \sum_{m=0}^{\infty} \frac{\Gamma(\alpha-m-1)}{m!} \left(-z_1^2\right)^m (s)^{m-\alpha}} \right) < \frac{\alpha}{s} \tag{4.13}$$

Then, with the help of (4.11) and (4.13) we easily obtain

$$0 < k_1(t) < (\alpha - 1), \alpha > 1, \text{ and } \alpha \neq 2, 3, ..., \forall t \in (-\infty, \infty) \tag{4.14}$$

(Since $k_1(t)$ is independent of t.)

Thus the relations (4.6)-(4.8) hold and $k_1(t)$ is non-negative for $\alpha > 1$,and $\alpha \neq 2, 3, ..., \forall t \in (-\infty, \infty)$ and thus the density function $g_1(v)$ is infinitely divisible (See also [3], [8], [9], [10] [11]) and therefore the distribution with density function (1.8) is also infinitely divisible.

Again, there exists

$$\lim_{n\to\infty} f_1(x) = \lim_{n\to\infty} \lambda_1 \frac{\prod_{k=1}^{n} (z_k^2)}{\prod_{k=1}^{n} (x^2 + z_k^2)} \to \frac{\lambda_1}{\prod_{k=1}^{\infty} \left(1 + \frac{x^2}{z_k^2}\right)} \tag{4.15}$$

where γ_1 is an arbitrary constant, $-\infty < x < \infty$, and $z_k \neq 0, \forall k = 1, 2....$

Therefore, the distribution with density (1.6) is also infinitely divisible.

The probability distribution with density (1.6) converges in the sense: (See, [7], [10])

$$\Phi_1(t) \to \Psi_1(t) = \int_{-\infty}^{\infty} e^{itx} \frac{\lambda_1}{\prod_{k=1}^{\infty} \left(1 + \frac{x^2}{z_k^2}\right)} dx, \ \lambda_1$$

is an arbitrary constant, $-\infty < t < \infty$, and $z_k \neq 0, \ \forall \ k = 1, 2...$
(4.16)

(B)- In the similar manner, from (1.8) we write the equality

$$f_2(x) = \lambda_2 \frac{\prod_{k=1}^{n} (1 - \varsigma_k^2)}{\prod_{k=1}^{n} (x^2 + \varsigma_k^2)} = \lambda_2 \sum_{k=1}^{n} \frac{\prod_{t==1}^{n} (1 - \varsigma_k^2)}{\prod_{l=1, l\neq k}^{n} (-\varsigma_k^2 + \varsigma_l^2)(x^2 + \varsigma_k^2)},$$

$-\infty < x < \infty,\ \varsigma_k \neq 0 \quad \text{and} \quad -1 < \varsigma_k < 1,\ \forall\, k = 1, 2, ...$
(4.17)

Again, with the help of the relation given in (4.2) and (4.17) we find

$$f_2\,(x) = \int_0^\infty \frac{1}{\sqrt{\pi (v)^\alpha}}\ e^{-x^2/v}\ \sum_{k=1}^n \frac{\lambda_2 \sqrt{\pi} \prod_{l==1}^n \left(1-\varsigma_l^2\right)}{\prod_{l=1, l\neq k}^n \left(-\varsigma_k^2 + \varsigma_l^2\right)}\ e^{-\varsigma_k^2/v}\ (v)^{\alpha-2}\ dv,\ \alpha > 1$$
$$\text{and } \alpha \neq 2, 3, ..., \varsigma_k \neq 0 \text{ and } -1 < \varsigma_k < 1,\ \forall\, k = 1, 2, ...$$

(4.18)

Now, from (4.18) we set that

$$g_2\,(v) = \lambda_2\,\sqrt{\pi}\,\sum_{k=1}^n \frac{\prod_{l==1}^n \left(1-\varsigma_l^2\right)}{\prod_{l=1, l\neq k}^n \left(-\varsigma_k^2 + \varsigma_l^2\right)}\ e^{-\varsigma_k^2/v}\,(v)^{\alpha-2}\,,$$

