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NONEXISTENCE OF NONTRIVIAL SOLUTIONS FOR AN ASYMMETRIC PROBLEM WITH WEIGHTS

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Abstract

In this paper we establish a nonexistence result for an elliptic problem involving the one-dimentional p-Laplacian operator with asymmetric second member of the equation.

1. Introduction

Let us to consider the one-dimentional asymmetric problem

$$(1.1) \qquad -\Delta_p u = m_1(x)\ u_+^{p-1} - m_2(x)\ u_-^{p-1} \quad \text{in }]0,\mathrm{T}[,$$

where $1<p<\infty$, $(\ m_1, m_2) \in (L^\infty]0,\mathrm{T}[)^2$, $u_\pm = \max(\pm u, 0)$, and -$\Delta_p u = (|u'|^{p-2}u')'$ denotes the one-dimentional p-Laplacian operator .

In this paper we study the nonexistence of nontrivial solutions for (1.1) when the pair $(\ m_1, m_2)$ is in some appropriate sense "between two consicutive curves of the Fucik spectrum"; we investigate three situations: The Dirichlet, Neumann and periodic boundary conditions. We recall that the Fucik spectrum for the corresponding boundary conditions wich is noted by Θ_D (resp.Θ_{Ne}), (resp.Θ_{pe}) is defined as the set of those $(\mu, \nu) \in \mathbf{R}^2$ such that the problem

$$(1.2) \qquad -\Delta_p u = \mu\ u_+^{p-1} - \nu\ u_-^{p-1} \quad \text{in }]0,\mathrm{T}[,$$

under Dirichlet (resp. Neumann) (resp. periodic) boundary conditions has a nontrivial solution. Several works have been devoted to the study of this spectrum and its applications especially in the higher dimension, cf. [3], [4], [5]. Of particular interest for our purposes are those of P.Drábek [4] where Θ_D is competely determined.

In order to prove our results we use some arguments like the property of the nodal set for eigenfunctions and the simplicity of the first eigenvalue λ_1 of the p-Laplacian on $W_0^{1,p}$ (cf.[1]) . We will give more details in the next section.

According to the particular quasilinear case where $m_1 = m_2$, one should refer to the interesting work of Del Pino, El Gueta and Manasevich [6], the authors establish a similar result by using the Sturm's comparaison theorem. I suggest that our envisagement is completely different from their approach. Finally I beleive the related result for periodic case is contained in [7], and one should make an appropriate comparaison.

2. Statements and proofs.

2.1. Curves of the Fucik spectrum and Dirichlet problem.

In this subsection we shall consider the problems (1.1) and (1.2) under the Dirichlet bounary conditions $u(0) = u(T) = 0$. We start by giving the result of P.Drábek [4] on the one-dimentional Fucik spectrum of the p-Laplacian on $W_0^{1,p}(]0,T[)$. According to the definition of Θ_D given in the introduction, directly we have $\Theta_D = \mathbf{R} \times \{\lambda_1\} \cup \{\lambda_1\} \times \mathbf{R}$ $\cup \left(\cup_{n=1}^{\infty}[C_{2n} \cup C_{2n+1}^{-} \cup C_{2n+1}^{+}]\right)$; where

$$C_{2n} = \left\{(\mu,\nu) \in {\mathbf{R}_+^*}^2 \ / \ n\left[\left(\tfrac{\lambda_1}{\mu}\right)^{1/p} + \left(\tfrac{\lambda_1}{\nu}\right)^{1/p}\right] = 1\right\}$$

$$C_{2n+1}^{-} = \left\{\begin{array}{r} (\mu,\nu) \in {\mathbf{R}_+^*}^2 \ / \ n\left[\left(\tfrac{\lambda_1}{\mu}\right)^{1/p} + \left(\tfrac{\lambda_1}{\nu}\right)^{1/p}\right] + \\ + \min\left(\left(\tfrac{\lambda_1}{\mu}\right)^{1/p}, \left(\tfrac{\lambda_1}{\nu}\right)^{1/p}\right) = 1 \end{array}\right\}$$

$$C_{2n+1}^{+} = \left\{\begin{array}{r} (\mu,\nu) \in {\mathbf{R}_+^*}^2 \ / \ n\left[\left(\tfrac{\lambda_1}{\mu}\right)^{1/p} + \left(\tfrac{\lambda_1}{\nu}\right)^{1/p}\right] + \\ + \max\left(\left(\tfrac{\lambda_1}{\mu}\right)^{1/p}, \left(\tfrac{\lambda_1}{\nu}\right)^{1/p}\right) = 1 \end{array}\right\}$$

