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Some new classes of vertex-mean graphs

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Abstract

A vertex-mean labeling of a (p,q) graph $G=(V,E)$ is defined as an injective function $f: E \longrightarrow \{0,1,2,...,q_*\}$, $q_* = max(p,q)$ such that the function $f^V : V \longrightarrow \mathbf{N}$ defined by the rule

$$f^V(v) = Round\left(\frac{\sum\limits_{e\in E_v} f(e)}{d(v)}\right)$$ *satisfies the property that*

$f^V(V) = \left\{f^V(u) : u \in V\right\} = \{1,2,...,p\}$, where E_v denotes the set of edges in G that are incident at v, $\mathbf{N}$ denotes the set of all natural numbers and Round is the nearest integer function. A graph that has a vertex-mean labeling is called vertex-mean graph or V-mean graph. In this paper, we study V-mean behaviour of certain new classes of graphs and present a method to construct disconnected V-mean graphs.

MSC (2010) : *Primary: 05C78; Secondary: 52C30.*

Keywords : *Mean labeling, edge labeling, vertex-mean labeling, vertex-mean graphs.*

1. Introduction

A vertex labeling of a graph G is an assignment f of labels to the vertices of G that induces a label for each edge xy depending on the vertex labels. An *edge labeling* of a graph G is an assignment f of labels to the edges of G that induces a label for each vertex v depending on the labels of the edges incident on it. Vertex labelings such as *graceful labeling, harmonious labeling*and *mean labeling* and edge labelings such as *edge-magic labeling, (a,d)-anti magic labeling* and *vertex-graceful labeling* are some of the interesting labelings found in the dynamic survey of graph labeling by Gallian[2]. In fact Acharya and Germina [1] has introduced *vertex-graceful graphs*, as an edge-analogue of *graceful graphs*.

A *mean labeling* f is an injective function from V to the set $\{0, 1, 2, ..., q\}$ such that the set of edge labels defined by the rule $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil$ for each edge uv is $\{1, 2, ..., q\}$. The *mean labeling* was introduced by Somasundaram and Ponraj [4]. Observe that, in a variety of practical problems, the arithmetic mean, X, of a finite set of real numbers $\{x_1, x_2, ..., x_n\}$ serves as a better estimate for it, in the sense that $\sum(x_i - X)$ is zero and $\sum(x_i - X)^2$ is the minimum. If it is required to use a single integer in the place of X then $Round(X)$ does this best, in the sense that $\sum(x_i - Round(X))$ and $\sum(x_i - Round(X))^2$ are minimum, where $Round(Y)$, *nearest integer function* of a real number, gives the integer closest to Y; to avoid ambiguity, it is defined to be the nearest integer greater than Y if the fraction of y is 0.5. Motivated by this and the concept of *vertex-graceful graphs*, Lourdusamy and Seenivasan [3] introduced *vertex-mean labeling* as an edge analogue of *mean labeling* as follows:

A *vertex-mean labeling* of a (p, q) graph $G = (V, E)$ is defined as an injective function $f : E \longrightarrow \{0, 1, 2, ..., q_*\}$, $q_* = max(p, q)$ such that the function $f^V : V \longrightarrow \mathbf{N}$ defined by the rule $f^V(v) = Round\left(\frac{\sum\limits_{e \in E_v} f(e)}{d(v)}\right)$ satisfies the property that $f^V(V) = \left\{f^V(u) : u \in V\right\} = \{1, 2, ..., p\}$, where E_v denotes the set of edges in G that are incident at v and $\mathbf{N}$ denotes the set of all natural numbers. A graph that has a vertex-mean labeling is called *vertex-mean graph* or *V-mean graph*. They, obtained necessary conditions for a graph to be a V-mean graph, and proved that any 3-regular graph of order $2m$, $m \geq 4$ is not a V-mean graph. They also proved that the path P_n, where $n \geq 3$ and the cycle C_n , the Corona $P_n \odot K_m^C$, where $n \geq 2$ and $m \geq 1$, the star graph $K_{1,n}$ if and only if $n \equiv 0(mod\ 2)$, and the crown

$C_n \odot K_1$ are V-mean graphs. A *dragon* is a graph obtained by identifying an end point of a path P_m with a vertex of the cycle C_n and mP_n denotes the disjoint union of m copies of the path P_n. For $3 \leq p \leq n-r$, $C_n(p,r)$ denotes the graph obtained from the cycle C_n with consecutive vertices v_1, v_2, ..., v_n by adding the r chords $v_1v_p, v_1v_{p+1}, ..., v_1v_{p+r-1}$. In this paper we present the V-mean labeling of the following graphs:

1. The graph $S(K_{1,n})$, obtained by subdividing every edge of $K_{1,n}$,
2. Dragon graph,
3. The graph obtained by identifying one vertex of the cycle C_3 with the central vertex of $K_{1,n}$,
4. The graph $C_n(3,1)$,
5. The graph obtained from the two cycles C_n and C_m by adding a new edge joining a vertex of C_n and C_m where $m \in \{n, n+1, n+2\}$,
6. The graph $C_n \cup C_m$,
7. The graph obtained by identifying one vertex of the cycle C_m with a vertex of C_n when $m = 3$ or 4.