$\alpha > 1$ and $\alpha \neq 2, 3, ..., \varsigma_k \neq 0$ and $-1 < \varsigma_k < 1,\ \forall\, k = 1, 2, ..., v > 0$
(4.19)

When $n = 1$ in (4.19), we have

$$g_2\,(v) = \lambda_2\,\sqrt{\pi}\left(1-\varsigma_1^2\right)e^{-\varsigma_1^2/v}\ \ (v)^{\alpha-2}\,,\ \varsigma_1 \neq 0,\ -1 < \varsigma_1 < 1,\ \alpha > 1$$

and $\alpha \neq 2, 3, ..., v > 0$
(4.20)

When $n = 2$ in (4.19), we have

$$g_2\,(v) = \lambda_2\,\sqrt{\pi}\left[\frac{1-\varsigma_1^2}{\left(-\varsigma_1^2+\varsigma_2^2\right)}\ e^{-\varsigma_1^2/v} + \frac{1-\varsigma_1^2}{\left(-\varsigma_2^2+\varsigma_1^2\right)} e^{-\varsigma_2^2/v}\right](v)^{\alpha-2}\,,\ \varsigma_1,\ \varsigma_2 \neq 0$$

(4.21)

and $-1 < \varsigma_1 < 1,\ -1 < \varsigma_2 < 1,\ \alpha > 1$, and $\alpha \neq 2, 3, ..., v > 0.$

The above results (4.19)-(4.21) show that $g_2(v)$ is a mixing probability density function of normal distribution and it is positive for $\alpha > 1$ and $\alpha \neq 2, 3, ..., v > 0$. Again the distribution with the density $g_2(v)$ is infinitely divisible and for that following relations hold:

$$-\chi_2'(s) = \chi_2(s) \int_0^\infty e^{-st} k_2(t)\, dt, \tag{4.22}$$

where

$$\chi_2(s) = \int_0^\infty e^{-sv} g_2(v)\, dv. \tag{4.23}$$

The $k_2(t)$ is a non-negative bounded function and there exists

$$k_2(t) = \lim_{R\to\infty} \frac{1}{2\pi i} \int_{\xi-iR}^{\xi+iR} e^{ts} \left(\frac{\chi_2'(s)}{\chi_2(s)}\right) ds,\ \xi > 0,\ \forall t \in (-\infty, \infty).$$

(4.24)

From the techniques applied in (4.9)-(4.14), we examine and conclude that $k_2(t)$ is non-negative and bounded function for $\alpha > 1$ and $\alpha \neq 2, 3, ..., \forall t \in (-\infty, \infty)$ and thus the density function $g_2(v)$ is infinitely divisible (See also [3], [8], [9], [10] [11]) and therefore the distribution with density function (1.9) is also infinitely divisible.

Again there exists

$$\lim_{n\to\infty} f_2(x) = \lim_{n\to\infty} \lambda_2 \frac{\prod_{k=1}^n (1-\varsigma_k^2)}{\prod_{k=1}^n (x^2+\varsigma_k^2)} \to \frac{\lambda_2}{\prod_{k=1}^\infty \left(\frac{x^2+\varsigma_k^2}{1-\varsigma_k^2}\right)}, \tag{4.25}$$

where γ_2 is an arbitrary constant, $-\infty < x < \infty$, and $\varsigma_k \neq 0$ and $-1 < \varsigma_k < 1$, $\forall k = 1, 2...$ Therefore, the distribution with density (1.7) is also infinitely divisible.

The probability distribution with density (1.7) converges in the sense: (See, [7], [10])

$$\Phi_2(t) \to \Psi_2(t) = \int_{-\infty}^\infty e^{itx} \frac{\lambda_2}{\prod_{k=1}^\infty \left(\frac{x^2+\varsigma_k^2}{1-\varsigma_k^2}\right)} dx,\ \lambda_2$$

is an arbitrary constant, $-\infty < t < \infty$, and $\varsigma_k \neq 0,\ -1 < \varsigma_k < 1$, $\forall\ k = 1, 2, ...$
(4.26)

is an arbitrary constant, $-\infty < t < \infty$, and $\varsigma_k \neq 0$, $-1 < \varsigma_k < 1$, $\forall k = 1, 2...$