Where λ_1 represent the first eigenvalue of the p-Laplacian on $W_0^{1,p}(]0,T[)$. If we represent $C_{2n}, C_{2n+1}^{-}, C_{2n+1}^{+}$ as the curves $\nu = f(\mu)$ so we have the following comparaison:

$$C_{2n} < C_{2n+1}^{-} \leq C_{2n+1}^{+} < C_{2n+2}.$$

Theorem 2.1. *Assume that $m_1,\ m_2 \in L^{\infty}(]0,T[)$ and there exists $(\mu_1,\nu_1),\ (\mu_2,\nu_2) \in \mathbf{R}^2$ such that $(\mu_1,\nu_1) \underset{\neq}{\leq} (m_1(x), m_2(x)) \underset{\neq}{\leq} (\mu_2,\nu_2)$. Then no nontrivial solutions for (1.1) exists, if one of the following asymptions is true:*

1. $\mu_1 = \nu_1 = \lambda_1$, *and* $(\mu_2,\nu_2) \in C_2$.
2. $(\mu_1,\nu_1) \in C_{2n}$, *and* $(\mu_2,\nu_2) \in C_{2n+1}^{-}$.
3. $(\mu_1,\nu_1) \in C_{2n+1}^{-} \cap \Delta^{\pm}$, *and* $(\mu_2,\nu_2) \in C_{2n+1}^{+} \cap \Delta^{\pm}$; *where* $\Delta^{+}(resp.\Delta^{-}) = \left\{(\mu,\nu) \in \mathbf{R}^2 \ / \ \mu < \nu \ (resp.\mu > \nu)\right\}.$

4. $(\mu_1, \nu_1) \in C^+_{2n+1}$, *and* $(\mu_2, \nu_2) \in C_{2n+2}$.

Remark 2.1. We define the quasilinear problem associated to (1.1) by substituting m_1 and m_2 in (1.1) by m , so (1.1) is wraten as

$$(1.3) \qquad -\Delta_p u = m(x)|u|^{p-2}u \quad \text{in }]0,\mathrm{T}[.$$

In this particular case the result of [6] remain included in theorem 2.1.

Proof of theorem 2.1. Suppose that there exists $u \neq 0$ solution of (1.1). We apply Anane's L^∞ estimation [2] and Tolksdorf's regularity [8] we have $u \in C^1(]0,\mathrm{T}[)$; and by using the maximum principle of Vasquez [9] it easy to see that if $u(a) = 0$ then $u'(a) \neq 0$, then we deduce $Z(u) = \{x \in [0, \mathrm{T}] \ / \ u(x) = 0\}$ is a finite subset. We take

$$Z(u) = \{ \ x_0 = 0 < x_1 < ... < x_k = \mathrm{T} \ \}.$$

Put $u_i = u/_{[x_i \ , \ x_{i\,+\,1}]}$, $m_{j,i} = m_j/_{[x_i \ , \ x_{i\,+\,1}]}$ for $j = 1, 2$ and $i = 0, ..., k-1$. Since u is a solution of (1.1) under the Dirichlet boundary conditions then u_i is a nontrivial solution for some eigenvalue problem with weight like (1.3) with some simple modifications, and by definition u_i does not changing sign, then from this fact we have by using the simplicity of λ_1 :

