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FIXED POINT PARAMETERS FOR MÖBIUS GROUPS *

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Abstract

Let Γ_n (respectively, Γ_∞) be a free group of rank n (respectively, a free group of countable infinite rank). We consider the space of algebraic representations of the group Γ_n (respectively, Γ_∞) $Hom(\Gamma_n, PGL(2, \mathbf{C}))$ (respectively, $Hom(\Gamma_\infty, PGL(2, \mathbf{C}))$). Inside each of these spaces we consider a couple of open and dense subsets. These subsets contain non-discrete groups of Möbius transformations. We proceed to find complex analytic parameters for these spaces given by fixed points.

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1. Introduction

In [3] we have considered a real parameterization of the Teichmüller space of a closed Riemann surface. This parameterization is real analytic and given by a collection of fixed points of a particular set of generators of a Fuchsian group acting on the upper-half plane $\mathbf{H}$. One of the main ideas used in that paper is a geometric configuration of axis for a particular set of generators (inequalities of real numbers). In this note, we produce parameterizations of the deformation space of finitely generated groups of Möbius transformations by a collection of fixed points of a particular set of generators. We do not use axis configurations and it is important to note that, in this general situation, we work with groups which may not be discrete ones nor Kleinian groups nor Fuchsian groups (this, including the complex nature of the parameters, is the main difference with the above work). This parameterization can be used in particular for describing (fixed points) complex analytic parameters for the deformation space of a Kleinian group. We also compute explicit (real analytic) models of some fuchsian groups (including an example of genus two). To describe this parameterization we start with some basic definitions.

A Möbius transformation B is a conformal automorphism of the Riemann sphere $\widehat{\mathbf{C}}$. In particular, $B(z) = \frac{az+b}{cz+d}$, where a, b, c, d are complex numbers satisfying $ad - bc \neq 0$. There is a natural isomorphism between the group of Möbius transformations and the projective linear group $PGL(2, \mathbf{C})$ given by

$$B(z) = \tfrac{az+b}{cz+d} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

For each Möbius transformation B, we denote its set of fixed points by $F(B)$. If B is neither the identity nor elliptic of order two, we can define the values $a(B), r(B) \in F(B)$ as follows.

(1) If B is Loxodromic, then $a(B)$ and $r(B)$ are the attracting and repelling fixed points of B.

(2) If B is parabolic, then $a(B) = r(B)$ is its unique fixed point.

(3) If B is elliptic, then $a(B)$ and $r(B)$ are its fixed points for which there exists a Möbius transformation H such that $H \circ B \circ H^{-1}(z) = \lambda z$, where $\|\lambda\| = 1$, imaginary part of λ is positive and $H(a(B)) = \infty$, $H(r(B)) = 0$.

The space of infinitely generated marked groups $(G, A_1, ..., A_n, ...))$ can be identified to the set $Hom(\Gamma_\infty, PGL(2, \mathbf{C}))$, where Γ_∞ is a free group of infinite rank. This is a infinite dimensional complex manifold (isomorphic to $PGL(2, \mathbf{C})^{\mathbf{N}}$). Similarly, the space of finitely generated marked groups $(G, A_1, ..., A_n))$ can be identified the set $Hom(\Gamma_n, PGL(2, \mathbf{C}))$, where Γ_n is a free group of rank n. This is a $3n$-dimensional complex manifold (isomorphic to $PGL(2, \mathbf{C})^n$).

Two marked groups $(G_1, (A_1, ..., A_n, ...))$ and $(G_2, (B_1, ..., B_n, ...))$ are said equivalent if and only if there is a Möbius transformation $H \in PGL(2, \mathbf{C})$ satisfying $H \circ A_i \circ H^{-1} = B_i$, for $i = 1, ..., n, ...$ The respective spaces of equivalence classes of marked groups are the (algebraic) deformation spaces $Def(\Gamma_\infty, PGL(2, \mathbf{C}))$ (an infinitely complex dimensional space) and $Def(\Gamma_n, PGL(2, \mathbf{C}))$ (a complex analytic space of dimension $3(n-1)$).

The sets $\mathcal{F}_\infty \subset Def(\Gamma_\infty, PGL(2, \mathbf{C}))$ and $\mathcal{F}_n \subset Def(\Gamma_n, PGL(2, \mathbf{C}))$ consist of equivalence classes of marked groups $[(G, (A_1, ..., A_n, ...))]$ of Möbius transformations (non necessarily discrete ones) satisfying the following.

(1) $A_i^2 \neq I$, for all $i \geq 1$;

(2) $(A_j \circ A_1)^2 \neq I$, for all $j \geq 2$; and

(3) $F(A_1) \cap F(A_j) = \emptyset$, for all $j \geq 2$.

Analogously, we define the subsets $\mathcal{G}_\infty$ and $\mathcal{G}_n$ consisting of the equivalence classes of marked (non necessarily discrete ones) groups $[(G, (A_1, ..., A_n, ...))]$ of Möbius transformations satisfying the following.

(1) $A_i^2 \neq I$, for all $i \geq 1$;

(2) $(A_j \circ A_{j-1} \cdots A_2 \circ A_1)^2 \neq I$, for all $j \geq 2$; and

(3) $F(A_1) \cap F(A_j) = \emptyset$, for all $j \geq 2$.

The sets $\mathcal{F}_\infty$, $\mathcal{G}_\infty$, $\mathcal{F}_n$ and $\mathcal{G}_n$ are open dense subsets of the respective deformation spaces. In particular, they are complex analytic spaces of the same respective dimensions.

In each class $[(G,(A_1,...,A_n))]$ in either $\mathcal{F}_n$ or $\mathcal{G}_n$ (respectively, $[(G,(A_1,...,A_n,...))]$ in either $\mathcal{F}_\infty$ or $\mathcal{G}_\infty$), there exists a unique representative $(G,(A_1,...,A_n))$ (respectively, $(G,(A_1,...,A_n,...)))$ normalized by $a(A_1) = \infty$, $a(A_2) = 0$ and $a(A_2 \circ A_1) = 1$. In this way, we may think of the elements of $\mathcal{F}_n$ and $\mathcal{G}_n$ (respectively, $\mathcal{F}_\infty$ and $\mathcal{G}_\infty$) as normalized marked groups satisfying the above respective properties.