5. Discussions

Takano [10, section-2, eqn. (2.4)] has taken a polynomial

$$(5.1)\quad h\left(u\right) = \frac{1}{n}\sum_{k=1}^{n}\left(-1\right)^{k}\ \frac{2n!}{(n-k)!\ (n+k)!}u^{k}, \frac{d}{du}h\left(u\right) = -P_{n-1}\left(u\right)$$

Then, from (5.1) for $u = e^{i\theta}, (0 \leq \theta \leq 2\pi)$ and using following identities

$$\sum_{k=1}^{n}\left(-1\right)^{k}\ \frac{2n!}{(n-k)!\ (n+k)!}\ \cos k\theta = 2^{2n-1}\ \sin^{2n}\left(\frac{\theta}{2}\right) - \frac{1}{2}\left(\begin{array}{c}2n\\ n\end{array}\right),$$

(5.2)

and

$$\sum_{k=1}^{n}\left(-1\right)^{k}\ \frac{2n!}{(n-k)!\ (n+k)!}\ \sin k\theta = -\sin\theta\sum_{r=0}^{n-1}\frac{(2n-2r-2)!}{\{(n-r-1)!\}^{2}}$$

$2^{2r}\left(\sin\frac{\theta}{2}\right)^{2r}$
(5.3)

Takano [10] has guessed the curve of trigonometric sum given in the equations (5.2) and(5.3).

Now consider a polynomial

$$(5.4)\qquad D_1^{(n)}\ (u) = \sum_{k=1}^{n}\ (-1)^{k}\ \frac{(2n)!\ (n-1)!}{(n-k)!\ (n+k)!}\ u^{k}$$

Then by equations (2.9), (2.10) and (5.4) we have

$$\frac{d}{du} D_1^{(n)}(u) = -(n)!\, P_{n-1}(u) = -Q_1^{(n-1)}(u)\,,\ n \geq 1. \tag{5.5}$$

Now in (5.4) take $u = e^{i\theta}$, $(0 \leq \theta \leq 2\pi)$ and use the identities (5.2) and (5.3) and suppose that

$D_1^{(n)}\left(e^{i\theta}\right) = U(\theta) + iV(\theta)$ we get

$U(\theta) = \frac{1}{n}\left[2^{2n-1}\ (n)!\ \sin^{2n}\left(\frac{\theta}{2}\right) - \frac{1}{2}\frac{(2n)!}{(n)!}\right]$

and

$\mathrm{V}(\theta) = -(n-1)!\ \sin\theta\ \sum_{r=0}^{n-1} \frac{(2n-2r-2)!}{\{(n-r-1)!\}^2}\ 2^{2r}\left(\sin\frac{\theta}{2}\right)^{2r},\ n \geq 1$

Thus from right hand side of (5.6) we may guess the curve whose slope has non-zero zeros of Bessel function.

Further put $u = e^{im\theta}$, $(0 \leq \theta \leq \frac{2\pi}{m},\ m > k)$ in (2.14) we get

$$H_{n-1}\left(e^{im\theta}\right) = -\frac{1}{n}\left[\sum_{k=1}^{n} \frac{(2n)!\ (-1)^k\ k}{(n-k)!\ (n+k)!}\ \cos\left(\frac{m}{k}\theta\right)\right.$$

$+i\sum_{k=1}^{n} \frac{(2n)!\ (-1)^k\ k}{(n-k)!\ (n+k)!}\ \sin\left(\frac{m}{k}\theta\right)\Big],\ n \geq 1, (5.6)$

At $\theta = 0$ from (5.7) we have

$$H_{n-1}\left(e^{im\theta}\right) = \frac{(2n-2)!}{(n-1)!\ (n)!},\ n \geq 1 \tag{5.7}$$

Thus using the relations (2.15) and (3.10) in (5.7) and (5.8) respectively we may find the curves which have the non-zero zeros (in rational form) of Bessel and Legendre functions. Due to lack of space we omit them.

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