$\lambda_1(\ m_{1,i}\ ) = 1$ or $\lambda_1(\ m_{2,i}\ ) = 1$, $\forall\ i = 0, ..., k-1$. More pricisely, if $u/_{]x_0 \ , \ x_1[} > 0$ (this hypothesis will be assumed in all the rest of the proof; the similar argument could be adapted to the case $u/_{]x_0 \ , \ x_1[} < 0$); then

$$\lambda_1(m_{1,2q}) = 1 \quad \text{ and } \quad \lambda_1(m_{2,2q+1}) = 1, ...\forall q \geq 0.$$

by using $(\mu_1, \nu_1) \leq (m_{1,i}(x) \ , \ m_{2,i}(x)) \leq (\mu_2, \nu_2)$ a.e $x \in]0,\mathrm{T}[\ \forall\ i$; the fact that there exists i_0, i_1 such that $(\mu_1, \nu_1) < (m_{1,i_0}(x) \ , \ m_{2,i_0}(x))$ and $(m_{1,i_1}(x) \ , \ m_{2,i_1}(x)) < (\mu_2, \nu_2)$ on some subsets of positive measures; and the monotonicity de λ_1 we get:

$$\lambda_1(\mu_2/_{]\,x_{2q}\,,\,x_{2q+1}\,[}) \underset{1\text{st}}{\leq} \lambda_1(m_{1,2q}) = 1 \underset{2\text{nd}}{\leq} \lambda_1(\mu_1/_{]\,x_{2q}\,,\,x_{2q+1}\,[})$$
$$\lambda_1(\nu_2/_{]\,x_{2q+1}\,,\,x_{2q+2}\,[}) \underset{1\text{st}}{\leq} \lambda_1(m_{2,2q+1}) = 1 \underset{2\text{nd}}{\leq} \lambda_1(\nu_1/_{]\,x_{2q+1}\,,\,x_{2q+2}\,[}),$$

and there exists some integers q such that one of two last firsts (1st) and one of two last seconds (2nd) inequalitys was stricts.

From [4] and for some positive constant α we get,

$$\lambda_1(\alpha/_{]a,b[}) = \frac{1}{\alpha}\left(\frac{\pi_p}{b-a}\right)^p;\ \pi_p = 2(p-1)^p \int_0^1 \frac{ds}{(1-s^p)^{1/p}},$$

then by using the last inequalitys we obtain :

$$\frac{\pi_p}{(\mu_2)^{1/p}} \underset{1\text{st}}{\leq} x_{2q+1} - x_{2q} \underset{2\text{nd}}{\leq} \frac{\pi_p}{(\mu_1)^{1/p}}, \qquad \frac{\pi_p}{(\nu_2)^{1/p}} \underset{1\text{st}}{\leq} x_{2q+2} - x_{2q+1} \underset{2\text{nd}}{\leq} \frac{\pi_p}{(\nu_1)^{1/p}}\ ,$$

and one of those 1st and one of those 2nd last inequalitys was stricts , then by according to this point and by sommons in the last inequalitys from $q = 0$ to $\frac{k-1}{2}$ we give

$$E(\tfrac{k}{2})\left[\tfrac{\pi_p}{(\mu_2)^{1/p}} + \tfrac{\pi_p}{(\nu_2)^{1/p}}\right] + \varepsilon\ \tfrac{\pi_p}{(\mu_2)^{1/p}} < T < E(\tfrac{k}{2})\left[\tfrac{\pi_p}{(\mu_1)^{1/p}} + \tfrac{\pi_p}{(\nu_1)^{1/p}}\right] + \\ +\varepsilon\ \tfrac{\pi_p}{(\mu_1)^{1/p}},$$

where $\varepsilon = 0$ if k is even and $\varepsilon = 1$ if not. So we obtain:

$$E(\tfrac{k}{2})\left[\left(\tfrac{\lambda_1}{\mu_2}\right)^{1/p} + \left(\tfrac{\lambda_1}{\nu_2}\right)^{1/p}\right] + \varepsilon\ \left(\tfrac{\lambda_1}{\mu_2}\right)^{1/p} < 1 < E(\tfrac{k}{2})\left[\left(\tfrac{\lambda_1}{\mu_1}\right)^{1/p} + \left(\tfrac{\lambda_1}{\nu_1}\right)^{1/p}\right] + \\ +\varepsilon\ \left(\tfrac{\lambda_1}{\mu 1}\right)^{1/p}.$$

By using the hypothesis 2) (for exemple) of therem 2.1; the definition of C_{2n}, C^-_{2n+1} and the last inequalitys it was easy to see that $n < E(\frac{k}{2}) < n+1$; wich gives a contradiction. The same conclusion will be established if we consider the hypothesis 1) or 3) or 4) of theorem 2.1. $\square$

2.2. Neumann Problem.

In this subsection we shall consider the problems (1.1) and (1.2) under the Neumann bounary conditions $u'(0) = u'(T) = 0$.