Using the unique normalized representative, we may construct functions

$$\begin{gathered}
\Phi_n : \mathcal{F}_n \to \widehat{\mathbf{C}} \times \mathbf{C}^{3n-4}, \\
\Psi_n : \mathcal{G}_n \to \widehat{\mathbf{C}} \times \mathbf{C}^{3n-4}, \\
\Phi_\infty : \mathcal{F}_\infty \to \widehat{\mathbf{C}} \times \mathbf{C}^{N}, \\
\Psi_\infty : \mathcal{G}_\infty \to \widehat{\mathbf{C}} \times \mathbf{C}^{N},
\end{gathered}$$

defined by:

$$\begin{gathered}
\Phi_n([(G,(A_1,...,A_n))]) = \\
(r(A_1),...,r(A_n),a(A_3),...,a(A_n),r(A_2 \circ A_1),...,r(A_n \circ A_1)), \\
\Psi_n([(G,(A_1,...,A_n))]) = (r(A_1),...,r(A_n),a(A_3),...,a(A_n),r(A_2 \circ \\
A_1),...,r(A_n \circ A_{n-1} \cdots A_2 \circ A_1)). \\
\Phi_\infty([(G,(A_1,...,A_n,...))]) = \\
(r(A_1),...,r(A_n),...,a(A_3),...,a(A_n),...,r(A_2 \circ A_1),...,r(A_n \circ A_1),...), \\
\Psi_\infty([(G,(A_1,...,A_n,...))]) = \\
(r(A_1),...,r(A_n),...,a(A_3),...,a(A_n),...,r(A_2 \circ A_1),...,r(A_n \circ \\
A_{n-1} \cdots A_2 \circ A_1),...).
\end{gathered}$$

Theorem 1. *The functions Φ_n, Ψ_n, ϕ_∞ and Ψ_∞ are one-to-one complex analytic map.*

For each normalized marked group $(G,(A_1,...,A_n,...))$ in either $\mathcal{F}_n$, $\mathcal{G}_n$, $\mathcal{F}_\infty$ and $\mathcal{G}_\infty$, we write down explicit matrices in $PGL(2,\mathbf{C})$

representing all the transformations A_i. The entries of such matrices are rational functions in the corresponding fixed point coordinates given by the above theorem.

2. A Couple of Applications

2.1. Deformation Spaces of Möbius groups

Let G be a group of Möbius transformations, maybe infinitely generated. The algebraic deformations of G are defined in similar fashion as it was done for Γ_n and Γ_∞. More precisely, we consider the space $Hom(G, PGL(2, \mathbf{C}))$ of representations of G into $PGL(2, \mathbf{C})$. Two representations are said equivalents if they are conjugate by some Möbius transformation. The set of equivalence classes $Def(G, PGL(2, \mathbf{C}))$ is the algebraic deformation space of G. Another deformation space associated to G is the quasiconformal deformation space. A quasiconformal homeomorphism $w : \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ is called a deformation of G if $w \circ G \circ w^{-1}$ is again a group (necessarily Kleinian) of Möbius transformations. Two deformations of G, say w_1 and w_2, are equivalent if there exists a Möbius transformation A so that $w_2 \circ g \circ w_2^{-1} = A \circ w_1 \circ g \circ w_1^{-1} \circ A^{-1}$, for all $g \in G$. The set of equivalence classes of deformations of G is called the deformation space of G and denoted by $T(G)$. In each class there is a unique representative deformation w_n satisfying $w_n(x) = x$, for $x \in \{\infty, 0, 1\}$. In the case that G is a geometrically finite Kleinian group, then the above two deformation spaces are the same. In general, we have $T(G) \subset Def(G, PGL(2, \mathbf{C}))$. In [8] the following is proved.

Theorem (Kra–Maskit). *If G is a finitely generated Kleinian group, then $T(G)$ is biholomorphically equivalent to a domain in $\mathbf{C}^n$.*

The proof of such a theorem is a consequence of the existence of certain points called stratification points. Our coordinates are a kind of stratification points for Möbius groups (non necessarily discrete ones and maybe infinitely generated). In particular, we have the following concerning the above Kra-Maskit's result. Let G be a Möbius group which can be generated by Möbius transformations A_1,..., A_n,...so that the following hold:

(1) $A_i^2 \neq I$, $(A_j \circ A_1)^2 \neq I$, for all $i \geq 1$ and all $j \geq 2$.

(2) $F(A_1) \cap F(A_j) = \emptyset$, for all $j \geq 2$.

(3) $a(A_1) = \infty$, $a(A_2) = 0$ and $a(A_2 \circ A_1) = 1$.

We denote by a_i the attracting fixed point of A_i, r_i the repelling fixed point of A_i, and s_k the repelling fixed point of $A_k \circ A_1$. Then theorem 1 implies the following:

Corollary 1. *If G is a Möbius group finitely generated by Möbius transformations A_1,..., A_n, so that they satisfy conditions (1), (2) and (3) as above, then the map $\Phi : T(G) \to \widehat{\mathbf{C}} \times \mathbf{C}^{3n-4}$, defined by*

$$\Phi([w]) = (w_n(r_1), ..., w_n(r_n), w_n(a_3), ..., w_n(a_n), w_n(s_2), ..., w_n(s_n))$$

turns out to be a one-to-one holomorphic map. Similarly, if G is a Möbius group infinitely generated by Möbius transformations A_1,..., A_n,..., so that they satisfy conditions (1), (2) and (3) as above, then the map $\Phi : T(G) \to \widehat{\mathbf{C}}\times\mathbf{C}^N$, defined by

$$\Phi([w]) = (w_n(r_1), ..., w_n(r_n), ..., w_n(a_3), ..., w_n(a_n), ..., w_n(s_2), ..., w_n(s_n), ...)$$

is a one-to-one holomorphic map.

The above is in really true at the level of the algebraic deformation spaces by theorem 1.

2.2. Models of Teichmüller spaces

Let $F < PGL^+(2, \mathbf{R})$ be a finitely generated Fuchsian group acting on the hyperbolic plane $\mathbf{H}$. A Fuchsian representation of F is a monomorphism $\theta : F \to PGL^+(2, \mathbf{R})$ such that there is a quasiconformal homeomorphism $\phi : \mathbf{H} \to \mathbf{H}$ satisfying $\theta(\gamma) = \phi \circ \gamma \circ \phi^{-1}$, for all $\gamma \in F$.

Two Fuchsian representations θ_1 and θ_2 are said Fuchsian equivalent if and only if there exists a Möbius transformation $A \in PGL(2, \mathbf{R})$ such that $\theta_2(\gamma) = A \circ \theta_1(\gamma) \circ A^{-1}$, for all $\gamma \in F$. The set of equivalence classes of Fuchsian representations of the Fuchsian group F is called the Teichmüller space of F and denoted as $\mathcal{T}(F)$. This set is a simply connected real analytic manifold of dimension $6g - 6 + 2k + 3l$, where $\mathbf{H}/F$ is a Riemann surface of genus g with k punctures and l holes (see [1]). We say that F has signature or type (g, k, l).

The Fuchsian group F has a presentation of the form:

$$F = \langle A_1^*, B_1^* ..., A_g^*, B_g^*, P_1^*, ..., P_k^*, L_1^*, ..., L_l^* :$$
$$\prod_{m=1}^{l} L_{l-m+1}^* \prod_{j=1}^{k} P_{k-j+1}^* \prod_{i=1}^{g} [A_{g-i+1}^*, B_{g-1+1}^*] = I \rangle,$$

where $[A_j^*, B_j^*]$ denotes the commutator between the hyperbolic transformations A_j^* and B_j^*, the transformations L_j^* are also hyperbolic and the transformations P_j^* are parabolic. We also may assume:

(1) if $g \geq 1$, then $a(A_1^*) = \infty$, $a(B_1^*) = 0$ and $a(B_1^* A_1^*) = 1$;

(2) if $g = 0$ and $k \geq 2$, then $a(P_1^*) = \infty$, $a(P_2^*) = 0$ and $a(P_2^* P_1^*) = 1$;

(3) if $g = 0$, $k = 1$, then $a(P_1^*) = \infty$, $a(L_1^*) = 0$ and $a(L_1^* P_1^*) = 1$;

(4) if $g = 0$, $k = 0$, then $a(L_1^*) = \infty$, $a(L_2^*) = 0$ and $a(L_2^* L_1^*) = 1$.

