

The total detour monophonic number of a graph

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Abstract

For a connected graph $G = (V, E)$ of order at least two, a chord of a path P is an edge joining two non-adjacent vertices of P . A path P is called a monophonic path if it is a chordless path. A longest $x - y$ monophonic path is called an $x - y$ detour monophonic path. A set S of vertices of G is a detour monophonic set of G if each vertex v of G lies on an $x - y$ detour monophonic path for some x and y in S . The minimum cardinality of a detour monophonic set of G is the detour monophonic number of G and is denoted by $dm(G)$. A total detour monophonic set of a graph G is a detour monophonic set S such that the subgraph induced by S has no isolated vertices. The minimum cardinality of a total detour monophonic set of G is the total detour monophonic number of G and is denoted by $dm_t(G)$. A total detour monophonic set of cardinality $dm_t(G)$ is called a dm_t -set of G . We determine bounds for it and characterize graphs which realize the lower bound. It is shown that for positive integers r, d and $k \geq 6$ with $r < d$ there exists a connected graph G with monophonic radius r , monophonic diameter d and $dm_t(G) = k$. For positive integers a, b such that $4 \leq a \leq b$ with $b \leq 2a$, there exists a connected graph G such that $dm(G) = a$ and $dm_t(G) = b$. Also, if p, d and k are positive integers such that $2 \leq d \leq p - 2$, $3 \leq k \leq p$ and $p - d - k + 3 \geq 0$, there exists a connected graph G of order p , monophonic diameter d and $dm_t(G) = k$.

Key Words: Detour monophonic set, detour monophonic number, total detour monophonic set, total detour monophonic number.

AMS Subject Classification: 05C12.

1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops and multiple edges. The order and size of G are denoted by p and q , respectively. For basic graph theoretic terminology we refer [1, 3]. The *distance* $d(x, y)$ between two vertices x and y in a connected graph G is the length of a shortest $x - y$ path in G . An $x - y$ path of length $d(x, y)$ is called an $x - y$ *geodesic*. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v . A vertex v is an *extreme vertex* if the subgraph induced by its neighbors is complete. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with $V_1 \cap V_2 = \phi$, then the join $G_1 + G_2$ is a graph $G = (V, E)$, where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$ together with all the edges joining vertices of V_1 to vertices of V_2 and $m_j K_j$ denotes m_j -copies of the complete graph K_j .

A *chord* of a path P is an edge joining two non-adjacent vertices of P . A path P is called a *monophonic path* if it is a chordless path. A longest $x - y$ monophonic path is called an $x - y$ *detour monophonic path*. A set S of vertices of G is a *detour monophonic set* of G if each vertex v of G lies on an $x - y$ detour monophonic path for some x and y in S . The minimum cardinality of a detour monophonic set of G is the *detour monophonic number* of G and is denoted by $dm(G)$. The detour monophonic number of a graph was introduced in [7] and further studied in [6].

A *connected detour monophonic set* of G is a detour monophonic set S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected detour monophonic set of G is the *connected detour monophonic number* of G and is denoted by $dm_c(G)$. The connected detour monophonic number of a graph was introduced and studied in [8].

The detour monophonic concepts have interesting applications in Channel Assignment Problem in radio technologies. Also, there are useful applications of these concepts to security based communication network design.

For any two vertices u and v in a connected graph G , the *monophonic distance* $d_m(u, v)$ from u to v is defined as the length of a longest $u - v$ monophonic path in G . The *monophonic eccentricity* $e_m(v)$ of a vertex v in G is $e_m(v) = \max \{d_m(v, u) : u \in V(G)\}$. The *monophonic radius*, $rad_m(G)$ of G is $rad_m(G) = \min \{e_m(v) : v \in V(G)\}$ and the *monophonic diameter*, $diam_m(G)$ of G is $diam_m(G) = \max \{e_m(v) : v \in V(G)\}$. The monophonic distance was introduced in [4] and further studied in [5].

The following theorems will be used in the sequel.

Theorem 1.1. [7] Each extreme vertex of a connected graph G belongs to every detour monophonic set of G . Moreover, if the set S of all extreme vertices of G is a detour monophonic set, then S is the unique minimum detour monophonic set of G .