Theorem 2.2. *Assume that* $m_1,\ m_2 \in L^\infty(]0,T[)$ *and there exists* $(\mu_1,\nu_1),\ (\mu_2,\nu_2) \in \mathbf{R}^2$ *such that* $(\mu_1,\nu_1) \underset{\neq}{\leq} (m_1(x),m_2(x)) \underset{\neq}{\leq} (\mu_2,\nu_2)$. *Then no nontrivial solutions for (1.1) exists, if one of the following asymptions is true:*

1. $\mu_1 = \nu_1 = 0$, *and* $(\mu_2,\nu_2) \in \overline{C}_1$.
2. $(\mu_1,\nu_1) \in \overline{C}_n$, *and* $(\mu_2,\nu_2) \in \overline{C}_{n+1} \quad (n \geq 1)$.

Where $\overline{C}_n = \left\{(\mu,\nu) \in \mathbf{R}_+^{*\,2} \ / \ n\left[\left(\frac{\lambda_1}{\mu}\right)^{1/p} + \left(\frac{\lambda_1}{\nu}\right)^{1/p}\right] = 2\right\}$.

Proof. Suppose that there exists $u \neq 0$ solution de (1.1). Then by according to the beginning of the proof of theorem 2.1 we get $u(0) \neq 0$, $u(T) \neq 0$ and $Z(u)$ is a finite subset. Let $Z(u) = \{x_1 < ... < x_k\}$. We define

$\overline{u}(x) = u(-x)$ if $-x_1 \leq x \leq 0$,
$\overline{u}(x) = u(x)$ if $0 \leq x \leq$T,
$\overline{u}(x) = u(2\mathrm{T}-x)$ if T$\leq x \leq$ 2T-x_k ,

and the analogous definition for $\overline{m}_1$ and $\overline{m}_2$. It easy to see that $\overline{u}$ is a solution of a Dirichlet problem like (1.3) with domain $]-x_1, 2\mathrm{T}-x_k[$ and weights $(\overline{m}_1, \overline{m}_2)$.

Without loss of generaliy we can assume that $u/_{]0\ ,\ x_1[} > 0$; and in order to simplify this proof we shall us simply the following hypothesis (of therem 2.2):

$$\mu_1 < m_1(x) < \mu_2 \quad \text{and} \quad \nu_1 < m_1(x) < \nu_2 \quad \text{a.e } x \in]0, \mathrm{T}[.$$

We proceed as in the proof of theorem 2.1, then we obtain:

1. $\frac{\pi_p}{(\mu_2)^{1/p}} < 2x_1 < \frac{\pi_p}{(\mu_1)^{1/p}}$

2. $\frac{\pi_p}{(\mu_2)^{1/p}} < x_2 - x_1 < \frac{\pi_p}{(\mu_1)^{1/p}}$ ....

k. $\frac{\pi_p}{(\nu_2)^{1/p}} < x_{\text{k}} - x_{\text{k-1}} < \frac{\pi_p}{(\nu_1)^{1/p}}$ if k is even ; $\frac{\pi_p}{(\mu_2)^{1/p}} < x_{\text{k}} - x_{\text{k-1}} < \frac{\pi_p}{(\mu_1)^{1/p}}$ if not

k+1. $\frac{\pi_p}{(\mu_2)^{1/p}} < 2\text{T} - 2x_{\text{k}} < \frac{\pi_p}{(\mu_1)^{1/p}}$ if k is even ; $\frac{\pi_p}{(\nu_2)^{1/p}} < 2\text{T} - 2x_{\text{k}} < \frac{\pi_p}{(\nu_1)^{1/p}}$ if not.

