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A COMPACTNESS EMBEDDING LEMMA, A PRINCIPLE OF SYMMETRIC CRITICALITY AND APPLICATIONS TO ELLIPTIC PROBLEMS *

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Abstract

In this paper, we present a compactness embedding theorem of a subspace of a the Sobolev space $W^{1,p}(\mathbf{R}^{\mathbf{N}})$ *into* $L^s(\mathbf{R}^{\mathbf{N}})$ *spaces, formed by functions* $u(x,y)$ *which are spherically symmetry with respect to first variable and* $\int a(z)|u|^p dz < \infty$*, where* a *is a continuous function satisfying* $a(z) > a_0 > 0$ *if* $|z| \geq R$*, for a large* $R > 0$ *and such that* $\lim\limits_{|y|\to\infty} a(x,y) = +\infty$ *uniformly for* x*. Moreover, we give two applications to elliptic problems.*

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1. Introduction

In this article we will prove a compactness embedding theorem of a certain subspace of the Sobolev space $W^{1,p}(\mathbf{R})^N$ into $L^s(\mathbf{R})^N$ spaces, for some $s > 0$. These results are very important and appear when one deals with some kind of elliptic problems, two of which we present in Section 4. The obstacle one faces when dealing these specific kind of theorems is the lack of compactness, since we shall work with functions defined in the whole $\mathbf{R})^N$ and with the critical Sobolev exponent $p^* = Np/(N-p)$ of the immersion $W^{1,p}(\mathbf{R}^N) \hookrightarrow L^P(\mathbf{R}^N)$.

In order to state more precisely our result, we consider a non-negative continuous function $a : \mathbf{R}^N = \mathbf{R}^L \times \mathbf{R}^M \to \mathbf{R}(L \geq 2)$ satisfying the following assumptions:

$(a_o) a(x,y) \geq a_o > 0$ if $|(x,y)| \geq R$,for a large $R > 0$;

$(a_1) a(x,y) \to +\infty$ when $|y| \to +\infty$ uniformly for $x \in \mathbf{R}^L$;

(a_2) $a(x,y) = a(x',y)$ for all $x,\ x' \in \mathbf{R}^L$ with $|x| = |x'|$ and all $y \in \mathbf{R}^M$.

Let us define the real $W^{1,p}-$subspace

$$\widetilde{E} = \{u \in W^{1,p}(\mathbf{R}^L \times \mathbf{R}^M) : u(x,y) = u(x',y)\, , x, x' \in \mathbf{R}^L,\ |x| = |x'|, \\ y \in \mathbf{R}^M\}.$$

and the reflexive space

$$E_a = \{u \in \widetilde{E} : \int_{\mathbf{R}^L \times \mathbf{R}^M} a(z)|u|^p dz < \infty\},\ z = (x,y),$$

endowed with the corresponding norm

$$||u||_a^p = \int_{\mathbf{R}^L \times \mathbf{R}^M} |\nabla u|^p dz + \int_{\mathbf{R}^L \times \mathbf{R}^M} a(z)|u|^p dz$$

The main result of this paper is the following

Theorem 1.1. If $(a_0) - (a_2)$ hold then the Banach space E_a is continuously immersed in $L^s(\mathbf{R}^N)$ if $p \leq s \leq p^*$, and compactly immersed if $p < s < p^*$.

Lions in [18], extends a compactness result due to Strauss (see [24]). He showed that if $\Omega \subset \mathbf{R}^M$ is a bounded domain and $q \in (p, pN/(N-p))$, then the restriction to $W^{1,p}_{0,r}(\mathbf{R}^L \times \Omega)$ of the Sobolev embedding from $W^{1,p}_0(\mathbf{R}^L \times \Omega)$ to $L^q(\mathbf{R}^L \times \Omega)$ is compact, where $W^{1,p}_{0,r}(\mathbf{R}^L \times \Omega)$ is the closed subspace of $W^{1,p}_0(\mathbf{R}^L \times \Omega)$ consisting of functions $u \in W^{1,p}_0(\mathbf{R}^L \times \Omega)$ which are spherically symmetric with respect to the first variable.