We also explain a method to obtain disconnected V-mean graphs from V-mean graphs. Following are some of the graphs so obtained: the graph $\bigcup_{i=1}^{k} P_{n_i}$, where $n_i \geq 3$, the graph mP_n where $m \geq 1$ and $n \geq 3$, the graph $C_n \cup \left(\bigcup_{i=1}^{k} P_{n_i}\right)$, where $n_i \geq 3$, the graph $C_n \cup kP_m$ where $m \geq 3$, the graph $C_n \cup C_m \cup \left(\bigcup_{i=1}^{k} P_{n_i}\right)$,where $n_i \geq 3$, the graph $C_n \cup C_m \cup kP_t$ where $t \geq 3$.

2. New classes of V-mean graphs

Theorem 2.1. *The graph $S(K_{1,n})$, obtained by subdividing every edge of $K_{1,n}$, exactly once, is a V-mean graph.*

Proof. Let $V = \{u, v_i, w_i : 1 \leq i \leq n\}$ and $E = \{uv_i, v_iw_i : 1 \leq i \leq n\}$ be the vertex set and edge set of $S(K_{1,n})$ respectively. Then G has order $2n+1$ and size $2n$.

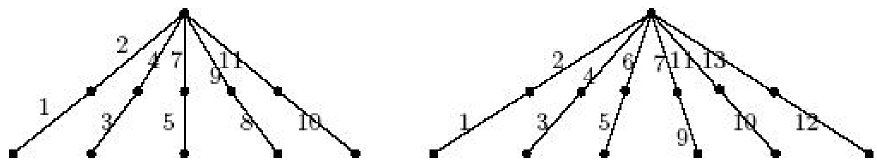

FIGURE 1. V-mean labeling of $S(K_{1,5})$ and $S(K_{1,6})$

case 1: n is odd.

Let $n = 2m + 1$. Define $f : E \longrightarrow \{0, 1, 2, ..., 4m + 3\}$ as follows:

$$f(uv_i) = \begin{cases} 2i & \text{if } \ 1 \le i \le m \\ 2i+1 & \text{if } \ m+1 \le i \le n \end{cases}, \text{ and}$$

$$f(u_iv_i) = \begin{cases} 2i-1 & \text{if } \ 1 \le i \le m+1 \\ 2i & \text{if } \ m+2 \le i \le n \end{cases}.$$

Then it is easy to verify that

$$f^V(u) = 2m + 3,$$

$$f^V(u_i) = \begin{cases} 2i & \text{if } \ 1 \le i \le m+1 \\ 2i+1 & \text{if } \ m+2 \le i \le n \end{cases}, \text{ and}$$

$$f^V(w_i) = \begin{cases} 2i-1 & \text{if } \ 1 \le i \le m+1 \\ 2i & \text{if } \ m+2 \le i \le n \end{cases}$$

Hence

$$f^V(V) = \{1, 2, 3, ..., 4m + 3\}.$$

case 2: n is even.

Let $n = 2m$. Define $f : E \longrightarrow \{0, 1, 2, ..., 4m + 1\}$ as follows:

$$f(uv_i) = \begin{cases} 2i & \text{if } \ 1 \le i \le m \\ 2i-1 & \text{if } \ i = m+1 \\ 2i+1 & \text{if } \ m+2 \le i \le n \end{cases}, \text{ and}$$

$$f(u_iw_i) = \begin{cases} 2i-1 & \text{if } \ 1 \le i \le m \\ 2i+1 & \text{if } \ i = m+1 \\ 2i & \text{if } \ m+2 \le i \le n \end{cases}$$

Then, it is easy to verify that

$$f^V(u) = 2m+1,$$
$$f^V(u_i) = \begin{cases} 2i & \text{if} \quad 1 \le i \le m+1 \\ 2i+1 & \text{if} \quad m+2 \le i \le n \end{cases} \text{, and}$$
$$f^V(w_i) = \begin{cases} 2i-1 & \text{if} \quad 1 \le i \le m+1 \\ 2i+1 & \text{if} \quad i = m+1 \\ 2i & \text{if} \quad m+2 \le i \le n \end{cases}$$

Hence
$$f^V(V) = \{1, 2, 3, ..., 4m+1\}.$$

Thus, $S(K_{1,n})$ is V-mean. □ V-mean labeling of $S(K_{1,5})$ and $S(K_{1,6})$ are shown in Figure 1.