We multiply the equations 2 to k by 2 and we sum from 1 to $k+1$, wich gives:

$$k\left[\frac{\pi_p}{(\mu_2)^{1/p}} + \frac{\pi_p}{(\nu_2)^{1/p}}\right] < 2T < k\left[\frac{\pi_p}{(\mu_1)^{1/p}} + \frac{\pi_p}{(\nu_1)^{1/p}}\right],$$

so

$$k\left[\left(\frac{\lambda_1}{\mu_2}\right)^{1/p} + \left(\frac{\lambda_1}{\nu_2}\right)^{1/p}\right] < 2 < k\left[\left(\frac{\lambda_1}{\mu_1}\right)^{1/p} + \left(\frac{\lambda_1}{\nu_1}\right)^{1/p}\right];$$

since $(\mu_1, \nu_1) \in \overline{C}_n$, and $(\mu_2, \nu_2) \in \overline{C}_{n+1}$ then it was easy to see that $n < k < n+1$, wich gives a contradiction. □

Corollary 2.1. The one-dimensional Fucik spectrum for the p-Laplacian with Neumann boundary conditions is: $\Theta_{\text{Ne}} = \mathbf{R} \times \{0\} \cup \{0\} \times \mathbf{R} \cup \left(\cup_{n=1}^{\infty} \overline{C}_n\right)$.

So we deduce the usuel one-dimensional spectrum for the p-Laplacian with Neumann boundary conditions is given by :

$\sigma_{\text{Ne}}(-\Delta_p) = \{n^p \lambda_1 \ / \ n \geq 0\}$.

Remark 2.2. If $(\mu, \nu) \in \overline{C}_1$ then $\lim\limits_{\mu \to +\infty} \nu(\mu) = \frac{\lambda_1}{2^p} > 0$; (here 0 represent the first eigenvalue of the p-Laplacian with Neumann boundary conditions). In [3] the authors show that there exists a connection between this asymptotic behaviour of $\overline{C}_1$ and the antimaximum principle; but if the dimension is larger then 2 and $p = 2$ so we have $\overline{C}_1$ is asymptotic to 0 (cf.[5]). In the Dirichlet case the first curve of the Fucik spectrum is always asymptotic to λ_1.

2.3. Periodic Problem.

In this subsection we shall consider the problems (1.1) and (1.2) under periodic bounary conditions: $u(0) - u(T) = u'(0) - u'(T) = 0$.

Theorem 2.3. *Assume that* m_1, $m_2 \in L^\infty(]0,T[)$ *and there exists* (μ_1,ν_1), $(\mu_2,\nu_2) \in \mathbf{R}^2$ *such that* $(\mu_1,\nu_1) \underset{\neq}{\leq} (m_1(x), m_2(x)) \underset{\neq}{\leq} (\mu_2,\nu_2)$ *a.e* $x \in]0,\mathrm{T}[$. *Then no nontrivial solutions for (1.1) exists, if one of the following asymptions is true:*

1. $\mu_1 = \nu_1 = 0$, *and* $(\mu_2,\nu_2) \in \widetilde{C}_1$.
2. $(\mu_1,\nu_1) \in \widetilde{C}_n$, *and* $(\mu_2,\nu_2) \in \widetilde{C}_{n+1} \quad (n \geq 1)$.

Where $\widetilde{C}_n = \left\{ (\mu,\nu) \in \mathbf{R}_+^{*\,2} \;/\; n\left[\left(\frac{\lambda_1}{\mu}\right)^{1/p} + \left(\frac{\lambda_1}{\nu}\right)^{1/p} \right] = 1 \right\}$.

Proof. Suppose that there exists $u \neq 0$ solution of (1.1) . Put $Z(u) = \{x_1 < ... < x_k\}$. We define:

$\tilde{u}(x) = u(x+\mathrm{T})$ if $x_k - \mathrm{T} \leq x \leq 0$,
$\tilde{u}(x) = u(x)$ if $0 \leq x \leq \mathrm{T}$,
$\tilde{u}(x) = u(2\mathrm{T} - x)$ if $\mathrm{T} \leq x \leq \mathrm{T} + x_1$

and the analogous definition for $\widetilde{m}_1$ and $\widetilde{m}_2$. It easy to see that $\tilde{u}$ is a nontrivial solution for a Dirichlet problelm like (1.3) with domain $]x_k - \mathrm{T}\ ,\ \mathrm{T} + x_1[$ and weights ($\widetilde{m}_1$, $\widetilde{m}_2$).