Recently, Costa in [12] (see also [13, 21]) proved a result like the above theorem under the assumption that the function a is coercive, i.e., $a(z) \to +\infty$as $\mid z \mid \to +\infty$. Therefore, Theorem 1.1 complements these compactness results mentioned above.

2. A compactness embedding result - Proof of Theorem 1

In what follows C will denote a generic positive constant.

First of all, we are going to prove that if condition (a_0) holds then the Banach space E_a is continuously immersed in $L^s(\mathbf{R}^N)$ for all $s \in [p, p^*]$. Notice that (a_0) yields that

$$\begin{aligned}\int_{\mathbf{R}^L \times \mathbf{R}^M} |u|^p dz \leq \int_{\{|(x,y)| \leq R\}} |u|^p dz + \\ a_o^{-1} \int_{\{|(x,y)| \geq R\}} a(z)|u|^p dz \leq C||u||_a^p,\end{aligned} \tag{2.1}$$

where we have used the continuity of the Sobolev embedding for bounded domains. On the other hand, the Sobolev-Gagliardo-Nirenberg inequality asserts that there exists positive constant S such that

$$\int_{\mathbf{R}^L \times \mathbf{R}^M} |u|^{p^*} dz \leq S \int_{\mathbf{R}^L \times \mathbf{R}^M} |\nabla u|^p dz. \tag{2.2}$$

Therefore, from inequalities (2.1) and (2.2) we have the continuity of the embedding for $s = p$ and $s = p^*$. The continuity of the immersion for a fixed $s \in (p, p^*)$, follows from the following interpolation inequality

$$||u||_{L^s} \leq ||u||_{L^p}^{1-t} ||u||_{L^{p^*}}^t \tag{2.3}$$

where $0 \leq t \leq 1$ is such that $1/s = (1-t)/p + t/p^*$.

Now, assuming $(a_0) - (a_2)$, we are going to prove the compactness of the embedding of the spaces $E_a \hookrightarrow L^s(\mathbf{R}^N)$ for $s \in (p, p^*)$. For this purpose it suffices to show that for any weakly convergent sequence $u_k \rightharpoonup 0$ in E_a, as $k \to \infty$, we have that, up to subsequences, the convergence holds strongly in $L^s(\mathbf{R}^L \times \mathbf{R}^M)$. Since (a_1) holds we may choose an unbounded increasing sequence of positive real numbers (r_n) such that for all $x \in \mathbf{R}^L$ we have $a(x,y) > n$, if $|y| \geq r_n$. So,

$$\begin{aligned} \int_{\mathbf{R}^L \times \{|y| \geq r_\mathbf{n}\}} |u_k|^p dz &\leq \tfrac{1}{n} \int_{\mathbf{R}^L \times \{|y| \geq r_\mathbf{n}\}} a(x,y)|u_k|^p dz \\ &\leq \tfrac{\|u_k\|_a^p}{n} \leq \tfrac{C}{n}, \end{aligned} \tag{2.4}$$

for all natural numbers k and n. This last inequality together with the interpolation inequality (2.3) yields that

$$\int_{\mathbf{R}^\mathbf{L} \times \{|y| \geq r_\mathbf{n}\}} |u_k|^s dz \leq \frac{C}{n^{s(1-t)/p}}. \tag{2.5}$$

On other hand, using a result due to Lions [18] (Lemma IV.2), for each natural number j we have a subsequence (u_{k_j}) of (u_k) such that

$$\int_{\mathbf{R}^\mathbf{L} \times \{|y| \leq r_j\}} |u_{k_j}|^s dz \leq \frac{1}{j}. \tag{2.6}$$

The proof follows by estimates (2.5), (2.6) and a diagonal type argument. □

3. A principle of symmetric criticality

Now we state a version for reflexive Banach spaces, of the well known principle of symmetric criticality due to Palais [22]. Let E be a reflexive Banach space (endowed with the norm $|\cdot|$) such that: "for each $S \in E^*$,