Theorem 2.2. *A dragon graph is V-mean.*

Proof. Let G be a dragon consisting of the path $P_m : v_1v_2...v_m$ and the cycle $C_n : u_1u_2...u_n$. Let v_m be identified with u_n and $r = \lceil \frac{n}{2} \rceil$. Let $e_i = v_iv_{i+1}, 1 \le i \le m-1, e'_{i+1} = u_iu_{i+1}, 1 \le i \le n-1$ and $e'_1 = u_nu_1$ be the edges of G. Observe that G has order and size both equal to $m+n-1$. The edges of G are labeled as follows:
For $1 \le i \le m-1$, the integer i is assigned to the edge e_i. The odd and even integers from 1 to n are respectively arranged in increasing sequences $\alpha_1, \alpha_2, ..., \alpha_r$ and $\beta_1, \beta_2, ..., \beta_{n-r}$ and $m-1+\alpha_k$ is assigned to e'_k and $m-1+\beta_k$ is assigned to e'_{n-k+1}.

vertex	induced edge label
$v_i, 1 \le i \le m-1$	i
$u_k, 1 \le k \le n-r$	$m-1+\beta_k$
$u_{n-k+1}, 1 \le i \le r$	$m-1+\alpha_k$

Table 1. Induced vertex labels

Clearly the edges of G receive distinct labels from $\{0, 1, 2, ..., m+n-1\}$ and the vertex labels induced are $1, 2, ..., m+n-1$ as illustrated in Table 1. Thus G is V-mean. □

For example V-mean labelings of dragons obtained from P_5 and C_7 and P_6 and C_7 are shown in Figure 2.

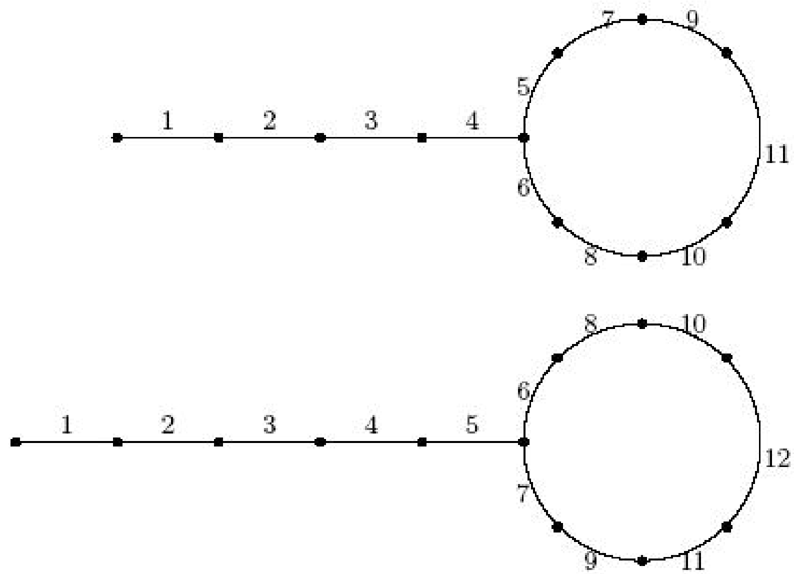

FIGURE 2. V-mean labelings of dragons

Theorem 2.3. *Let G be a graph obtained by identifying one vertex of the cycle C_3 with the central vertex of $K_{1,n}$. Then G is V-mean.*

Proof. Let u_1, u_2, u_3 be the consecutive vertices of C_3. Let $V(K_{1,n}) = \{w, w_1, w_2, ..., w_n\}$ with $deg\ w = n$ and u_1 be identified with w. Then G is of order and size both equal to $n+3$. Let $r = \lfloor \frac{n}{2} \rfloor$. Define $f : E(G) \longrightarrow \{0, 1, 2, ..., n+3\}$ as follows:

$$f(wu_2) = r+2,\ f(wu_3) = r+4,\ f(u_2u_3) = r+3,\ \textit{and}$$

$$f(ww_i) = \begin{cases} i & if\ 1 \leq i \leq r+1 \\ i+3 & if\ r+2 \leq i \leq n \end{cases}.$$

