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# ON THE HYPERBOLIC DIRICHLET TO NEUMANN FUNCTIONAL IN ABELIAN LIE GROUPS *

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**Abstract**

*We prove the injectivity of the linearization of the hyperbolic Dirichlet to Neumann functional in a "small" compact neighborhood of the identity element $e$ of an abelian Lie group $G$, under some suitable transversality condition.*

**AMS subject classification :** *35L20, 58G20.*

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## 1. Introduction and Statement of the Result

Let $\overline{\Omega}$ be a compact manifold of dimension $n$ with smooth boundary $\partial\Omega$ and let $\mathrm{Met}(\overline{\Omega})$ denote the set of all Riemannian metrics $g$ on $\overline{\Omega}$.

We consider the ansitropic wave equation

$$\begin{aligned} \square_g u &= \frac{\partial^2 u}{\partial t^2} - \Delta_g u = 0 \text{ in } \Omega \times (0,T), \\ u &= f \text{ on } \Gamma = \partial\Omega \times (0,T),\ f \in C_0^\infty(\Gamma), \\ u &= \frac{\partial u}{\partial t} = 0 \text{ in } \Omega \times \{0\}. \end{aligned} \tag{1.1}$$

There is a unique solution to (1.1); hence we may define the hyperbolic Dirichlet to Neumann map as the linear operator $\Lambda_g : C_0^\infty(\Gamma) \longrightarrow C^\infty(\Gamma)$, given by

$$\Lambda_g f = du \cdot \nu_g \Big|_\Gamma = \frac{\partial u}{\partial \nu_g}\Big|_\Gamma \tag{1.2}$$

where $u$ is the unique solution to (1.1) and $v_g$ is the $g$-interior unit normal to $\partial\Omega$. The hyperbolic Dirichlet to Neumann Functional:

$$\begin{aligned} \Lambda\colon \quad \mathrm{Met}(\overline{\Omega}) &\longrightarrow \mathrm{O}_p(\Gamma) \\ g &\mapsto \Lambda_g, \end{aligned} \tag{1.3}$$

where $\mathrm{O}_p(\Gamma)$ denotes the space of all linear operators from $C_0^\infty(\Gamma)$ into $C^\infty(\Gamma)$, is known to be invariantly defined on the orbit obtained by the action over $\mathrm{Met}(\overline{\Omega})$ of the group $\mathcal{D}$ of all diffeomorphism $\psi$ of $\overline{\Omega}$, each of which restricts to the identity on $\partial\Omega$. A natural conjecture is that this is the only obstruction to the uniqueness of $\Lambda$. From the point of view of applications, an even more important problem is to give a method to reconstruct $g$ from $\Lambda_g$.

The elliptic Dirichlet to Neumann map was treated by several authors and is closely related to the physical problem referred as Electrical Impedance Tomography, of determining the conductivity of a body from measurement of voltage potential and corresponding current fluxes at the boundary (see [4]).

For fixed $g$, we consider the following map:

$$\psi \in \mathcal{D} \xrightarrow{A_g} \psi_g^* \in \mathrm{Met}(\overline{\Omega}) \tag{1.4}$$

It is easy to see that the tangent space $T_I\mathcal{D}$ of $\mathcal{D}$ at the identity mapping $I$ is the vector space $\Gamma_0(T\overline{\Omega})$ of all smooth vector fields on $\overline{\Omega}$ which vanish on $\partial\Omega$. On the other hand, the tangent space $T_g(\mathrm{Met}(\overline{\Omega}))$ of $\mathrm{Met}(\overline{\Omega})$ at $g$ is the vector space $\Gamma(S^2\overline{\Omega})$ of all smooth sections of symmetric 2-tensors on $\overline{\Omega}$. We introduce on the spaces $\Gamma_0(T\overline{\Omega})$ and $\Gamma(S^2\overline{\Omega})$ the inner products

$$\langle X, Y\rangle = \int_{\overline{\Omega}} g(X,Y)v_g, \qquad X, Y \in \Gamma_0(T\overline{\Omega}), \tag{1.5}$$

$$\langle\langle m, l\rangle\rangle = \frac{1}{n}\int_{\overline{\Omega}} tr(\hat{m}\circ\hat{l})v_g, \qquad m, l \in \Gamma(S^2\overline{\Omega}), \tag{1.6}$$

where $v_g$ (resp., $tr$), denote the volume element (resp., the trace) associated to $g$ and $\hat{m}$ is the unique linear map defined by

$$g(\hat{m}u, v) = m(u,v) \text{ for all } u, v \in \Gamma(T\overline{\Omega}). \tag{1.7}$$