(3.1) there exists a unique $u_o \in E$ such that $(fu) = |u|^2 = |S|^2_{E^*}$"

(here $(\cdot,\cdot)$ denotes the bracket duality in $E^* \times E$ and E^* denotes the dual space of E). For example: Hilbert spaces, uniformly convex spaces and Sobolev spaces $W^{1,p}$, with $1 < p < +\infty$, satisfy unicity condition (3.1). Suppose that G is a subgroup of isometries $g : E \to E$. Consider the $G-$invariant closed subspace $\Sigma = \{u \in E : \mathrm{g}u = \mathrm{g}(u) = u$, for all $g \in G\}$.

Let f be a continuous functional on E such that

$$(f, u) = 0, \ u \in \Sigma$$

and f is invariant under the action of G, that is, $f \circ g = f$, for all $g \in G$ In this case $\Sigma \subset \mathrm{Ker} f$. We claim that under the above assumptions $f \equiv 0$. On the contrary way, since (3.1) holds, there exists a unique $v_o \in E \setminus \{0\}$ such that $(f, v_o) = |v_o|^2$ and $|f|_{E^*} = |v_o|$. Notice that

$$(f, gv_o) = (f \circ g, v_o) = (f, v_o) = |v_o|^2 = |gv_o|^2$$

and $|f|_{E^*} = |v_o| = |gv_o|$ for all isometry $g \in G$. By the unicity of v_o we have $gv_o = v_o$ for all $g \in G$ which implies that $v_o \in \Sigma \subset \mathrm{Ker}\ f$, contrary to the fact that $v_o \neq 0$.

We are now able to prove the proposition.

Proposición 3.1. Let E, G and Σ be as above and I be a C^1-functional defined on E such that $I \circ g = I$, for all $g \in G$. Then $u \in \Sigma$ is a critical point of I if and only if u is a critical point of $I\mid_\Sigma$ (the restriction of the functional f to the set Σ).

Proof. Observe that linear functional $I'(u)$ is invariant under the action of G. Taking $f = I'(u)$, the proof follows from the above remarks . $\square$

4. Application to a p-Laplacian equation

Our purpose in this section is obtaining a solution for the global semilinear elliptic problem:

$$\begin{cases} -\Delta_p u + a(z)u^{p-1} = \lambda u^q + u^{p^*-1}, \ \mathbf{R}^N \\ u \geq 0, \ u \neq 0, \ \int_{\mathbf{R^N}} |\nabla u|^p < \infty. \end{cases} \tag{4.1}$$

where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, a satisfies (a_o), (a_1), (a_2), $p^* = Np/(N-p)$; $p - 1 < q < p^* - 1$, $\lambda > 0$ and $N = L + M > p$, $L \geq 2$.

In the last years, several researchers have been studying variants of problem (4.1). Among others, in bounded domains, we can cite the pioneering article due to Brezis & Nirenberg [10] which treats the case $a \equiv 0$ and $p = 2$. Also in bounded domains, Azorero & Alonzo in [4], [5] and [6] generalize some similar results for the $p-$Laplacian operator, and Egnell in [14]generalizes some results of [11]. In the unbounded domains case, we cite Rabinowitz [23] and [12]. Rabinowitz considers a more general non-linearity, however he does not treat the Sobolev critical exponent case. Benci & Cerami [7] consider the problem (4.1) when $\lambda = 0$, and [2] deals with the case where λ is replaced by an integrable function.

Theorem 4.1. Suppose that $\lambda > 0$ and that (a_o), (a_1) and (a_2) are satisfied. If one of the following inequality holds

(i) $N \geq p^2$ and $p-1 < q < p^* - 1$;
(ii) $p < N < p^2$ and $q > p^* - \frac{p}{p-1} - 1$
(ii) $p < N < p^2$ and $p - 1 < q < p^* - \frac{p}{p-1} - 1$, and large λ.