Theorem 1.2. [7] Let G be a connected graph with a cut-vertex v and let S be a detour monophonic set of G . Then every component of $G - v$ contains an element of S .

Theorem 1.3. [8] Each extreme vertex of a connected graph G belongs to every connected detour monophonic set of G .

Theorem 1.4. [8] Every cut-vertex of a connected graph G belongs to every connected detour monophonic set of G .

Theorem 1.5. [8] For the complete graph K_p ($p \geq 2$), $dm_c(K_p) = p$.

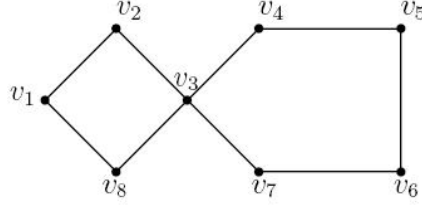
Throughout this paper G denotes a connected graph with at least two vertices.

2. Total detour monophonic number

Definition 2.1. A *total detour monophonic set* of a graph G is a detour monophonic set S such that the subgraph $G[S]$ induced by S has no isolated vertices. The minimum cardinality of a total detour monophonic set of G is the *total detour monophonic number* of G and is denoted by $dm_t(G)$. A total detour monophonic set of cardinality $dm_t(G)$ is called a dm_t -set of G .

Obviously, any connected detour monophonic set of G is a total detour monophonic set of G .

Example 2.2. For the graph G in Figure 2.1, it is clear that $S = \{v_1, v_4, v_5\}$ is a minimum detour monophonic set of G so that $dm(G) = 3$. It is easily verified that the set $S_1 = \{v_1, v_2, v_4, v_5\}$ is a minimum total detour monophonic set of G so that $dm_t(G) = 4$. Also, it is clear that $S_2 = \{v_1, v_2, v_3, v_4, v_5\}$ is a minimum connected detour monophonic set of G and so $dm_c(G) = 5$. Thus the detour monophonic number, total detour monophonic number and connected detour monophonic number of a graph all different.

Figure 2.1: G

It was shown in [2] that determining the monophonic number of a graph is an NP-hard problem. Since every total detour monophonic set of G is a monophonic set, determining the total detour monophonic number of a graph is also an NP-hard problem.

Definition 2.3. A vertex v of a connected graph G is called a *support vertex* of G if it is adjacent to an end-vertex of G .

Theorem 2.4. *Each extreme vertex and each support vertex of a connected graph G belongs to every total detour monophonic set of G . If the set S of all extreme vertices and support vertices form a total detour monophonic set, then it is the unique minimum total detour monophonic set of G .*

Proof. Since every total detour monophonic set of G is a detour monophonic set of G , by Theorem 1.1, each extreme vertex belongs to every total detour monophonic set of G . Since a total detour monophonic set of G contains no isolated vertices, it follows that each support vertex of G also belongs to every total detour monophonic set of G . Thus, if S is the set of all extreme vertices and support vertices of G , then $dm_t(G) \geq |S|$. On the other hand, if S is a total detour monophonic set of G , then $dm_t(G) \leq |S|$. Therefore $dm_t(G) = |S|$ and S is the unique minimum total detour monophonic set of G . \square

Corollary 2.5. *For the complete graph K_p ($p \geq 2$), $dm_t(G) = p$.*

Theorem 2.6. *Let G be a connected graph with cut-vertices and let S be a total detour monophonic set of G . If v is a cut-vertex of G , then every component of $G - v$ contains an element of S .*

Proof. Since every total detour monophonic set of G is a detour monophonic set of G , the result follows from Theorem 1.2. \square

Theorem 2.7. For a connected graph G of order p , $2 \leq dm(G) \leq dm_t(G) \leq dm_c(G) \leq p$.

Proof. Any detour monophonic set of G needs at least two vertices and so $dm(G) \geq 2$. Since every total detour monophonic set of G is also a detour monophonic set of G , it follows that $dm(G) \leq dm_t(G)$. Also, since every connected detour monophonic set of G is a total detour monophonic set of G , it follows that $dm_t(G) \leq dm_c(G)$. Since $V(G)$ is a connected detour monophonic set of G , it is clear that $dm_c(G) \leq p$. Hence $2 \leq dm(G) \leq dm_t(G) \leq dm_c(G) \leq p$. \square

Corollary 2.8. Let G be a connected graph. If $dm_t(G) = 2$, then $dm(G) = 2$.