Then, it follows easily that

$$f^V(w) = r+2,\ f^V(u_2) = r+3,\ f^V(u_3) = r+4,\ \textit{and}$$

$$f^V(w_i) = \begin{cases} i & if\ 1 \le i \le r+1 \\ i+3 & if\ r+2 \le i \le n \end{cases}.$$

Hence $f^V(V(G)) = \{1, 2, 3, ...n+3\}$. Thus G is a V-mean graph. □
V-mean labeling of the graphs obtained from $K_{1,5}$ and $K_{1,6}$ by idenntifying the central vertex of each with a vertex of C_3 as shown in Figure 3.

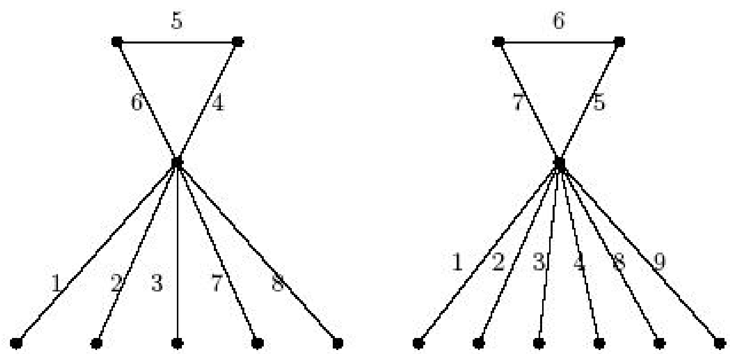

FIGURE 3. V-mean labeling of G obtained from $K_{1,5}$ and $K_{1,6}$

Theorem 2.4. *The graph $C_n(3,1)$ is a V-mean graph.*

Proof. Let $G = C_n(3,1)$. Let $V(G) = \{v_1, v_2, ..., v_n\}$ and $E(G) = \{v_1v_3, v_nv_1, v_iv_{i+1} : 1 \le i \le n-1\}$. Then G has order n and size $n+1$. Let $r = \lceil \frac{n}{2} \rceil$. The edges of G are assigned labels as follows: Define $f : E(G) \longrightarrow \{0, 1, 2, ..., n+1\}$ as follows : The integers 0, 1, 2, 3 are respectively assigned to the edges v_1v_2, v_2v_3, v_1v_3, and v_nv_1. The odd and even integers of $\{4, 5, 6, ..., n\}$ are respectively arranged in increasing sequences $\alpha_1, \alpha_2, ..., \alpha_{r-2}$ and $\beta_1, \beta_2, ..., \beta_{n-r-1}$ and α_k is assigned to $v_{k+2}v_{k+3}$, and β_k is assigned to $v_{n-k}v_{n-k+1}$.

Vertex	Induced edge label
v_1	2
v_2	1
v_3	3
v_n	$\beta_1 = 4$
$v_{k+3}, 1 \leq k \leq n-r-2$	β_{k+1}
$v_{n-k}, 1 \leq k \leq r-2$	α_k

Table 2. Induced vertex labels

Clearly the assignment is an injective function and the set of induced vertex labels is $\{1, 2, ..., n\}$, as illustrated in Table 2. Thus G is a V-mean graph. □ V-mean labeling of $C_8(3,1)$ and $C_9(3,1)$ are shown in Figure 4.

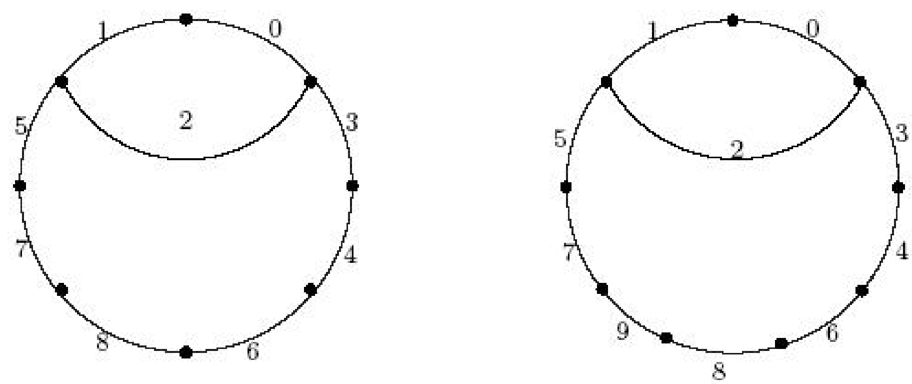

FIGURE 4. V-mean labeling of $C_n(3,1)$ for $n = 8, 9$.