Then problem (4.1) possesses a nontrivial classical solution $u \in E_a$.
Proof. The proof consists in using variational methods to get critical points of the Euler-Lagrange functional associated to (4.1). We follow the same steps made in [19] and [2].

Define on E_a the functional:

$$I(u) = \frac{1}{p}||u||_a^p - \frac{\lambda}{q+1}\int_{\mathbf{R}^N}\left(u^+\right)^{q+1} - \frac{1}{p^*}\int_{\mathbf{R}^N}\left(u^+\right)^{p^*}$$

where $u^+(z) = \max\{u(z), 0\}$ and $u^-(x) = \min\{-u(x), 0\}$.

The critical points of I are precisely the weak solutions of (4.1). These solutions may be regularized.

Using Theorem 1.1 one can check that I is a well-defined $C^1(E_a)$ functional. It is easy to verify that

$$\frac{\lambda}{q+1}\int_{\mathbf{R}^N}(u^+)^{q+1} + \frac{1}{p^*}\int_{\mathbf{R}^N}(u^+)^{p^*} = o(||u||_a^p)\,,\ \text{as } u \to 0, \tag{4.2}$$

and hence that I has a local minimum at the origin. This is not a global minimum. Indeed, if $u \in E\backslash\{0\}$, $u \geq 0$, we have that

$$I(tu) = \frac{t^p}{p}||u||_a^p - \frac{\lambda t^{q+1}}{q+1}\int_{\mathbf{R}^N}(u^+)^{q+1} - \frac{t^{p^*}}{p^*}\int_{\mathbf{R}^N}(u^+)^{p^*}.$$

Choosing $u \in E_a$ such that $\int_{\mathbf{R}^\mathbf{N}} (u^+)^{p^*} \neq 0$, we conclude that $I(tu) \to -\infty$ as $t \to \infty$.

So, we have just seen that I has the *Mountain Pass Theorem Geometry.*

Let $e \in E_a$ such that $I(e) < 0$, and define

$$\Gamma = \{g : [0,1] \to E_a, \text{ continuous} : g(0) = 0\,,\ g(1) = e\} \tag{4.3}$$

and

$$c = \inf_{g\in\Gamma} \max_{0\le t\le 1} I(g(t)). \tag{4.4}$$

Thus c is the mountain pass minimax value associated to I. Assertion (4.2) implies that $c > 0$. At this moment, it is important to notice that c is not the minimax value associated to the Euler Lagrange functional of problem (4.1) defined in the whole $W^{1,p}(\mathbf{R}^N)$.

Using an application of the Ekeland Variational Principle (Theorem 4.3 of [20]) we may prove the Mountain Pass Theorem without (PS) condition, this is, there exists a sequence $(u_m) \subset E_a$ such that

$$I(u_m) \to c,\ \ I'(u_m) \to 0. \tag{4.5}$$

A standard argument proves that the above sequence (u_m) is bounded.

The following lemma shows that we can choose a vector $e \in E_a \backslash \{0\}$ in the definition of Γ, such that $I(e) < 0$ and

$$0 < c < \frac{1}{N} S_p^{\frac{N}{p}}, \tag{4.6}$$

where S_p is the best constant of the Sobolev immersion $W^{1,p}(\mathbf{R}^N) \subset L^{p^*}(\mathbf{R}^N)$. Precisely,

$$S_p = \inf_{u\in W^{1,p}(\mathbf{R}^N)} \frac{\int_{\mathbf{R}^N} |\nabla u|^p}{\left(\int_{\mathbf{R}^N} |u|^{p^*}\right)^{\frac{p}{p^*}}}.$$

Using the above facts and arguments due to Brezis & Nirenberg [11], we will show that the choice in (4.6) implies in obtaining a non-trivial solution of (4.1).