For any non-trivial path of order at least 4, the detour monophonic number is 2 and the total detour monophonic number is 4. This shows that the converse of the Corollary 2.8 need not be true.

Remark 2.9. The bounds in Theorem 2.7 are sharp. For the complete graph $G = K_2$, $dm_t(G) = 2$ and for the complete graph K_p , $dm_t(K_p) = p$. For the graph G given in Figure 2.1, $dm(G) = 3$, $dm_t(G) = 4$, $dm_c(G) = 5$ and $p = 8$ so that $2 < dm(G) < dm_t(G) < dm_c(G) < p$. Hence all the parameters in Theorem 2.7 are distinct.

Theorem 2.10. For any non-trivial tree T , the set of all end-vertices and support vertices of T is the unique minimum total detour monophonic set of G .

Proof. Since the set of all end-vertices and support vertices of T forms a total detour monophonic set, the result follows from Theorem 2.4. \square

Now we proceed to characterize graphs G for which the lower bound in Theorem 2.7 is attained.

Theorem 2.11. For any connected graph G , $dm_t(G) = 2$ if and only if $G = K_2$.

Proof. If $G = K_2$, then $dm_t(G) = 2$. Conversely, let $dm_t(G) = 2$. Let $S = \{u, v\}$ be a minimum total detour monophonic set of G . Then uv is an edge. It is clear that a vertex different from u and v cannot lie on a $u - v$ detour monophonic path and so $G = K_2$. \square

Theorem 2.12. Let G be a connected graph with at least 2 vertices. Then $dm_t(G) \leq 2 dm(G)$.

Proof. Let $S = \{v_1, v_2, \dots, v_k\}$ be a minimum detour monophonic set of G . Let $u_i \in N(v_i)$ for $i = 1, 2, \dots, k$ and let $T = \{u_1, u_2, \dots, u_k\}$. Then $S \cup T$ is a total detour monophonic set of G so that $dm_t(G) \leq |S \cup T| \leq 2k = 2 dm(G)$. \square

Theorem 2.13. If $G = C_p$ or $G = \overline{K}_2 + H$ ($p \geq 3$), where H is a graph of order $p - 2$, then $dm_t(G) = 3$.

Proof. First, suppose that $G = C_p$. It is easily verified that any three consecutive vertices of C_p is a minimum total detour monophonic set of C_p and so $dm_t(G) = dm_t(C_p) = 3$. Next, suppose that $G = \overline{K}_2 + H$, where H is a graph of order $p - 2$. Let $V(\overline{K}_2) = \{u_1, u_2\}$. Then for any vertex v of H , the set $S = \{v, u_1, u_2\}$ is a minimum total detour monophonic set of G and so $dm_t(G) = 3$. \square

Theorem 2.14. For the complete bipartite graph $G = K_{r,s}$ ($2 \leq r \leq s$),

$$dm_t(G) = \begin{cases} 3 & \text{if } 2 = r \leq s \\ 4 & \text{if } 3 \leq r \leq s. \end{cases}$$

Proof. Let $U = \{u_1, u_2, \dots, u_r\}$ and $W = \{w_1, w_2, \dots, w_s\}$ be the bipartition of G . We prove this theorem by considering four cases.

Case 1. $2 = r = s$. Then G is the cycle C_4 and by Theorem 2.13, $dm_t(G) = 3$.

Case 2. $2 = r < s$. Then the minimum total detour monophonic set of G is obtained by choosing the two elements from U and any one element from W and so $dm_t(G) = 3$.

Case 3. $3 \leq r \leq s$. If $3 = r = s$, then any minimum total detour monophonic set of G is of the following forms: (i) $U \cup \{w_j\}$ for some $j(1 \leq j \leq 3)$, (ii) $W \cup \{u_i\}$ for some $i(1 \leq i \leq 3)$, or (iii) a set containing any two elements from each of U and W . If $3 = r < s$, then any minimum total detour monophonic set of G is either $U \cup \{w_j\}$ for some $j(1 \leq j \leq s)$,

or any set containing any two elements from each of U and W . Hence in both cases, we have $dm_t(G) = 4$.