Theorem 2.5. *If $m \in \{n, n+1, n+2\}$, the graph obtained from the two cycles C_n and C_m by adding a new edge joining a vertex of C_n and C_m is a V-mean graph.*

Proof. Let G be the graph consisting of two cycles $C_n : v_1 v_2 ... v_n$ and $C_m : u_1 u_2 ... u_m$ and $e_0 = v_n u_1$ be the bridge connecting them. Then G has order $m+n$ and size $m+n+1$. Let $e_i = v_i v_{i+1}$, $1 \leq i \leq n-1$ and $e_n = v_n v_1$ and $e_i^{'} = u_i u_{i+1}$, $1 \leq i \leq m-1$, and $e_m^{'} = u_m u_1$. Let $r = \lceil \frac{n}{2} \rceil$. Define $f : E(G) \longrightarrow \{0, 1, 2, ..., m+n+1\}$ as follows:

$$f(e_i) = \begin{cases} 1 & if\ i = 0 \\ 2i+1 & if\ 1 \leq i \leq r-1 \\ 2(n-i) & if\ r \leq i \leq n \end{cases},$$

$$f(e_i^{'}) = n + i\ if\ 1 \leq i \leq m.$$

Then

$$f^V(v_i) = \begin{cases} 2i & if\ 1 \leq i \leq r-1 \\ 2(n-i)+1 & if\ r+1 \leq i \leq n \\ 2(n-i) & if\ n\ is\ even\ and\ i = r \\ 2(n-i)+1 & if\ n\ is\ odd\ and\ i = r \end{cases},$$

$$f^V(u_i) = n + i\ if\ 1 \leq i \leq m.$$

Clearly f is an injective function and the set of induced vertex labels is $\{1, 2, ..., n+m\}$. Hence the theorem. $\square$

A V-mean labeling of the graph obtained from C_7 and C_9 is shown in Figure 5.

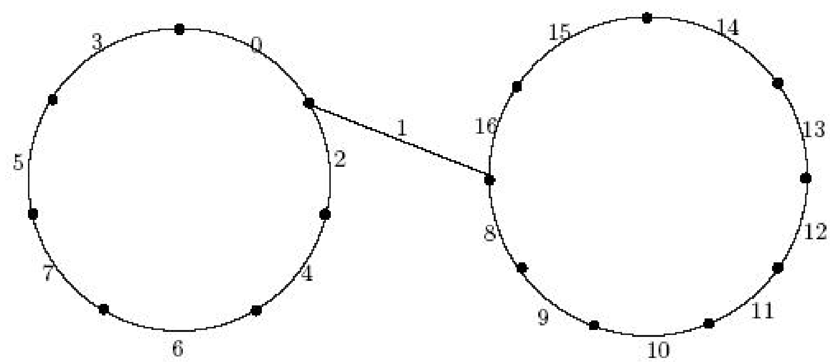

FIGURE 5. A V-mean labeling of a graph obtained from C_7 and C_9

Theorem 2.6. *The graph $C_n \cup C_m$ is a V-mean graph.*

Proof. Let $\{e_1, e_2, ..., e_n\}$ be the edge set of C_n such that $e_i = v_i v_{i+1}$, $1 \le i \le n-1$, $e_n = v_n v_1$ and $\left\{e_1^{'}, e_2^{'}, ..., e_m^{'}\right\}$ be the edge set of C_m such that $e_i^{'} = u_i u_{i+1}$, $1 \le i \le m-1$, $e_m^{'} = u_m u_1$. Then the graph $G = C_n \cup C_m$ has order and size both equal to $m+n$. Let $m \ge n$. Define $f : E(G) \longrightarrow \{0, 1, 2, ..., m+n\}$ as follows:

$$f(e_i) = \begin{cases} i-1 & if\ 1 \le i \le \lfloor \frac{n}{2} \rfloor \\ i & if\ \lfloor \frac{n}{2} \rfloor + 1 \le i \le n-1 \\ n+1 & if\ i = n \end{cases},$$

$$f(e_i^{'}) = \begin{cases} n+2i+1 & if\ 1 \le i \le \lceil \frac{m}{2} \rceil - 1 \\ n+2(m-i) & if\ \lceil \frac{m}{2} \rceil \le i \le m \end{cases}.$$

Then

$$f^V(v_i) = \begin{cases} \lfloor \frac{n}{2} \rfloor + 1 & if\ i = 1 \\ i-1 & if\ 2 \le i \le \lfloor \frac{n}{2} \rfloor + 1 \\ i & if\ \lfloor \frac{n}{2} \rfloor + 2 \le i \le n \end{cases},$$

$$f^V(u_i) = \begin{cases} n+2i & if\ 1 \le i \le \lfloor \frac{m}{2} \rfloor \\ n+2(m-i)+1 & if\ \lfloor \frac{m}{2} \rfloor + 1 \le i \le m \end{cases}.$$

Clearly f is an injective function and the set of induced vertex labels is $\{1, 2, ..., n+m\}$. Hence the theorem. □

A V-mean labeling of $C_8 \cup C_{12}$ is shown in Figure 6.