We can identify the Teichmüller space of F with the set of marked groups $(G, (A_1, B_1, ..., A_g, B_g, P_1, ..., P_k, L_1, ..., L_l))$ for which:

(1) G is a fuchsian group acting on $\mathbf{H}$ of same type as F;

(2) there is an isomorphism $\psi : F \to G$, so that $\psi(A_j^*) = A_j$, $\psi(B_j^*) = B_j$, $\psi(P_i^*) = P_i$ and $\psi(L_r^*) = L_r$;

(3) if $g \geq 1$, then $a(A_1) = \infty$, $a(B_1) = 0$ and $a(B_1 A_1) = 1$;

(4) if $g = 0$ and $k \geq 2$, then $a(P_1) = \infty$, $a(P_2) = 0$ and $a(P_2 P_1) = 1$;

(5) if $g = 0$, $k = 1$, then $a(P_1) = \infty$, $a(L_1) = 0$ and $a(L_1 P_1) = 1$;

(6) if $g = 0$, $k = 0$, then $a(L_1) = \infty$, $a(L_2) = 0$ and $a(L_2 L_1) = 1$.

If we use the function Ψ_n, with $n = 2g+k+l$, then theorem 1 implies it is a one-to-one real analytic map into $\mathbf{R}^{3n-3}$. The image is contained inside an algebraic variety defined by $(3+k)$ real polynomials. Let us set the following notation $a_j := a(A_j)$, $b_j := r(A_j)$, $c_j := a(B_j)$, $d_j := r(B_j)$, $p_j := a(P_j)$, $x_j := a(L_j)$, $y_j := r(L_j)$, $e_j := r(A_jA_1)$, $f_j := r(B_jA_1)$, $r_j := r(P_jA_1)$, $s_j := r(L_jA_1)$, $q_j := r(P_jP_{j-1}\cdots P_1)$, $w_j := r(L_jL_{j-1}\cdots L_1P_kP_{k-1}\cdots P_1)$ and $t_j := r(L_jL_{j-1}\cdots L_1)$.

2.3. Signature (g,0,0), g $\geq$ 2

In this case, the map $Q : T(F) \to W \subset \mathbf{R}^{6g-3}$, defined by $Q(G,(A_1,B_1,...,A_g,B_g)) = (a,b,c,d,e,f)$, where

$$\begin{array}{rcl} a & = & (a_2,...,a_g) \\ b & = & (b_1,...,b_g) \\ c & = & (c_2,...,c_g) \\ d & = & (d_1,...,d_g) \\ e & = & (e_2,...,e_g) \\ f & = & (f_1,...,f_g) \end{array}$$

turns out the to be a one-to-one real analytic map into the real affine variety W defined by three polynomials. These three polynomials are $E_{11} = 1$, $E_{12} = 0$ and $E_{22} = 1$, where $\prod_{j=1}^{g}[A_j,B_j]$ is represented by the matrix $(A_{11},A_{12},A_{21},A_{22})$.

2.4. Signature (g,k,l), g $\geq$ 1, l $\geq$ 1

The map $Q : T(F) \to \mathbf{R}^{6g-6+2k+3l}$, defined by

$Q(G,(A_1,B_1,...,A_g,B_g,P_1,...,P_k,L_1,...,L_l))$

$= (a, b, c, d, e, f, p, r, s, x, y)$, where

$$\begin{array}{rcl} a & = & (a_2, ..., a_g) \\ b & = & (b_1, ..., b_g) \\ c & = & (c_2, ..., c_g) \\ d & = & (d_1, ..., d_g) \\ e & = & (e_2, ..., e_g) \\ f & = & (f_1, ..., f_g) \\ p & = & (p_1, ..., p_k) \\ r & = & (r_1, ..., r_k) \\ s & = & (s_1, ..., s_{l-1}) \\ x & = & (x_1, ..., x_{l-1}) \\ y & = & (y_1, ..., y_{l-1}) \end{array}$$

turns out the to be a one-to-one real analytic map.

2.5. Signature (g,k,0), $g \geq 1$, $k \geq 1$

The map $Q : T(F) \to W \subset \mathbf{R}^{6g-5+2k}$, defined by $Q(G, (A_1, B_1, ..., A_g, B_g, P_1, ..., P_k)) = (a, b, c, d, e, f, q, r)$, where

$$\begin{array}{rcl} a & = & (a_2, ..., a_g) \\ b & = & (b_1, ..., b_g) \\ c & = & (c_2, ..., c_g) \\ d & = & (d_1, ..., d_g) \\ e & = & (e_2, ..., e_g) \\ f & = & (f_1, ..., f_g) \\ q & = & (q_1, ..., q_{k-1}) \\ r & = & (r_1, ..., r_{k-1}) \end{array}$$

turns out the to be a one-to-one real analytic map into the real affine variety W defined by a polynomial $E = 0$, where E is defined by the following observation. The above data determines uniquely the transformations P_1,..., P_{k-1}. Since the transformation P_k is the inverse of the compositions of these transformations and it is parabolic, then the polynomial corresponds to have square of the trace of P_k equal to 4.

2.6. Signature (0,k,0), k $\geq$ 4

The map $Q : T(F) \to W \subset \mathbf{R}^{2k-5}$, defined by $Q(G,(P_1,...,P_k)) = (p,q)$, where

$$\begin{aligned} p &= (p_3,...,p_k) \\ q &= (q_2,...,q_{k-2}) \end{aligned}$$

turns out the to be a one-to-one real analytic map into the affine real variety defined by one polynomial obtained in the same way as in the case above.

2.7. Signature (0,k,l), k $\geq$ 2, l $\geq$ 1

The map $Q : T(F) \to \mathbf{R}^{2k+3l-6}$, defined by $Q(G,(P_1,...,P_k,L_1,...,L_l)) = (p,q,w,x,y)$, where

$$\begin{aligned} p &= (p_3,...,p_k) \\ q &= (q_2,...,q_k) \\ w &= (w_1,...,w_{l-2}) \\ x &= (x_1,...,x_{l-1}) \\ y &= (y_1,...,y_l) \end{aligned}$$

turns out the to be a one-to-one real analytic map.

2.8. Signature (0,1,l), l $\geq$ 2

The map $Q : T(F) \to \mathbf{R}^{3l-4}$, defined by $Q(G,(P_1,L_1,...,L_l)) = (w,x,y)$, where

$$\begin{aligned} w &= (w_1,...,w_{l-2}) \\ x &= (x_2,...,x_{l-1}) \\ y &= (y_1,...,y_l) \end{aligned}$$

turns out the to be a one-to-one real analytic map.