Case 4. $4 \leq r \leq s$. Then any minimum total detour monophonic set of G contains any two elements from each of U and W , and hence $dm_t(G) = 4$. \square

Theorem 2.15. *If $G = K_1 + \bigcup m_j K_j$, where $j \geq 2, \sum m_j \geq 2$, then $dm_t(G) = p - 1$.*

Proof. Let $G = K_1 + \bigcup m_j K_j$, where $j \geq 2$ and $\sum m_j \geq 2$. It is clear that G has exactly one cut-vertex which is not the support vertex and all the remaining vertices are extreme vertices. Hence by Theorem 2.4, we have $dm_t(G) = p - 1$. \square

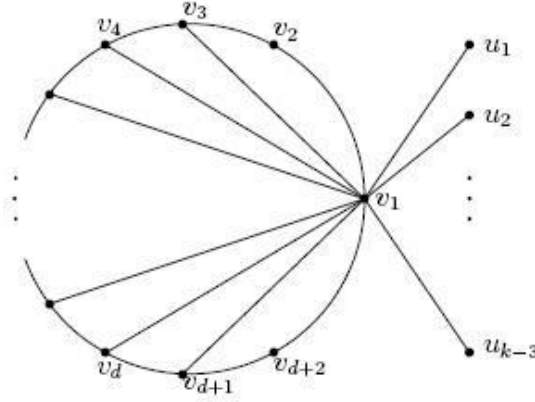
3. Some realization results on the total detour monophonic number

For any connected graph G , $rad_m(G) \leq diam_m(G)$. It is shown in [4] that every two positive integers a and b with $a \leq b$ are realizable as the monophonic radius and monophonic diameter, respectively, of some connected graph. This theorem can also be extended so that the total detour monophonic number can be prescribed when $rad_m(G) < diam_m(G)$.

Theorem 3.1. *For positive integers r, d and $k \geq 6$ with $r < d$, there exists a connected graph G such that $rad_m(G) = r, diam_m(G) = d$ and $dm_t(G) = k$.*

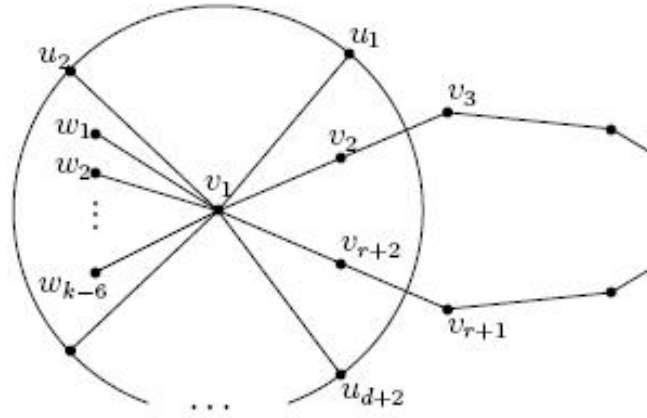
Proof. We prove this theorem by considering two cases.

Case 1. $r = 1$. Then $d \geq 2$. Let $C_{d+2} : v_1, v_2, \dots, v_{d+2}, v_1$ be a cycle of order $d + 2$. Let G be the graph obtained by adding $k - 3$ new vertices u_1, u_2, \dots, u_{k-3} to C_{d+2} and joining each of the vertices $u_1, u_2, \dots, u_{k-3}, v_3, v_4, \dots, v_{d+1}$ to the vertex v_1 . The graph G is shown in Figure 3.1. It is easily verified that $1 \leq e_m(x) \leq d$ for any vertex x in G and $e_m(v_1) = 1, e_m(v_2) = d$. Then $rad_m(G) = 1$ and $diam_m(G) = d$.

Figure 3.1: G

Let $S = \{u_1, u_2, \dots, u_{k-3}, v_2, v_{d+2}, v_1\}$ be the set of all extreme vertices and support vertex of G . By Theorem 2.4, S is a subset of any total detour monophonic set of G . It is clear that S is the total detour monophonic set of G and so $dm_t(G) = k$.