Considerer as in [3], the formal linearizations of $A_g$ at $I$ and of $\Lambda$ at $g$, respectively :

$$A'_g[I] := A'_g : \Gamma_0(T\overline{\Omega}) \longrightarrow \Gamma(S^2\overline{\Omega}), \tag{1.8}$$

and

$$\Lambda'_g : \Gamma(S^2\overline{\Omega}) \longrightarrow \mathrm{O}_p(\Gamma). \tag{1.9}$$

Let $(A'_g)^*$ denote the formal adjoint of $A'_g$ with respect to the inner products (1.5) and (1.6) and $diam_g(\Omega)$ the diameter of $\Omega$ in the metric $g$. In [3], the authors stated the following :

**Conjecture 1.1** *Let $\Omega_0 \subset \Omega$ be a submanifold, $m \in \Gamma(S^2\overline{\Omega})$ have support in $\overline{\Omega}_0$ i.e., $m \in \Gamma_0(S^2\overline{\Omega})$ and assume that a) $\Lambda'_g(m) = 0$, b) $(A'_g)^*(m) = 0$ and c) $\mathrm{diam}_g(\Omega_0) < T$ is sufficiently small that the exponential map for $g$ is a global diffeomorphism. Then $m$ is identically zero.*

As in [3] we refer to condition b) as the Transversality Condition. We remind that condition c) is necessary to avoid the appearance of caustics. The Tranversality Condition replace the harmonic hypothese used in [4].

The main results of this paper is:

**Theorem 1.1** *Conjecture 1.1 holds if* $G$ *be an abelian Lie group and* $\overline{\Omega}$ *is a compact neighborhood of* $G$ *at the identity element* $e$*, and* $g$ *is an invariant metric on* $G$ *and* $m \in \Gamma_0(S^2\overline{\Omega})$.

We are able to obtain an generalization Cardozo-Mendoza Theorem 1.1 [3] as corollary of Theorem 1.1.

**Corollary 1.1** *Conjecture 1.1 holds if* $\overline{\Omega}$ *is a bounded domain of* $\mathbb{R}^{\mathrm{n}}$*, and* $g$ *is a metric on* $\mathbb{R}^{\mathrm{n}}$ *and* $m \in \Gamma_0(S^2\overline{\Omega})$.

**Corollary 1.2** *Conjecture 1.1 holds if* $\overline{\Omega}$ *is a bounded domain of the Torus* $T^n$*, and* $g$ *is an invariant metric on* $T^n$ *and* $m \in \Gamma_0(S^2\overline{\Omega})$.

**Corollary 1.3** *Conjecture 1.1 holds if* $\overline{\Omega}$ *is a bounded domain of the product* $\mathbb{R}^{\mathrm{n}} \times \mathrm{T}^{\mathrm{m}}$ *and* $g$ *is an invariant metric on* $\mathbb{R}^{\mathrm{n}} \times \mathrm{T}^{\mathrm{m}}$ *and* $m \in \Gamma_0(S^2\overline{\Omega})$.

**Remark 1.1** *In* [3]*, the authors proved that Conjecture 1.1 holds if* $\overline{\Omega}$ *is a bounded domain of* $\mathbb{R}^{\mathrm{n}}$, $\mathrm{n} \geq 2$ *and* $g$ *is the Euclidean metric. They also proved the conjecture when n=2 and* $g$ *is near the Euclidean metric in the* $C^3$*-topology. In* [1] *they prove that Conjecture 1.1 holds if* $\overline{\Omega}$ *is a bounded domain of the hyperbolic space (resp.,n-sphere) and* $g$ *is the canonical metric in this spaces. In* [2]*, the authors proved the uniqueness conjecture for the case when the manifold is a sufficiently small bounded domain of* $\mathbb{R}^3$*, under suitable geometric conditions and the metric* $g$ *is* $C^3$*-close to Euclidean metric. We shall make use of the invariant formulas for* $A'_g, (A'_g)^*$ *and* $\Lambda'_g$ *proved in Section 2* [3].