2.9. Signature (0,0,l), l $\geq$ 3

The map $Q : T(F) \to \mathbf{R}^{3l-6}$, defined by $Q(G,(L_1,...,L_l)) = (t,x,y)$, where

$$\begin{aligned} t &= (t_2,...,t_{l-2}) \\ x &= (x_3,...,x_l) \\ y &= (y_1,...,y_{l-1}) \end{aligned}$$

turns out the to be a one-to-one real analytic map.

Remarks. In the last section we do more explicit computations for types $(0,4)$, $(1,1)$ and $(2,0)$.We obtain parameter spaces related to the ones given by Maskit in [4], [5], [6] and Min in [9]. We must remark that Min's parameters use multipliers and ours (also Maskit's ones) only use fixed points. Unfortunately, our parameters look more difficult to Maskit's ones. Application to the Schottky space and noded Riemann surfaces can be found in [2].

For infinitely generated Fuchsian groups, we may also use the results in this note to construct models of the respective Teichmüller spaces.

3. Proof of Theorem 1

In this section we prove theorem 1. For this, we need the following lemmas.

Lemma 1. $\widehat{\Phi}_2 : \mathcal{F}_2 \to \widehat{\mathbf{C}} \times \mathbf{C}^2$ *is one-to-one.*

Lemma 2. *Let A and A be two Möbius transformations such that, $F(A) \cap F(B) = \emptyset$. If A, $a(B)$, $r(B)$ and $r(B \circ A)$ are known, then the transformation B is uniquely determined.*

Proof of Theorem 1. The proof is a direct consequence of lemmas 1 and 2 as follows.

(1) Lemma 1 implies that A_1 and A_2 are uniquely determined. Now, apply Lemma 2 to the pair $A = A_1$ and $B = A_j$ to obtain A_j uniquely, for every $j \geq 3$. The complex analyticity of the map Φ_n and Φ_∞ follows easily from the explicit matrix description of the generators in $PGL(2, \mathbf{C})$.

(2) Apply Lemma 1 to obtain uniquely the elements A_1 and A_2. Next, apply Lemma 2 to the pair $A = A_2 \circ A_1$ and $B = A_3$ to obtain A_3 uniquely. We continue inductively applying Lemma 2 to the pairs $A = A_{j-1} \circ \cdots \circ A_1$ and $B = A_j$ to obtain A_j uniquely.

Proof of Lemma 1. We decompose the set $\mathcal{F}_2$ as the disjoint union of four subsets, say $\mathcal{F}_2 = \cup_{i=1}^4 L_i$, where

$$L_1 = \{[(G,(A_1,A_2))] \in \mathcal{F}_2; A_1 \text{ and } A_2 \text{ are not parabolics}\};$$

$$L_2 = \{[(G,(A_1,A_2))] \in \mathcal{F}_2; A_1 \text{ is parabolic and } A_2 \text{ is not parabolic}\};$$

$$L_3 = \{[(G,(A_1,A_2))] \in \mathcal{F}_2; A_2 \text{ is parabolic and } A_1 \text{ is not parabolic}\};$$

$$L_4 = \{[(G,(A_1,A_2))] \in \mathcal{F}_2; A_1 \text{ and } A_2 \text{ are both parabolic}\}.$$

It is easy to see that the images under Φ_2 of L_i and L_j are disjoint if $i \neq j$. This is a consequence of the fact that, T is parabolic if and only if $a(T) = r(T)$. The injectivity of Φ_2 then follows from the injectivity of Φ_2 restricted to each L_i, $i = 1,2,3,4$. In what follows, we denote by x, y and z the points $r(A_1)$, $r(A_2)$ and $r(A_2 \circ A_1)$, respectively.

(I) Φ_2 is injective on L_1. The transformations A_1 and A_2 are not parabolics. The matrix representation of these transformations is given by

$$A_1 = \begin{pmatrix} k_1^2 & x(1-k_1^2) \\ 0 & 1 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} y & 0 \\ 1-k_2^2 & yk_2^2 \end{pmatrix},$$

where either $\|k_j^2\| > 1$ or $k_j^2 = e^{2\pi i\theta_j}$, $\theta_j \in (0, 1/2)$, for $j = 1, 2$. In this case, the product $A_2 \circ A_1$ has the matrix representation

$$A_2 \circ A_1 = \begin{pmatrix} yk_1^2 & xy(1-k_1^2) \\ k_1^2(1-k_2^2) & x(1-k_1^2)(1-k_2^2)+yk_2^2 \end{pmatrix}.$$

Since 1 is a fixed point of $A_2 \circ A_1$, we have the equation

$$k_2^2(k_1^2 + x(1-k_1^2) - y) = (1-y)(k_1^2 + x(1-k_1^2)).$$

The facts $y \neq 1$, $y \neq 0$ and $k_2 \neq 0$ imply that $k_1^2+x(1-k_1^2)-y \neq 0$. In particular,

$$(*) \qquad k_2^2 = \tfrac{(1-y)(k_1^2+x(1-k_1^2))}{k_1^2+x(1-k_1^2)-y}.$$

The fixed points of $A_2 \circ A_1$ are the roots of the quadratic polynomial in w:

$$k_1^2(1-k_2^2)w^2 + [x(1-k_1^2)(1-k_2^2) + yk_2^2 - yk_1^2]w - xy(1-k_1^2) = 0.$$

In particular, $z = \frac{-xy(1-k_1^2)}{k_1^2(1-k_2^2)}$ and (since $z \neq 0$)

$$(**) \qquad k_2^2 = \tfrac{xy(1-k_1^2)+zk_1^2}{k_1^2 z}.$$

The equality of the RHS of $(*)$ and $(**)$ gives us the following equation to be satisfied by k_1^2:

$$k_1^4 - k_1^2\left(\tfrac{xz(y-1)+xy(1-x)+(x-y)(z-xy)}{y(1-x)(z-x)}\right) + \tfrac{x(x-y)}{(1-x)(z-x)} = 0.$$

The solutions to this equations are by $1, -1, k_1 and - k_1$. ¿From that one obtain:

$$k_1^2 = \tfrac{x(x-y)}{(1-x)(z-x)}, \quad \text{and} \quad k_2^2 = \tfrac{x(y-1)(y-z)}{z(x-y)}.$$

(II) Φ_2 **is injective on** L_2. In this case A_1 and A_2 have the following representation

$$A_1 = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} y & 0 \\ 1-k_2^2 & yk_2^2 \end{pmatrix},$$

where either $\|k_2^2\| > 1$ or $k_2^2 = e^{2\pi i\theta}$, $\theta \in (0,1/2)$ and $a \neq 0$. In this case the product $A_2 \circ A_1$ has the following matrix representation

$$A_2 \circ A_1 = \begin{pmatrix} y & ay \\ 1-k_2^2 & a(1-k_2^2)+yk_2^2 \end{pmatrix}.$$