Case 2. $r \geq 2$. Let $C : v_1, v_2, \dots, v_{r+2}, v_1$ be a cycle of order $r + 2$ and let $W = K_1 + C_{d+2}$ be the wheel with $V(C_{d+2}) = \{u_1, u_2, \dots, u_{d+2}\}$. Let H be the graph obtained from C and W by identifying v_1 of C and the central vertex K_1 of W . If d is odd, then add $k - 6$ new vertices w_1, w_2, \dots, w_{k-6} to the graph H and join each $w_i (1 \leq i \leq k - 6)$ to the vertex v_1 and obtain the graph G of Figure 3.2. It is easily verified that $r \leq e_m(x) \leq d$ for any vertex x in G and $e_m(v_1) = r$ and $e_m(u_1) = d$. Thus $rad_m(G) = r$ and $diam_m(G) = d$. Let $S = \{w_1, w_2, \dots, w_{k-6}, v_1\}$ be the set of all extreme vertices and support vertex of G . By Theorem 2.4, every total detour monophonic set of G contains S . It is clear that S is not a total detour monophonic set of G . Also, $S \cup \{x_1, x_2, x_3, x_4\}$ where $x_j (1 \leq j \leq 4) \in V(G) - S$, is not a total detour monophonic set of G . Let $T = S \cup \{u_1, u_2, u_3, v_2, v_3\}$. It is easily verified that T is a total detour monophonic set of G and so $dm_t(G) = k$. If d is even, then add $k - 5$ new vertices w_1, w_2, \dots, w_{k-5} to the graph H and join each $w_i (1 \leq i \leq k - 5)$ to the vertex v_1 and obtain the graph G . Similar to the above argument, $T' = \{w_1, w_2, \dots, w_{k-5}, v_1, v_2, v_3, u_1, u_{\frac{d+4}{2}}\}$ is a minimum total detour monophonic set of G and so $dm_t(G) = k$. \square

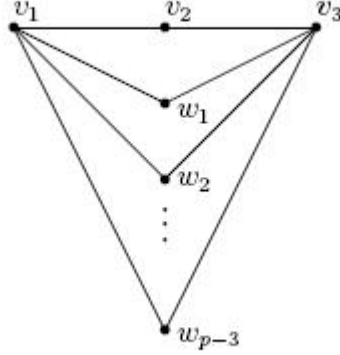
Figure 3.2: G

Problem 3.2. For any three positive integers r , d and $k \geq 6$ with $r = d$ does there exist a connected graph G with $\text{rad}_m(G) = r$, $\text{diam}_m(G) = d$ and $\text{dm}_t(G) = k$?

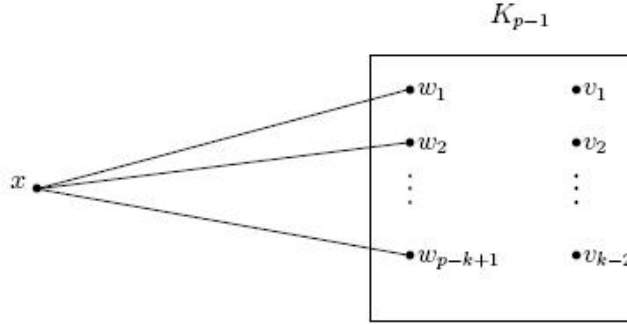
Theorem 3.3. If p, d and k are positive integers such that $2 \leq d \leq p - 2$, $3 \leq k \leq p$ and $p - d - k + 3 \geq 0$, then there exists a connected graph G of order p , monophonic diameter d and $\text{dm}_t(G) = k$.

Proof. We prove this theorem by considering two cases.

Case 1. $d = 2$. First, let $k = 3$. Let $P_3 : v_1, v_2, v_3$ be the path of order 3. Now, add $p - 3$ new vertices w_1, w_2, \dots, w_{p-3} to P_3 . Let G be the graph obtained by joining each w_i ($1 \leq i \leq p - 3$) to v_1 and v_3 . The graph G is shown in Figure 3.3. Then G has order p and monophonic diameter $d = 2$. Clearly $S = \{v_1, v_2, v_3\}$ is a minimum total detour monophonic set of G so that $\text{dm}_t(G) = k = 3$.