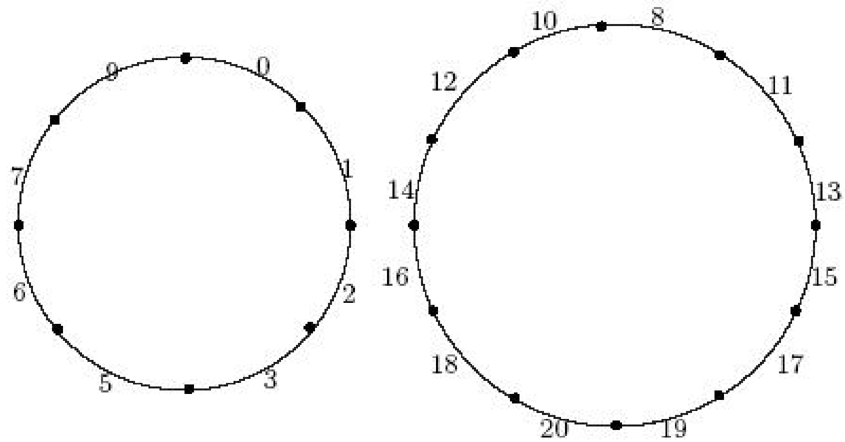

FIGURE 6. A V-mean labeling of $C_8 \cup C_{12}$

Theorem 2.7. *If $m \in \{3,4\}$, the graph G obtained by identifying one vertex of the cycle C_m with a vertex of C_n is a V-mean graph.*

Proof.

case 1 $m = 3$.

Let G be the graph consisting of two cycles $C_3 : v_1v_2v_3v_1$ and $C_n : v_3v_4...v_{n+2}v_3$. Let $r = \lceil \frac{n}{2} \rceil$. Define $f : E(G) \longrightarrow \{0, 1, 2, ..., n+3\}$ as follows: The integers 0, 1, 2, 3, 4 are assigned respectively to the edges $v_1v_2, v_3v_1, v_{n+2}v_3, v_2v_3, v_3v_4$. The odd and even integers of $\{5, 6, 7, ..., n+2\}$ are arranged respectively in increasing sequences $\alpha_1, \alpha_2, ..., \alpha_{r-1}$ and β_1, β_2, ..., β_{n-r-1} and α_k is assigned to $v_{k+3}v_{k+4}$, and β_k is assigned to $v_{n+2-k}v_{n+3-k}$.

Vertex	vertex label
$v_k,\ k = 1, 2, 3$	k
v_{n+2}	4
v_4	$\alpha_1(= 5)$
$v_{k+4}, 1 \leq k \leq n - r - 1$	β_k
$v_{n-k+2}, 1 \leq k \leq r - 2$	α_{k+1}

Table 3. Induced vertex labels

Vertex	vertex label
v_1	3
v_2	1
v_3	2
v_4	4
v_5	5
v_{n+3}	6
v_6	$\alpha_1(= 7)$
$v_{k+6}, 1 \leq k \leq n - r - 2$	β_k
$v_{n-k+3}, 1 \leq k \leq r - 2$	α_{k+1}

Table 4. Induced vertex labels

Clearly the assignment is an injective function and the set of induced vertex labels is $\{1, 2, ..., n + 2\}$, as illustrated in Table 3. Thus G is a V-mean graph.

case 2 $m = 4$.

Let G be the graph consisting of two cycles $C_4 : v_1v_2v_3v_4v_1$ and $C_n : v_4v_5...v_{n+3}v_4$. Let $r = \lfloor \frac{n}{2} \rfloor$. Define $f : E(G) \longrightarrow \{0, 1, 2, ..., n + 4\}$ as follows: The integers 0, 1, 2, 3, 4, 5, 6 are assigned respectively to the edges v_1v_2, v_2v_3, v_3v_4, $v_{n+3}v_4$, v_4v_5, v_4v_1, v_5v_6. The odd and even integers of $\{7, 8, ..., n+3\}$ are arranged respectively in increasing sequences $\alpha_1, \alpha_2, ..., \alpha_{r-1}$ and $\beta_1, \beta_2, ..., \beta_{n-r-2}$ and α_k is assigned to $v_{k+5}v_{k+6}$, and β_k is assigned to $v_{n+3-k}v_{n+4-k}$.