The fact that 1 is a fixed point of $A_2 \circ A_1$ gives us the equation

$$(1+a)(y-1) = k_2^2(y-1-a).$$

Since $k_2 \neq 0$ and $y \neq 1$, we must have that $1+a=0$ if and only if $y = 1+a$; in which case $y=0$, a contradiction. In particular, we get

$$(*) \qquad k_2^2 = \tfrac{(1+a)(y-1)}{y-1-a}.$$

The fixed points of the transformation $A_2 \circ A_1$ are the roots of the quadratic equation in w:

$$w^2 + (a-y)w - \tfrac{ay}{1-k_2^2} = 0.$$

It follows that $z = \frac{-ay}{1-k_2^2}$, and

$$(**) \qquad k_2^2 = \tfrac{z+ay}{z}.$$

The equality of the RHS of $(*)$ and $(**)$ implies the following equation to be satisfied by a:

$$a^2y + ay(z-y+1) = 0.$$

Since $a \neq 0$ and $y \neq 0$, we obtain

$$a = y-z-1, \quad \text{and} \quad k_2^2 = \tfrac{(y-1)(y-z)}{z}.$$

(III) Φ_2 is injective on L_3. In this case, A_1 and A_2 have the following matrix representation

$$A_2 = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} k_1^2 & x(1-k_1^2) \\ 0 & 1 \end{pmatrix},$$

where either $\|k_1^2\| > 1$ or $k_1^2 = e^{2\pi i\theta}$, $\theta \in (0, 1/2)$ and $b \neq 0$. The product $A_2 \circ A_1$ has the following matrix representation

$$A_2 \circ A_1 = \begin{pmatrix} k_1^2 & x(1-k_1^2) \\ bk_1^2 & bx(1-k_1^2)+1 \end{pmatrix}.$$

The fact that 1 is a fixed point of $A_2 \circ A_1$ implies the following equation

$$1 + x(b-1) = k_1^2(1 + x(b-1) - b).$$

Since $k_1 \neq 0$, we must have that $1 + x(b-1) = 0$ if and only if $1 + x(b-1) - b = 0$, in which case $b = 0$ a contradiction. In particular, we get

$$(*) \qquad k_1^2 = \tfrac{1+x(b-1)}{1+x(b-1)-b}.$$

The fixed points of the transformation $A_2 \circ A_1$ are the roots of the quadratic equation in w:

$$bk_1^2 w^2 + (bx+1)(1-k_1^2)w - x(1-k_1^2) = 0.$$

It follows that $z = \frac{-x(1-k_1^2)}{bk_1^2}$, and

$$(**) \qquad k_1^2 = \tfrac{x}{x-zb}.$$

The equality of the RHS of $(*)$ and $(**)$ gives the following equation to be satisfied by b:

$$zxb^2 + b(z(1-x) - x) = 0.$$

Since $b \neq 0$, $x \neq 0$ and $z \neq 0$, we obtain

$$b = \tfrac{x+z(x-1)}{xz} \quad \text{and} \quad k_1^2 = \tfrac{x^2}{(x-1)(x-z)}.$$

(IV) Φ_2 is injective on L_4. In this case, A_1 and A_2 have the following matrix representation

$$A_2 = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix},$$
$$A_1 = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix},$$

where $ab \neq 0$. In this case the product $A_2 \circ A_1$ has the following matrix representation

$$A_2 \circ A_1 = \begin{pmatrix} 1 & a \\ b & 1+ab \end{pmatrix}$$

Since 1 is a fixed point of $A_2 \circ A_1$, we have the following equation

$$a = b(1+a).$$

The fact that $a \neq 0$ implies $a \neq -1$, and we obtain the equation

$$(*) \qquad b = \tfrac{a}{1+a}.$$

The fixed points of the transformation $A_2 \circ A_1$ are the roots of the quadratic equation in w:

$$w^2 + aw - \tfrac{a}{b} = 0.$$

It follows that $z = -\frac{a}{b}$, and

$$(**) \qquad b = -\tfrac{a}{z}.$$

The equality of the RHS of $(*)$ and $(**)$ implies

$$a = -1 - z \quad \text{and} \quad b = \tfrac{1+z}{z}.$$

Proof of Lemma 2.

We normalize so that $a(A)=\infty$.

Case 1. Assume A and B to be parabolic elements. In this case, A and B have the following matrix representation

$$A=\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix},$$
$$B=\begin{pmatrix} 1+px & -px^2 \\ p & 1-px \end{pmatrix},$$

where $ap\neq 0$ and x is the fixed point of B. We want to obtain a unique value of p in function of a, x and $r(B\circ A)$.

In this case, the product $B\circ A$ has the following matrix representation

$$B\circ A=\begin{pmatrix} 1+px & a(1+px)-px^2 \\ p & 1+ap-px \end{pmatrix}.$$

Denote by z the point $r(B\circ A)$. The fact that z is fixed point of $B\circ A$ gives us the equation $p(x-z)(z+a-x)=-a$. Since $a\neq 0$, we obtain

$$p=\tfrac{-a}{(x-z)(z+a-x)}.$$

Case 2. Assume A to be parabolic and B to be non-parabolic. In this case, A and B have the following matrix representation

$$A=\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix},$$
$$B=\begin{pmatrix} x-yk^2 & xy(k^2-1) \\ 1-k^2 & xk^2-y \end{pmatrix},$$

where $a \neq 0$, $r(B) = x$ and $a(B) = y$ are the fixed point of B and either $\|k\| > 1$ or $k^2 = e^{2\pi i\theta}$, $\theta \in (0, 1/2)$. We want to obtain a unique value of k^2 in function of a, x, y and $r(B \circ A)$.

In this case, the product $B \circ A$ has the following matrix representation

$$B \circ A = \begin{pmatrix} x - yk^2 & a(x - yk^2) + xy(k^2 - 1) \\ 1 - k^2 & a(1 - k^2) + xk^2 - y \end{pmatrix}$$

Denote by z the point $r(B \circ A)$. The fact that z is fixed point of $B \circ A$ gives us the equation $k^2(y - z)(x - z - a) = (x - z)(y - z - a)$. Since $x \neq z$ and $y \neq z$, we have that $x - z - a = 0$ if and only if $y - z - a = 0$ in which case $x = y$ a contradiction. In particular,

$$k^2 = \frac{(x - z)(y - z - a)}{(y - z)(x - z - a)}.$$

Case 3. Assume A non-parabolic and B to be parabolic. Let $r(A) = r$, $a(B) = r(B) = x$ and $r(B \circ A) = z$. The transformations A and B have the following matrix representation

$$A = \begin{pmatrix} k^2 & r(1 - k^2) \\ 0 & 1 \end{pmatrix},$$

$$B = \begin{pmatrix} 1 + px & -px^2 \\ p & 1 - px \end{pmatrix},$$

where $p \neq 0$. In this case we want to determine the value of p uniquely. The transformation $B \circ A$ has the matrix representation

$$B \circ A = \begin{pmatrix} k^2(1 + px) & (1 + px)r(1 - k^2) - px^2 \\ pk & pr(1 - k^2) + 1 - px \end{pmatrix},$$