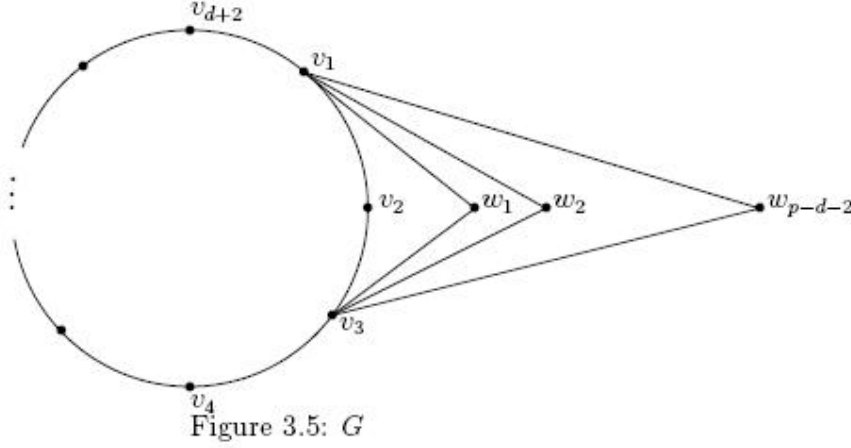
Figure 3.3: G

Now, let $4 \leq k \leq p$. Let K_{p-1} be the complete graph with the vertex set $\{w_1, w_2, \dots, w_{p-k+1}, v_1, v_2, \dots, v_{k-2}\}$. Now, add the new vertex x to K_{p-1} and let G be the graph obtained by joining x with each vertex w_i ($1 \leq i \leq p-k+1$). The graph G is shown in Figure 3.4. Then G has order p and monophonic diameter $d = 2$. Let $S = \{v_1, v_2, \dots, v_{k-2}, x\}$ be the set of all extreme vertices of G . By Theorem 2.4, every total detour monophonic set of G contains S . It is clear that S is a detour monophonic set of G . Since the induced subgraph $G[S]$ has an isolated vertex, $dm_t(G) \geq k$. For any vertex $v \in \{w_1, w_2, \dots, w_{p-k+1}\}$, it is clear that $S \cup \{v\}$ is a minimum total detour monophonic set of G and so $dm_t(G) = k$.

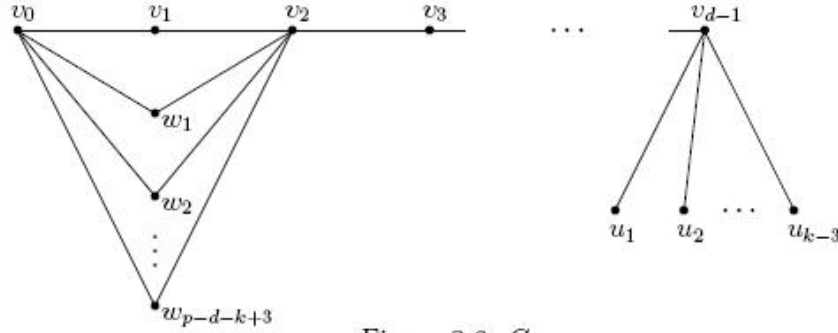
Figure 3.4: G

Case 2. $3 \leq d \leq p-2$. First, let $k = 3$. Let $C_{d+2} : v_1, v_2, \dots, v_{d+2}, v_1$ be the cycle of order $d+2$. Add $p-d-2$ new vertices $w_1, w_2, \dots, w_{p-d-2}$ to C and join each vertex w_i ($1 \leq i \leq p-d-2$) to both v_1 and v_3 , thereby producing

the graph G of Figure 3.5. Then G has order p and monophonic diameter d . It is clear that $S = \{v_3, v_4, v_5\}$ is a minimum total detour monophonic set of G and so $dm_t(G) = 3 = k$.

Figure 3.5: G

Now, let $k \geq 4$. Let $P_d : v_0, v_1, \dots, v_{d-1}$ be a path of order d . Add $p - d$ new vertices $w_1, w_2, \dots, w_{p-d-k+3}, u_1, u_2, \dots, u_{k-3}$ to P_d and join $w_1, w_2, \dots, w_{p-d-k+3}$ to both v_0 and v_2 ; and join u_1, u_2, \dots, u_{k-3} to v_{d-1} , thereby producing the graph G of Figure 3.6.