Lemma 4.2. Suppose that $\lambda > 0$ and one of the following conditions is satisfied:

(i) $N \geq p^2$ and $p-1 < q < p^* - 1$;
(ii) $p < N < p^2$ and $q > p^* - \frac{p}{p-1} - 1$; .
(ii) $p < N < p^2$ and $p - 1 < q < p^* - \frac{p}{p-1} - 1$, and large λ.

Then, there is a vector $e \in E_a \backslash \{0\}$, $e \geq 0$, $I(e) < 0$ such that

$$\sup_{t \geq 0} I(te) < \frac{1}{N} S_p^{\frac{N}{p}}, \tag{4.7}$$

Proof. See [1], page. 56.

In order to complete the proof of Theorem 4.1, let us consider $e \in E_a \backslash \{0\}$ given by Lemma 4.2. Let (u_m) be the sequence in E_a satisfying (4.5). We may assume that

$$\begin{gathered} u_m \rightharpoonup u \text{ in } E_a \\ u_m \to u \text{ in } L^s(\mathbf{R}^N), \;\; p < s < p^* \\ u_m(x) \to u(x) \text{ a.e. in } \mathbf{R}^N. \end{gathered}$$

The above limits yields that u must be a critical point of I in E_a (see [1]), that is,

$$I'(u) = 0.$$

We claim that $u \neq 0$. In fact, if $u \equiv 0$ and taking $l \geq 0$ such that

$$\int_{\mathbf{R}^N} (|\nabla u_m|^p + a|u_m|^p) \to l,$$

then

$$\int_{\mathbf{R}^N} \left(u_m^+\right)^{p^*} \to l,$$

since $I'(u_m) \to 0$ and $E \subset L^{q+1}(\mathbf{R}^N)$ compactly. Therefore using the fact that $I(u_m) \to c$, we get

$$Nc = l. \tag{4.8}$$

From the definition of S_p,

$$\begin{aligned}\textstyle\int_{\mathbf{R}^N}(|\nabla u_m|^p + a|u_m|^p) \geq \int_{\mathbf{R}^N}|\nabla u_m|^p \geq S_p\left(\int_{\mathbf{R}^N}|u_m|^{p^*}\right)^{\frac{2}{p^*}}\\ \geq S_p\left(\textstyle\int_{\mathbf{R}^N}(u_m^+)^{p^*}\right)^{\frac{2}{p^*}}.\end{aligned}$$

Passing to the limit in the last inequalities, we achieve that

$$l \geq S_p l^{\frac{2}{p^*}}$$

and by (4.8) that

$$c \geq \frac{1}{N} S_p^{\frac{N}{p}}$$

which contradicts the above choice of e and thus the claim is proved.

Observe that $I'(u)u^- = 0$ implies that $\int_{\mathbf{R^N}}|\nabla u^-|^p + a(x)\,(u^-)^p = 0$ and then $u^- \equiv 0$. Hence $u \geq 0$.

Notice that up to this moment we do not know if u satisfies (4.1) in the $W^{1,p}(\mathbf{R}^N)$ sense but, thanks the Proposition 3.1 used with

$$E = \{v \in W^{1,p}(\mathbf{R}^N) : \int_{\mathbf{R}^N} a(z)|v|^p dz < \infty\},$$

$G = \{g{:}E \to E : \ g(v) = v \circ \begin{pmatrix} R & 0 \\ 0 & Id \end{pmatrix}$, where R is a rotation in $\mathbf{R}^L$

(Id is the $M \times M$ identity matrix) we have $\Sigma = E_a$ and u is a critical point of I in whole E. Theorem 1 is proved. $\square$

5. Application to a system of elliptic equations

Our aim in this section is to apply Theorem 1.1 and Proposition 3.1 to find solutions for the following system of elliptic equations

$$(5.1)\qquad \begin{cases} -\Delta u + a(z)u &= \lambda\dfrac{2\gamma}{\gamma+\delta}u^{\gamma-1}v^{\delta} + \dfrac{2\alpha}{2^*}u^{\alpha-1}v^{\beta}, \quad \mathbf{R}^N \\ -\Delta v + b(z)v &= \lambda\dfrac{2\delta}{\gamma+\delta}u^{\gamma}v^{\delta-1} + \dfrac{2\beta}{2^*}u^{\alpha}v^{\beta-1}, \quad \mathbf{R}^N \\ & u, v > 0 \end{cases}$$

where $N = L + M \geq 3$ and $L \geq 2$.