The condition that z is a fixed point of $B \circ A$ gives us the equation

$$p((k^2 + 1)xz + r(1 - k^2)(x - z) - x^2 - k^2z^2) = (1 - k^2)(z - r).$$

Since $p \neq 0$, $k^2 \neq 1$ and $z \neq r$, both sides of the above equation are necessarily different from zero. In particular,

$$p = \frac{(1-k^2)(z-r)}{(k^2+1)xz+r(1-k^2)(x-z)-x^2-k^2z^2}.$$

Case 4. Assume A and B to be non-parabolic elements. In this case, A and B have the following matrix representation

$$A = \begin{pmatrix} k_1^2 & x(1-k_1^2) \\ 0 & 1 \end{pmatrix},$$

$$B = \begin{pmatrix} u - tk_2^2 & ut(k_2^2 - 1) \\ 1 - k_2^2 & uk_2^2 - t \end{pmatrix},$$

where $r(A) = x$, $r(B) = u$, $a(B) = t$, and either $\|k_j\| > 1$ or $k_j^2 = e^{2\pi i \theta_j}$, $\theta_j \in (0, 1/2)$. We want to obtain a unique value of k_2^2 in function of x, u, t, k_1^2 and $r(B \circ A)$.

In this case the product $B \circ A$ has the following matrix representation

$$B \circ A = \begin{pmatrix} k_1^2(u - tk_2^2) & x(1-k_1^2)(u-tk_2^2) + ut(k_2^2-1) \\ k_1^2(1-k_2^2) & x(1-k_1^2)(1-k_2^2) + uk_2^2 - t \end{pmatrix}.$$

Denote by z the point $r(B \circ A)$. The fact that z is fixed point of $B \circ A$ gives us the equation $k_2^2(z-t)(k_1^2z+x(1-k_1^2)-u) = (z-u)(k_1^2z+x(1-k_1^2)-t)$. Since $t \neq z$ and $u \neq z$, we have that $k_1^2z+x(1-k_1^2)-u = 0$ if and only if $k_1^2z + x(1-k_1^2) - t = 0$; in which case $u = t$, a contradiction. In particular,

$$k_2^2 = \frac{(z-u)(k_1^2z+x(1-k_1^2)-t)}{(z-t)(k_1^2z+x(1-k_1^2)-u)}.$$

4. Explicit Matrix Representation

For the case of normalized marked groups in $\mathcal{F}_n$ or $\mathcal{F}_\infty$, as a consequence of Theorem 1, we can write matrices in $PGL(2, \mathbf{C})$ representing the transformations A_1,..., A_n, ..., as follows.

(I) If $r_1 \neq \infty$, then

$$A_1 = \begin{pmatrix} k_1^2 & r_1(1-k_1^2) \\ 0 & 1 \end{pmatrix},$$

where $k_1^2 = \frac{r_1(r_1-r_2)}{(1-r_1)(s_2-r_1)}$.

(II) If $r_1 = \infty$, then

$$A_1 = \begin{pmatrix} 1 & r_2 - s_2 - 1 \\ 0 & 1 \end{pmatrix}.$$

(III) If $r_2 \neq 0$, then

$$A_2 = \begin{pmatrix} r_2 & 0 \\ (1-k_2^2) & r_2 k_2^2 \end{pmatrix},$$

where $k_2^2 = \frac{r_1(r_2-1)(r_2-s_2)}{s_2(r_1-r_2)}$.

(IV) If $r_2 = 0$, then

$$A_2 = \begin{pmatrix} 1 & 0 \\ \frac{r_1+s_2(r_1-1)}{r_1 s_2} & 1 \end{pmatrix}.$$

(V) If $r_j = a_j$ $(j = 3, ..., n)$ and $r_1 = \infty$, then

$$A_j = \begin{pmatrix} 1 + p_j r_j & -p_j r_j^2 \\ p_j & 1 - p_j r_j \end{pmatrix},$$

where $p_j = \frac{1+s_2-r_2}{(r_j-s_j)(s_j-r_j+r_2-s_2-1)}$.

(VI) If $r_j = a_j$ $(j = 3, ..., n)$ and $r_1 \neq \infty$, then

$$A_j = \begin{pmatrix} 1 + p_j r_j & -p_j r_j^2 \\ p_j & 1 - p_j r_j \end{pmatrix},$$

where $p_j = \frac{(1-k_1^2)(s_j-r_1)}{(k_1^2+1)r_j s_j + r_1(1-k_1^2)(r_j-s_j) - r_j^2 - k_1^2 s_j^2}$.

(VII) If $r_j \neq a_j$ $(j = 3, ..., n)$ and $r_1 = \infty$, then

$$A_j = \begin{pmatrix} r_j - a_j k_j^2 & r_j a_j (k_j^2 - 1) \\ 1 - k_j^2 & r_j k_j^2 - a_j \end{pmatrix},$$

where $k_j^2 = \frac{(r_j-s_j)(a_j-s_j-r_2+s_2+1)}{(a_j-s_j)(r_j-s_j-r_2+s_2+1)}$.

(VIII) If $r_j \neq a_j$ $(j = 3, ..., n)$ and $r_1 \neq \infty$, then

$$A_j = \begin{pmatrix} r_j - a_j k_j^2 & r_j a_j (k_j^2 - 1) \\ 1 - k_j^2 & r_j k_j^2 - a_j \end{pmatrix},$$

where $k_j^2 = \frac{(r_j-s_j)(a_j-k_1^2 s_j - r_1(1-k_1^2))}{(a_j-s_j)(r_j-k_1^2 s_j - r_1(1-k_1^2))}$.

For the case of normalized marked groups in $\mathcal{V}_n$ or $\mathcal{V}_\infty$, as a consequence of Theorem 1, we can write matrices in $PGL(2, \mathbf{C})$ representing the transformations A_1,..., A_n, ..., as follows.

$$A_1 = \begin{pmatrix} k_1^2 & r_1(1 - k_1^2 \\ 0 & 1 \end{pmatrix},$$

where $k_1^2 = \frac{r_1(r_1-r_2)}{(1-r_1)(t_2-r_1)}$.

$$A_2 = \begin{pmatrix} r_2 & 0 \\ 1 - k_2^2 & r_2 k_2^2 \end{pmatrix},$$

where $k_2^2 = \frac{r_1(r_2-1)(r_2-t_2)}{t_2(r_1-r_2)}$.

$$A_j = \begin{pmatrix} r_j - a_j k_j^2 & a_j r_j (k_j^2 - 1) \\ 1 - k_j^2 & r_j k_j^2 - a_j \end{pmatrix},$$

where $k_j^2 = \frac{(r_j-t_j)(m_{j-1}(a_j-t_{j-1}+t_{j-1}\mu_j^2-t_j\mu_j^2)-a_jt_j+t_{j-1}t_j-a_jt_{j-1}\mu_j^2+a_jt_j\mu_j^2)}{(a_j-t_j)(m_{j-1}(r_j-t_{j-1}+t_{j-1}\mu_j^2-t_j\mu_j^2)-r_jt_j+t_{j-1}t_j-r_jt_{j-1}\mu_j^2+r_jt_j\mu_j^2)}$, and m_{j-1} is the attracting fixed point of the transformation $A_{j-1} \circ A_{j-2} \cdots A_2 \circ A_1$ with multiplier μ_j^2, where $\left\| \mu_j^2 \right\| > 1$.