Clearly the assignment is an injective function and the set of induced vertex labels is $\{1, 2, ..., n+3\}$, as illustrated in Table 4. Thus G is a V-mean graph.

□

Fox example V-mean labeling of graphs obtained from C_3 and C_8 and C_4 and C_8 are shown in Figure 7.

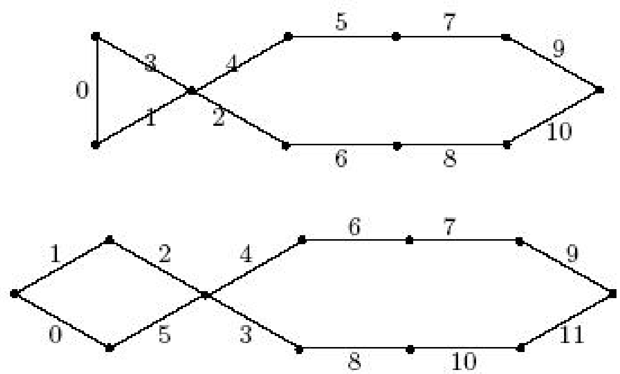

FIGURE 7.

3. Some disconnected V-mean graphs

In this section we present a method to construct disconnected V-mean graphs from V-mean graphs. The following observation is obvious from the definition of *V-mean labeling*.

Observation 3.1. *If f is any V-mean labeling of a (p, q) graph G, then $f(e) \geq p$ for some edge $e \in E(G)$. In particular, if $p \geq q$ then $f(e) \leq p$ for every edge $e \in E(G)$ and hence $f(e) = p$ for some edge $e \in E(G)$.*

Notation 3.2. *We call a V-mean labeling f of a graph $G(p, q)$ as type-A, if $f(e) \leq p$ for every edge $e \in E(G)$, type-B if $f(e) \geq 1$ for every edge $e \in E(G)$, and type-AB if $1 \leq f(e) \leq p$ for every edge $e \in E(G)$. For $S \in \{A, B, AB\}$, we call G as V-mean graph of type-S if it has a V-mean labeling f of type-S.*

Remark 3.3. *We observe that the V-mean graphs presented in [3] and Theorem 2.1 through Theorem 2.7 can be classified as given in Table 5.*

S.N0	V-mean Graph	Type
1	C_n	A
2	$C_n \odot K_1^C$	A
3	$C_n(3,1)$	A
4	The graph consisting of two cycles C_n and C_m connected by a bridge	A
5	The graph $C_n \cup C_m$ where $m \in \{n, n+1, n+2\}$	A
6	The graph obtained by identifying one vertex of the cycle C_m with a vertex of C_n when $m = 3$ or 4	A
7	P_n where $n \geq 3$	AB
8	$P_n \odot K_m^C$ *where* $n \geq 2$	AB
9	$K_{1,n}$ *if and only if* $n \equiv 0(mod\ 2)$	AB
10	The graph $S(K_{1,n})$, obtained by subdividing every edge of $K_{1,n}$	AB
11	Dragon graph	AB
12	The graph obtained by identifying one vertex of cycle C_m with the central vertex of $K_{1,n}$ when $m = 3$ *or* 4	AB

Table 5. V-mean graphs

Let f be a V-mean labeling of $G(p_1, q_1)$ and g be a V-mean labeling of $H(p_2, q_2)$. Observe that the graph $G \cup H$ has order $p = p_1 + p_2$ and size $q = q_1 + q_2$. Define $h : E(G \cup H) \longrightarrow \{0, 1, 2, ..., q_*\}$ as follows:

$$h(e) = \begin{cases} f(e) & if\ e \in E(G) \\ g(e) + p_1 & if\ e \in E(H) \end{cases}.$$

Suppose f is of type-A and g is of type-B. Then $f(e) \leq p_1$ for every edge $e \in E(G)$ and $g(e) \geq 1$ for every edge $e \in E(H)$. As f and g are injective functions, $f(e) \leq p_1$ for every edge $e \in E(G)$ and $g(e) + p_1 \geq p_1 + 1$ for every edge $e \in E(H)$, h is injective.