Our goal is to demonstrate the theorem.

Theorem 5.1. Suppose that the real functions a and b fulfill assumptions $(a_0), (a_1)$ and (a_2). If the following hypotheses hold

$$\gamma, \delta > 1 \text{ and } q = \gamma + \delta < 2^*, \tag{5.2}$$

$$\alpha, \beta > 1 \ \text{ and } \quad \alpha + \beta = 2^*, \tag{5.3}$$

$$q > 1 \text{ and } N \geq 4 \text{ or } 3 < q < 5 \text{ and } N = 3 \tag{5.4}$$

then for every $\lambda > 0$ system (5.1) has a solution.

Moreover, the same results is still valid if (5.4) is replaced by

$$N = 3,\ 1 < q \leq 3 \text{ and } \lambda > 0 \text{ is sufficiently large.} \tag{5.5}$$

We shall follow the same route of the previous section. For this purpose let us consider the cross product Banach space $E = E_a \times E_b$ endowed with the norm $||(u,v)||_E^2 = ||u||_a^2 + ||v||_b^2$.

The weak solutions of (5.1) are the critical points of the functional

$$I : E \to \mathbf{R}$$

defined by

$$I(u,v) = \frac{1}{2}||(u,v)||_E^2 - \tfrac{2\lambda}{\gamma+\delta} \textstyle\int_{\mathbf{R}^N} (u^+)^\gamma (v^+)^\delta - \tfrac{2}{2^*} \textstyle\int_{\mathbf{R}^N} (u^+)^\alpha (v^+)^\beta \tag{5.6}$$

Theorem 1.1 and the following lemma assure that I is a well defined functional, and indeed by a straightforward computation, a C^1- functional defined on E.

Lemma 5.2. For $\alpha' + \beta' = r \leq 2^*$, there exists a constant K such that

$$\left(\int_{\mathbf{R^N}} |u|^{\alpha'} |v|^{\beta'}\right)^{1/r} \leq K||(u,v)||_E \tag{5.7}$$

for all $(u, v) \in E$.

Proof. The proof results from the constants

$$S_a = \inf_{u \in Ea\setminus\{0\}} \frac{\int_{\mathbf{R}^N}(|\nabla u|^2 \ + a(z)|u|^2)dz}{(\int_{\mathbf{R}^N} |u| \ ^r dz)^{2/r}}$$

and S_b, defined likewise the above one, and the inequality

$$|u|^{\alpha'}|v|^{\beta'} \leq |u|^r + |v|^r.$$

$\square$

Remark 1. Taking $a = b$, $u = v$ in (5.1) we get the scalar case (4.1) with $p = 2$.

Lemma 5.2, hypotheses (5.2), (5.3) and Theorem 1.1 guarantee that the functional I has the Mountain Pass Theorem geometry. Hence, as made in the previous section, we may find a sequence $(u_n, v_n) \subset E$ such that

$$I(u_n, v_n) \to c \text{ in } \mathbf{R} \text{ and } I'(u_n,v_n) \to 0 \text{ in } E', \tag{5.8}$$

where c is defined likewise in (4.3) and (4.4), replacing E_a by E. The sequence in (5.8) is bounded in E.

It is standard to assume and prove that the sequence (u_n, v_n) weakly converges to $(u_o, v_o) \in E$ and then that we have

$$I'(u_o, v_o) = 0, \tag{5.9}$$

i.e, (u_o, v_o) is a weak solution for (5.1).