The above values of m_j and μ_j^2 are obtained in an inductive way, where $m_2 = 1$ and μ_2^2 is the multiplier of $A_2 \circ A_1$.

5. Computing Models for Some Teichmüller Spaces

5.1. Teichmüller Spaces of Riemann Surfaces of Type (0,4)

A Riemann surface S is said to be of type $(0, 4)$ if it is a Riemann surface of genus zero with exactly 4 boundary components. If some of the boundaries is a puncture, then S is called a parabolic Riemann surface of type $(0, 4)$; otherwise, it is called a hyperbolic Riemann surface of type $(0, 4)$.

Let $\Gamma < PGL^+(2, \mathbf{R})$ be a Fuchsian group acting on the hyperbolic plane such that $S = \mathbf{H}/\Gamma$ is a hyperbolic Riemann surface of type $(0, 4)$. Let α_1, α_2 and α_3 be simple loops on S (through the point z) as shown in figure 1.

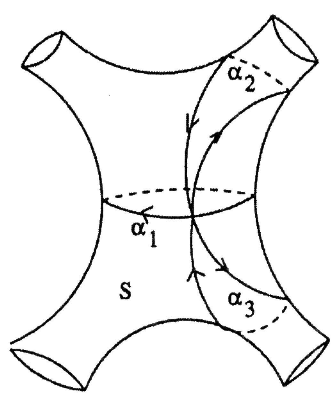

Figure 1.

The fundamental group of S, at the point $z \in S$, has a presentation

$$\Pi_1(S, z) = < \alpha_1, \alpha_2, \alpha_3 > \cong \Gamma_3.$$

In this way, we have that Γ is a free group of rank 3 generated by A_1, A_2 and A_3, so that the transformations A_i are hyperbolic and the axis of these transformations A_1, A_2, A_3, $A_2 \circ A_1$ and $A_3 \circ A_1$ are shown in figure 2.

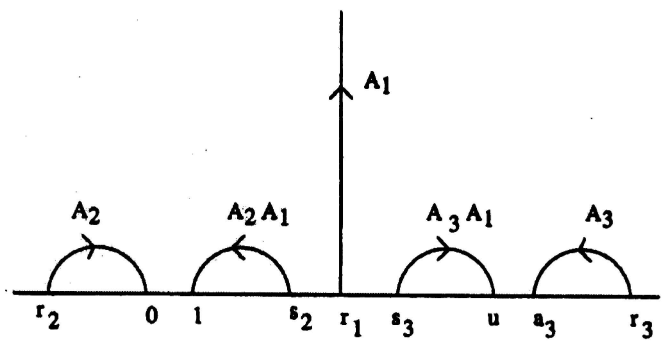

Figure 2.

The Teichmüller space of Γ (or S) $\mathcal{T}(\Gamma)$ is in this case given by the subset $V(3)$ consisting of those marked groups $(G, (B_1, B_2, B_3))$ satisfying the following.

(1) G is discrete subset of $PGL^+(2, \mathbf{R}) \subset PGL(2, \mathbf{C})$.

(2) $a(B_1) = \infty$, $a(B_2) = 0$ and $a(B_2 \circ B_1) = 1$.

(3) There exists a quasiconformal homeomorphism $F : \mathbf{H} \to \mathbf{H}$ such that $F \circ A_i \circ F^{-1} = B_i$, for $i = 1, 2, 3.$.

As a consequence of theorem 1, we have one-to-one real analytic map

$$\Phi_3 : \mathcal{T}(\Gamma) = V(3) \rightarrow \mathbf{R}^6,$$

defined by $\Phi_3((G,(B_1,B_2,B_3))) = (r_1,r_2,r_3,a_3,s_2,s_3)$, where r_i, a_3 and s_k are the repelling fixed point of B_i, the attracting fixed point of B_3 and the repelling fixed point of $B_k \circ B_1$, respectively.

If we denote by u the attracting fixed point of the transformation $B_3 \circ B_1$, then we have that the value u is a real analytic function on r_1, r_2, r_3, a_3, s_2 and s_3. In fact,

$$u = r_1 - \tfrac{(r_1-a_3)(r_1-r_3)(r_1-1)(r_1-s_2)}{r_1(r_1-r_2)(r_1-s_3)}.$$

The axis of the transformations B_1, B_2, B_3, $B_2 \circ B_1$ and $B_3 \circ B_1$ have the same topological configuration as the axis of the transformations A_1, A_2, A_3, $A_2 \circ A_1$ and $A_3 \circ A_1$, respectively. In particular, the fixed points of the above transformations satisfy the following inequalities.

(E1) $r_2 < 0$.

(E2) $1 < s_2 < r_1 < s_3 < u < a_3 < r_3$.

Let us consider the parameter space $\mathcal{R}$ as the open subset of $\mathbf{R}^6$ consisting of the tuples $(r_1,r_2,r_3,a_3,s_2,s_3)$ satisfying the inequalities given by (E1) and (E2). In particular, $\Phi_3(\mathcal{T}(\Gamma))$ is contained $\mathcal{R}$.

Corollary 2. $\Phi_3(\mathcal{T}(\Gamma)) = \mathcal{R}$.

Proof. We have to show that for any point $p = (r_1,r_2,r_3,a_3,s_2,s_3)$ contained in the region $\mathcal{R}$ there is a normalized marked group $(G,(B_1,B_2,B_3))$ in $V(3)$ so that $\Phi_3((G,(B_1,B_2,B_3))) = p$. For p as above we can construct Möbius transformations

$$B_1 = \begin{pmatrix} k_1^2 & r_1(1-k_1^2) \\ 0 & 1 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} r_2 & 0 \\ (1-k_2^2) & r_2 k_2^2 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} r_3 - a_3k_3^2 & r_3a_3(k_3^2-1) \\ 1-k_3^2 & r_3k_3^2 - a_3 \end{pmatrix},$$

where

$$\begin{aligned} k_1^2 &= \tfrac{r_1(r_1-r_2)}{(1-r_1)(s_2-r_1)}; \\ k_2^2 &= \tfrac{r_1(r_2-1)(r_2-s_2)}{s_2(r_1-r_2)}; \\ k_3^2 &= \tfrac{(r_3-s_3)(a_3-s_3-r_2+s_2+1)}{(a_3-s_3)(r_3-s_3-r_2+s_2+1)}. \end{aligned}$$

The inequalities (E1) and (E2) ensure that the transformations B_1, B_2, B_3, $B_2 \circ B_1$ and $B_3 \circ B_1$ are hyperbolic. We also have that $a(B_1) = \infty$, $r(B_1) = r_1$, $a(B_2) = 0$, $r(B_2) = r_2$, $a(B_3) = a_3$, $r(B_3) = r_3$, $a(B_2 \circ B_1) = 1$, $r(B_2 \circ B_1) = s_2$, $a(B_3 \circ B_1) = r_1 - \frac{(r_1-a_3)(r_1-r_3)(r_1-1)(r_1-s_2)}{r_1(r_1-r_2)(r_1-s_3)}$ and $r(B_3 \circ B_1) = s_3$. We denote the axis of the transformations B_1, B_2, B_3, $B_2 \circ B_1$ and $B_3 \circ B_1$ by N_1, N_2, N_3, $N_{2,1}$ and $N_{3,1}$, respectively, as shown in figure 3.