In order to simplify this proof we shall assume

$0 < x_1 < x_k < \mathrm{T}$, $(\mu_1,\nu_1) < (m_1(x), m_2(x)) < (\mu_2,\nu_2)$ a.e, and $u/_{]0\ ,\ x_1[} > 0$.

In the fact that $u'(0) - u'(T) = 0$ we obtain that k is even , so by a similar arguments of the proof of theorem 2.2 and by using the hypothesis 2) of theorem 2.3 we get:

1. $\frac{\pi_p}{(\mu_2)^{1/p}} < \mathrm{T} + x_1 - x_{\mathrm{k}} < \frac{\pi_p}{(\mu_1)^{1/p}}$

2. $\frac{\pi_p}{(\mu_2)^{1/p}} < x_2 - x_1 < \frac{\pi_p}{(\mu_1)^{1/p}}$ ....

k. $\frac{\pi_p}{(\nu_2)^{1/p}} < x_{\text{k}} - x_{\text{k-1}} < \frac{\pi_p}{(\nu_1)^{1/p}}$

k+1. $\frac{\pi_p}{(\mu_2)^{1/p}} < \mathrm{T} + x_1 - x_{\text{k}} < \frac{\pi_p}{(\mu_1)^{1/p}}$

we multiply the equations 2 to k by 2 and we sum from 1 to $k+1$, we have

$$\frac{k}{2}\left[\left(\frac{\lambda_1}{\mu_2}\right)^{1/p} + \left(\frac{\lambda_1}{\nu_2}\right)^{1/p}\right] < 1 < \frac{k}{2}\left[\left(\frac{\lambda_1}{\mu_1}\right)^{1/p} + \left(\frac{\lambda_1}{\nu_1}\right)^{1/p}\right];$$

Since $(\mu_1, \nu_1) \in \widetilde{C}_n$, and $(\mu_2, \nu_2) \in \widetilde{C}_{n+1}$ then it easy to see that $n < \frac{k}{2} < n+1$ wich gives a contradiction because k is even. $\square$

Corollary 2.2. The one-dimensional Fucik spectrum for the p-Laplacian with periodic boundary conditions is: $\Theta_{\text{pe}} = \mathbf{R} \times \{0\} \cup \{0\} \times \mathbf{R} \cup \left(\cup_{n=1}^{\infty} \widetilde{C}_n\right)$.

So we deduce the usuel one-dimensional spectrum for the p-Laplacian with periodic boundary conditions is given by :

$\sigma_{\text{pe}}(-\Delta_p) = \{(2n)^p \lambda_1 \ / \ n \geq 0\}$.

REFERENCES.

[1] **A. Anane**, Simplicité et isolation de la première valeure propre du p-laplacien avec poids, C. R. Acad. Sci. Paris , t. 305, pp 725-728, (1987).

[2] **A. Anane,** Etude des valeurs propres et de la résonance pour l'opérateur p-laplacien , thèse de Doctorat, Université Libre de Bruxelles, (1988).

[3] **M. Arias, J. Campos & J.-P. Gossez**, On the antimaximum principle and the Fucik spectrum for the Neumann p-Laplacien, (to appear in Diff. Int. Equa.).

[4] **P. Drábek**. Solvability and bifurcations of nonlinear equations, Pitman Resarch Notes in Mathematics, 264 (1992).

[5] **D. G. de Figueredo & J.-P. Gossez**, On the first curve of the Fucik spectrum of an elliptic operator, Diff. Int. Equa., volume 7, number 5, pp-1285-1302, (1994).

[6] **M. del Pino, M. Elgueta & R. Manasevich**, A homotopic deformation along p of a Leray-Shauder degree result and existence for $(|u'|^{p-2}u')' + f(t,u) = 0$, $u(0) = u(T) = 0$, $p > 1$. J. Diff. Eq. , 80, pp 1-13, (1989).

[7] **Del Pino, Manasevich & Murua. .....,**Nonlinear Analysis **18**, pp **79-92**, (1992)

[8] **P. Tolksdorf,** Regularity for more general class of quasilinear elliptic equation, J. Diff. Eq. , 8, pp 773-817, (1983).

[9] **J. L.Vasquez**, A strong maximum principle for quasilinear equations, Appl. Math. Optim., 12, pp 191-202, (1984).

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