Suppose f is of type-A and g is of type-AB. Then $f(e) \leq p_1$ for every edge $e \in E(G)$ and $1 \leq g(e) \leq p_2$ for every edge $e \in E(H)$. As f and g are injective functions, $f(e) \leq p_1$ for every edge $e \in E(G)$ and $p_1 + 1 \leq g(e) + p_1 \leq p_1 + p_2$ for every edge $e \in E(H)$, h is injective and $h(e) \leq p$ for every edge $e \in E(G \cup H)$.

Suppose f is of type-AB and g is of type-B. Then $1 \leq f(e) \leq p_1$ for every edge $e \in E(G)$ and $g(e) \geq 1$ for every edge $e \in E(H)$. As f and g are injective functions, $1 \leq f(e) \leq p_1$ for every edge $e \in E(G)$ and

$p_1 + 1 \leq g(e) + p_1$ for every edge $e \in E(H)$, h is injective and $h(e) \geq 1$ for every edge $e \in E(G \cup H)$.

Suppose, both f and g are of type-AB. Then $1 \leq f(e) \leq p_1$ for every edge $e \in E(G)$ and $1 \leq g(e) \leq p_2$ for every edge $e \in E(H)$. As, f and g are injective functions, $1 \leq f(e) \leq p_1$ for every edge $e \in E(G)$ and $p_1 + 1 \leq g(e) + p_1 \leq p_1 + p_2$ for every edge $e \in E(H)$, h is injective and $1 \leq h(e) \leq p$ for every edge $e \in E(G \cup H)$.

The set of induced vertex labels of $G \cup H$ in all four cases is as follows:

$$\begin{aligned} h^V(V(G \cup H)) &= \left\{ f^V(v) : v \in V(G) \right\} \cup \left\{ p_1 + g^V(u) : u \in V(H) \right\} \\ &= \{1, 2, ..., p_1\} \cup \{p_1 + 1, p_1 + 2, ..., p_1 + p_2\} \\ &= \{1, 2, ..., p_1 + p_2\}. \end{aligned}$$

Thus we have the following four theorems.

Theorem 3.4. *If $G(p_1, q_1)$ is a V-mean graph of type-A and $H(p_2, q_2)$ is a V-mean graph of type-B, then $G \cup H$ is V-mean.*

Theorem 3.5. *If $G(p_1, q_1)$ is a V-mean graph of type-A and $H(p_2, q_2)$ is a V-mean graph of type-AB, then $G \cup H$ is V-mean graph of type-A.*

Theorem 3.6. *If $G(p_1, q_1)$ is a V-mean graph of type-AB and $H(p_2, q_2)$ is a V-mean graph of type-B, then $G \cup H$ is V-mean graph of type-B.*

Theorem 3.7. *If both $G(p_1, q_1)$ and $H(p_2, q_2)$ are V-mean graphs of type-AB then the graph $G \cup H$ is V-mean graph of type-AB.*

Corollary 3.8. *Let G be a tree or a unicyclic graph or a two regular graph. If G is V-mean and H is a V-mean graph of type-B, then $G \cup H$ is V-mean.*

Corollary 3.9. *If $G(p, q)$ is V-mean graph of type-AB then, the graph mG is V-mean graph type-AB.*

Corollary 3.10. *If both $G(p_1, q_1)$ and $H(p_2, q_2)$ are V-mean graphs of type-AB, then the graph $mG \cup nH$ is V-mean graph of type-AB.*

Corollary 3.11. *If $G(p_1, q_1)$ is a V-mean graph of type-A and $H(p_2, q_2)$ is a V-mean graph of type-AB, then $G \cup mH$ is V-mean graph of type-A.*

It is interesting to note that a number of disconnected V-mean graphs can be obtained by applying Theorem 3.4 through Corollary 3.11 on V-mean graphs listed in Table 5. For example, the graph $\bigcup_{i=1}^{k} P_{n_i}$, where $n_i \geq 3$, the graph mP_n where $n \geq 3$, the graph $C_n \cup \left(\bigcup_{i=1}^{k} P_{n_i}\right)$, where $n_i \geq 3$, the graph $C_n \cup kP_m$ where $m \geq 3$, the graph $C_n \cup C_m \cup \left(\bigcup_{i=1}^{k} P_{n_i}\right)$, where $n_i \geq 3$, the graph $C_n \cup C_m \cup kP_t$ where $t \geq 3$ are some of such graphs. To illustrate this a V-mean labeling of $C_{10} \cup P_4 \cup P_5$ and a V-mean labeling of $(C_8 \cup C_{12}) \cup K_{1,8}$ are given in Figure 8 and Figure 9 respectively.

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