Figure 3.6: G

Then G has order p and monophonic diameter d . Let $S = \{u_1, u_2, \dots, u_{k-3}, v_{d-1}\}$ be the set of all end-vertices and support vertex of G . By Theorem 2.4, every total detour monophonic set of G contains S . It is clear that S is not a total detour monophonic set of G . Also, for any $x \notin S$, $S \cup \{x\}$ is

not a total detour monophonic set of G . It is easily seen that $S \cup \{v_0, v_1\}$ is a minimum total detour monophonic set of G and so $dm_t(G) = k$. \square

In view of Theorems 2.13, 2.14 and 3.3, we leave the following problem as an open question.

Problem 3.4. Characterize graphs G for which $dm_t(G) = 3$.

In view of Theorem 2.7, we have the following realization result.

Theorem 3.5. If a, b and p are positive integers such that $4 \leq a \leq b \leq p$, then there exists a connected graph G of order p , $dm_t(G) = a$ and $dm_c(G) = b$.

Proof. We prove this theorem by considering four cases.

Case 1. $4 \leq a = b = p$. Let $G = K_p$. Then by Corollary 2.5 and Theorem 1.5, we have $dm_t(G) = dm_c(G) = p$.

Case 2. $4 \leq a < b < p$. Let $P_{b-a+3} : u_1, u_2, \dots, u_{b-a+3}$ be a path of order $b - a + 3$. Add $p - b + a - 3$ new vertices $v_1, v_2, \dots, v_{p-b}, w_1, w_2, \dots, w_{a-3}$ to P_{b-a+3} and join v_1, v_2, \dots, v_{p-b} with both u_1 and u_3 ; and also join each of $w_i (1 \leq i \leq a - 3)$ with u_{b-a+3} , thereby producing the graph G of Figure 3.7. Then G has order p . Let $S = \{w_1, w_2, \dots, w_{a-3}, u_{b-a+3}\}$ be the set of all extreme vertices and support vertex of G . By Theorem 2.4, every total detour monophonic set of G contains S . It is clear that for any $x \notin S$, $S \cup \{x\}$ is not a total detour monophonic set of G . Since $S \cup \{u_1, u_2\}$ is a total detour monophonic set of G , we have $dm_t(G) = a$.



Figure 3.7: G

Let $S_1 = \{w_1, w_2, \dots, w_{a-3}, u_3, u_4, \dots, u_{b-a+3}\}$ be the set of all extreme vertices and cut-vertices of G . By Theorems 1.3 and 1.4, every connected

detour monophonic set of G contains S_1 . It is clear that for any $x \notin S_1$, $S_1 \cup \{x\}$ is not a connected detour monophonic set of G . Let $S'_1 = S_1 \cup \{u_1, u_2\}$. It is easily verified that S'_1 is a connected detour monophonic set of G and so $dm_c(G) = b$.

Case 3. $4 \leq a = b < p$. Let $P_3 : v_1, v_2, v_3$ be a path of order 3. Let H be the graph obtained from P_3 by adding $p - a$ new vertices w_1, w_2, \dots, w_{p-a} to P_3 and join each w_i ($1 \leq i \leq p - a$) to v_1 and v_3 . Also, add $a - 3$ new vertices $u_1, u_2, u_3, \dots, u_{a-3}$ to H and join each u_i ($1 \leq i \leq a - 3$) with v_2 , thereby producing the graph G in Figure 3.8 of order p . Let $S = \{u_1, u_2, \dots, u_{a-3}, v_2\}$ be the set of all extreme vertices and support vertex of G . By Theorems 2.4, 1.3 and 1.4, every total detour monophonic set and every connected detour monophonic set of G contains S . It is clear that for any $x \notin S$, $S \cup \{x\}$ is not a total detour monophonic set of G . Since $S \cup \{v_1, v_3\}$ is a connected detour monophonic set of G and also a total detour monophonic set of G , we have $dm_c(G) = dm_t(G) = a$.