## 2. Proof of Theorem 1.1

Let $G$ be an abelian Lie group, and Lie$(G)$ the algebra formed by the set of all left invariant vector fields on $G$, and $\overline{\Omega}$ is a compact neighborhood of $G$ at the identity element $e$, and $\Gamma_0(S^2\overline{\Omega})$ denote the vector space of all smooth sections of symmetric 2-tensors on $G$ which are supported on $\overline{\Omega}$.

**Proof of Theorem 1.1** Let $x \in \Omega$, $v \in Lie(G)$. The geodesic with initial tangent vector $(x, v) \in TG \cong G \times Lie(G)$ is given by $\gamma(t) = x\ exp(tv)$. Since $x \in \Omega$ there exists an element $A \in Lie(G)$ such that $x = exp(A)$.

Let $\{E_\mu\}_{\mu=1,\ldots,n}$ a orthonormal frame field in $\Gamma(TG)$ then we can define a orthonormal system $\{\tilde{E}_\mu, V_\mu\}_{\mu=1,\ldots,n}$ in $\Gamma(TTG)$ as follows. We consider the diagram

$$\begin{array}{ccc} TG & \xrightarrow{\tilde{E}_\mu, V_\mu} & TTG \\ \pi \downarrow & & \downarrow \pi \\ G & \xrightarrow{E_\mu} & TG \end{array}$$

where $\tilde{E}_\mu(x, v) = (x, v, E_\mu, 0)$ and $V_\mu(x, v) = (x, v, 0, E_\mu.)$
Let $x = (x_1, ..., x_n)$ the canonical coordinates system (of the first kind) of $G$ around the identity $e$ associated to the orthonormal frame field $\{E_\mu\}_{\mu=1,\ldots,n}$and $v = (v_1, ..., v_n)$ the canonical coordinates of Lie$(G)$.
Now we consider the function $F_t : TG \longrightarrow I\!R$ defined by

$$F_t(x, v) = m(x\ \exp(tv))(v, v)$$

and the operator

$$\mathcal{L} = \sum_{\mu=1}^{n} \tilde{E}_\mu V_\mu \in \mathrm{End}(C^\infty(TG)).$$

We obtain

$$\mathcal{L}F_t = (-t)\Delta_g F_t + 2 \sum_{\mu,\beta=1}^{n} v_\mu E_\beta m_{\mu\beta}(\gamma), \tag{2.1}$$

where $\Delta_g$ denotes the Laplace-Beltrami operator associated to the metric $g$ and $m_{\mu\beta} = m(E_\mu, E_\beta)$.

On the other hand, the Transversality Condition satisfied by $m$ means that

$$E_1(m_{\mu 1}) + ... + E_n(m_{\mu n}) = 0\ , \quad \mu = 1, ..., n \tag{2.2}$$

Using (2.1) and (2.2) we conclude that

(2.3) $$\mathcal{L}F_t = (-t)\Delta_g F_t.$$

We define

(2.4) $$H^k(x,v) = \int_{-\infty}^{\infty} (-t)^k F_t(x,v)\, dt,$$

for all $k$ = 0,1,...

Using (2.3), it is easy to show that

(2.5) $$\mathcal{L}H^k = \Delta_g H^{k+1}\ , \quad k = 0, 1, 2, ...$$

If we fix $v$, since $m$ is supported in $\overline{\Omega}$, it follows that $H^k(\cdot, v)$ for all $k$ = 1,2,... vanish on a non-empty open subset of $G$. From Proposition 3.1 in [3], $H^0 = 0$ and hence $H^1$, and in fact all $H^k$ are harmonic functions. Therefore $H^k$ is identically zero for all $k$ = 0,1,2,... .

Observe that $t \to F_t$ is supported in some closed interval $[a, b]$ containing zero. Since the subalgebra generated by the family of functions $\{(-t)^k\}_{k=0.1...}$ is dense in $C([a, b], I\!R)$, we obtain that $t \to F_t$ is identically zero. If we take $t = 0$ and recall that $m$ is symmetric, we conclude that $m$ is identically zero. ■

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