From now on we shall be concentrated in proving that u_o, $v_o > 0$. In order to proceed further in this direction, we need some results. Let us define

$$S_{\alpha+\beta}(\mathbf{R}^N) = \inf_{u \in W^{1,2}(R^N)\setminus\{0\}} \frac{\int_{R^N} |\nabla u| dz}{(\int_{R^N} |u| \ ^{\alpha+\beta} dz)^{\frac{2}{\alpha+\beta}}} \tag{5.10}$$

and

$$\widetilde{S}_{(\alpha,\beta)}(\mathbf{R}^N) = \inf_{(u,v)\in\{W^{1,2}(\mathbf{R}^N)\times W^{1,2}(\mathbf{R}^N)\}\setminus\{0\}} \frac{\int_{R^N}(|\nabla u|^2 \ +|\nabla v|^2 \ )dz}{\left(\int_{R^N}|u| \ ^\alpha|v| \ ^\beta dz\right)^{\frac{2}{\alpha+\beta}}}. \tag{5.11}$$

Two of the authors together with Alves in [3] proved that

$$\widetilde{S}_{(\alpha,\beta)}(\mathbf{R}^N) = \left[\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\alpha}{\beta}\right)^{\frac{-\alpha}{\alpha+\beta}}\right] S_{\alpha+\beta}(\mathbf{R}^N), \tag{5.12}$$

and that if z_o realizes $S_{\alpha+\beta}(\mathbf{R}^N)$, then (w_o, υ_o) realizes $\widetilde{S}_{(\alpha,\beta)}(\mathbf{R}^N)$,

$$\text{for all } w_o = Bz_o \text{ and } \upsilon_o = Cz_o \text{ with } \frac{B}{C} = \sqrt{\frac{\alpha}{\beta}}. \tag{5.13}$$

In what follows, for $\alpha+\beta = 2^*$,we shall denote

$$S_{\alpha+\beta}(\mathbf{R}^N) = S_*$$

and

$$\widetilde{S}_{(\alpha,\beta)}(\mathbf{R}^N) = \widetilde{S}_*.$$

Remark 2. In [25] it is proved that S_* is realized by the one-parameter family of functions

$$\omega_\epsilon(x) = \frac{[N(N-2)\epsilon]^{\frac{N-2}{4}}}{(\epsilon+|x|^2)^{\frac{N-2}{2}}}, \ \ \epsilon > 0.$$

Lemma 5.3. If (5.4) or (5.4) hold then there is a c in (5.8) such that

$$c < \frac{1}{N} 2 \left(\frac{\widetilde{S}_*}{2}\right)^{\frac{2}{N}} \tag{5.14}$$

Proof. Let $\varphi \in C_o^\infty(\mathbf{R}^N)$ be a cut-off function with support on the ball B_{2R}, centered at the origin with radius $2R$, and such that $\varphi \equiv 1$ on B_R and $0 \le \varphi \le 1$ on B_{2R}. Define $\psi_\epsilon(x) = \varphi(x)\omega_\epsilon(x)$ and $v_\epsilon \equiv \dfrac{\psi}{\left(\int_{\mathbf{B}_{2R}} \psi_\epsilon^{2^*} dz\right)^{\frac{1}{2^*}}}$ with

$$\int_{\mathbf{R}^N} |v_\epsilon^+|^{2^*}\ dz = 1.$$

Let us consider constants B and C such that $\dfrac{B}{C} = \sqrt{\dfrac{\alpha}{\beta}}$.