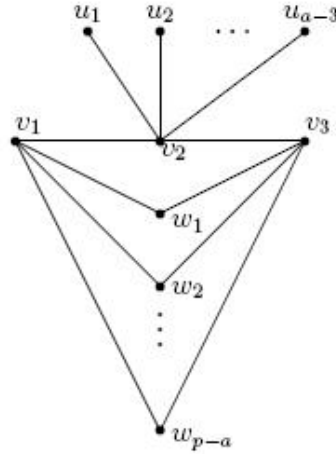


Figure 3.8: G

Case 4. $4 \leq a < b = p$. Let $P_{b-a+3} : u_1, u_2, \dots, u_{b-a+3}$ be a path of order $b - a + 3$. Add $a - 3$ new vertices v_1, v_2, \dots, v_{a-3} to P_{b-a+3} and join each v_i ($1 \leq i \leq a - 3$) with u_{b-a+3} , thereby producing the graph G in Figure 3.9 of order $p = b$. Let $S = \{u_1, v_1, v_2, \dots, v_{a-3}, u_2, u_{b-a+3}\}$ be the set of all extreme vertices and support vertices of G . By Theorem 2.4, every total detour monophonic set of G contains S . It is clear that S is the total detour monophonic set of G and so $dm_t(G) = a$.

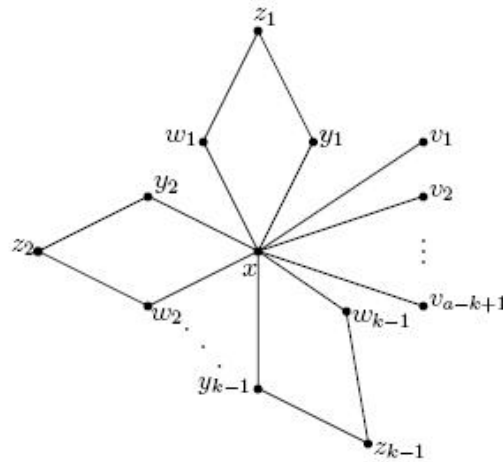
Figure 3.9: G

It is clear that $V(G)$ is the set of all extreme vertices and cut-vertices of G . Then by Theorems 1.3 and 1.4, $V(G)$ is the connected detour monophonic set of G and so $dm_c(G) = b = p$. \square

Theorem 3.6. For positive integers a, b such that $4 \leq a \leq b$ with $b \leq 2a$, there exists a connected graph G such that $dm(G) = a$ and $dm_t(G) = b$.

Proof. **Case 1.** $4 \leq a = b$. Then the complete graph K_a has the desired properties.

Case 2. $4 \leq a < b$. Let $b = a + k$, where $1 \leq k \leq a$. For $k = 1$, the star $K_{1,a}$ has the desired properties. Now, let $k \geq 2$. Let $C_i : x_i, y_i, z_i, w_i, x_i$ ($1 \leq i \leq k - 1$) be “ $k - 1$ ” copies of C_4 . Let H be the graph formed by identifying the vertices x_i from C_i ($1 \leq i \leq k - 1$) and let x be the identified vertex. Let G be the graph obtained from H by adding $a - k + 1$ new vertices $v_1, v_2, \dots, v_{a-k+1}$ and joining each v_i ($1 \leq i \leq a - k + 1$) with x . The graph G is shown in Figure 3.10. Let $S = \{v_1, v_2, \dots, v_{a-k+1}\}$ be the set of all extreme vertices of G . By Theorem 1.1, every detour monophonic set of G contains S . It is clear that $S_1 = S \cup \{z_1, z_2, \dots, z_{k-1}\}$ is the detour monophonic set of G and so $dm(G) = a$.

Figure 3.10: G

Let $S' = S \cup \{x\}$ be the set of all extreme vertices and support vertex of G . By Theorem 2.4, every total detour monophonic set of G contains S' . Also, by Theorem 2.6, every total detour monophonic set of G contains at least one vertex from each $V(C_i) - \{x\}$. Hence $S'' = S' \cup \{y_1, z_1, y_2, z_2, \dots, y_{k-1}, z_{k-1}\}$ is a minimum total detour monophonic set of G so that $dm_t(G) = b$. \square

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