Then for $t > 0$ and a fixed $\epsilon > 0$ we have

$$\begin{aligned}
&(5.15)\quad I(tBv_\epsilon, tCv_\epsilon) &=& \frac{t^2}{2}\left[B^2||v_\epsilon||_a^2 + C^2||v_\epsilon||_b^2\right] \\
&(5.16) && -\frac{2\lambda}{q}B^\gamma C^\delta t^q \int_{\mathbf{B_{2R}}} (v_\epsilon^+)^q - \frac{2}{2^*}t^{2^*}B^\alpha C^\beta
\end{aligned}$$

The $t-$function in the right hand side of the above equality has a maximum at the point t_ϵ such that

$$(5.17)\qquad 0 < t_\epsilon < \left\{\frac{1}{2}\frac{B^2||v_\epsilon||_a^2 + C^2||v_\epsilon||_b^2}{B^\alpha C^\beta}\right\}^{\frac{1}{2^*-2}} := t_o^{\frac{1}{2^*-2}}.$$

Hence

$$(5.18)\qquad \begin{aligned} I(t_\epsilon Bv_\epsilon, t_\epsilon Cv_\epsilon) = B^\alpha C^\beta \left[t_\epsilon^2 t_o^{2^*-2} - \tfrac{2}{2^*}t_\epsilon^{2^*}\right] \\ -\lambda C_\epsilon \textstyle\int_{\mathbf{B}_{2R}} (v_\epsilon^+)^q. \end{aligned}$$

Note that the function $t^2 t_o^{2^*-2} - \frac{2}{2^*}t^{2^*}$ is increasing in the interval $[0, t_o)$. This fact together with (5.17) and (5.18) yields that

$$I(t_\epsilon Bv_\epsilon, t_\epsilon Cv_\epsilon) \le \frac{2}{N}\left[\frac{1}{2}\frac{B^2||v_\epsilon||_a^2 + C^2||v_\epsilon||_b^2}{(B^\alpha C^\beta)^{\frac{2}{2^*}}}\right]^{\frac{N}{2}} - \lambda C_\epsilon \int_{\mathbf{B}_{2R}} (v_\epsilon^+)^q$$

and then that

$$I(t_\epsilon Bv_\epsilon, t_\epsilon Cv_\epsilon) \le \tfrac{2}{N}\left[\tfrac{1}{2}\tfrac{B^2+C^2}{(B^\alpha C^\beta)^{\frac{2}{2^*}}}\right]^{\frac{N}{2}} \left[\int_{\mathbf{R}^N} |\nabla v_\epsilon\ |^2\ dz + \int_{\mathbf{B}_{2R}} (a+b){v_\epsilon}^2\right]^{\frac{N}{2}}$$

$$(5.19) \qquad - \lambda C_\epsilon \int_{\mathbf{B}_{2R}} (v_\epsilon^+)^q$$

In [19] it is shown that

$$(5.20) \quad \left[\int_{\mathbf{R}^N} |\nabla v_\epsilon|^2 \, dz + \int_{\mathbf{B}_{2R}} (a+b) {v_\epsilon}^2\right]^{\frac{N}{2}} \leq S_*^{\frac{N}{2}} + O(\epsilon^{\frac{N-2}{2}}) + C_1 \int_{\mathbf{B}_{2R}} (a+b){v_\epsilon}^2$$

for some constant C_1, and that if (5.4) or (5.5) hold then

$$(5.21) \quad \lim_{\epsilon \to \infty} \frac{1}{\epsilon^{\frac{N-2}{N}}} \left(C_1 \int_{\mathbf{B}_{2R}} (a+b){v_\epsilon}^2 - \lambda C_\epsilon \int_{\mathbf{B}_{2R}} (v_\epsilon^+)^q \right) = -\infty.$$

Using (5.19), (5.13),(5.12), (5.21) and (5.20) for a sufficiently small ϵ, we deduce (5.14). □

In order to prove that $u_o, v_o \neq 0$, one observes that $u_o = 0$ if and only if $v_o = 0$. Taking in account the convergence of the sequence (u_n, v_n) and (5.11) and (5.14), the rest of the proof follows the same steps of the previous section .

Since $I'\,(u_n, v_n)(u_n^-, v_n^-) = ||(u_n^-, v_n^-)||_E^2$, passing to the limit in this equality and using (5.8) we may assume that $u_o, v_o \geq 0$. A Maximum Principle implies that $u_o, v_o > 0$, as we wanted to prove. The proof of Theorem 5.1 is finished. □

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