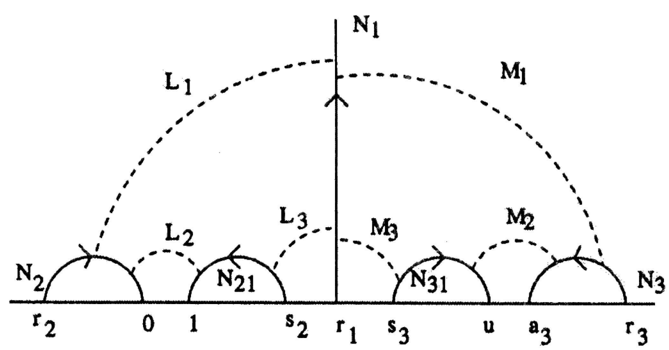

Figure 3.

Construct the geodesics:

L_1 (common perpendicular of N_1 and N_2),

L_2 (common perpendicular between N_2 and $N_{2,1}$),

L_3 (common perpendicular between N_1 and $N_{2,1}$),

M_1 (common perpendicular between N_1 and N_3),

M_2 (common perpendicular between N_3 and $N_{3,1}$)

and

M_3 (common perpendicular between N_1 and $N_{3,1}$).

Denote by R_i and S_i the reflection on L_i and M_i, respectively (see figure 3). Direct computations show that the hyperbolic distance d between L_1 and L_3 is the same as the hyperbolic distance between M_1 and M_3 ($d = \log(\sqrt{\frac{r_1(r_1-r_2)}{(r_1-s_2)(r_1-1)}})$). In particular, we have $B_1 = R_1 \circ R_3 = S_1 \circ S_3$, $B_2 = R_2 \circ R_1$ and $B_3 = S_2 \circ S_1$. We consider the groups $G_1 =< B_1, B_2 >$ and $G_2 =< B_1, B_3 >$. The group G_i uniformizes a pant P_i, both of them having a boundary (given by the axe N_1) of the same length. It is easy to see that we can apply the first combination theorem of Maskit [7] to these groups with common subgroup $J =< B_1 >$ (the discs used in such a theorem are the discs bounded by the axe N_1). As a consequence, the group G generated by B_1, B_2 and B_3 is a free group of rank three and $\mathbf{H}/G$ is a hyperbolic Riemann surface of type $(0,4)$ (see figure 4 for a fundamental domain of G).

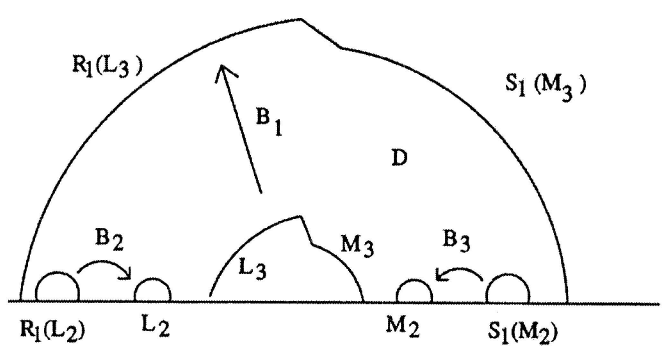

Figure 4.

The construction of a quasiconformal homeomorphism of the hyperbolic plane as required is standard using the fundamental domains for Γ and G which are topologically the same. $\square$

Remark. The angle involve in gluing the pants corresponding to the groups G_1 and G_2 in the above proof is given by

$$\theta = \theta(r_1, r_2, r_3, a_3, s_2, s_3) = \frac{\pi}{\log(\sqrt{\frac{r_1(r_1-r_2)}{(r_1-s_2)(r_1-1)}})} \log(\sqrt{\frac{(r_1-a_3)(r_1-r_3)}{r_1(r_1-r_2)}}).$$

We can either make $r_2 = 0$ or $s_2 = 1$ or $u = s_3$ or $a_3 = r_3$ to obtain explicit models for the Teichmüller space (as boundaries of the above model) of parabolic Riemann surfaces of type $(0, 4)$. We must remark that if we make $r_1 = \infty$, then we get only a model of the Teichmüller space of pants. In the particular case, $r_2 = 0$, $s_2 = 1$, $u = s_3$ and $a_3 = r_3$, we obtain Maskit's model for the Teichmüller space of marked surfaces of genus zero with four punctures (see [4]). In this case, the above formula for u gives us $u = r_1 + \frac{(r_1-1)(r_3-1)}{r_1} = r_3 - \frac{r_3-r_1}{r_1}$. The explicit model is $\mathcal{M} = \{(r_1, r_3) \in \mathbf{R}^2; 1 < r_1 < r_3\}$. In figure 5 we draw a fundamental domain for the group $G =< B_1, B_2, B_3 >$ such that $\Phi_3(G =< B_1, B_2, B_3 >) = (r_1, 0, r_3, r_3, 1, u)$.

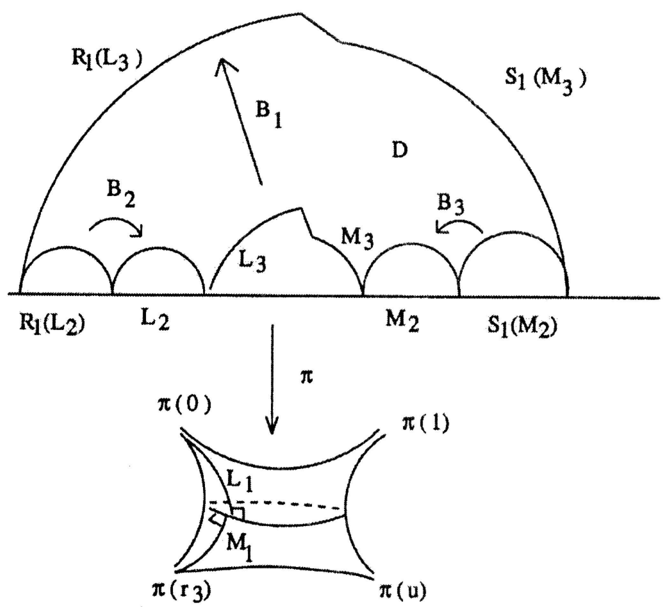

Figure 5.

In this case the angle $\theta = \theta(r_1, r_3)$ is $\theta = \theta(r_1, r_3) = \frac{\pi}{\log(\frac{r_1}{r_1 - 1})} \log(\frac{r_3 - r_1}{r_1})$. We have an explicit one-to-one real analytic diffeomorphism between $\mathcal{M}$ and the Fricke space (angle, length coordinates) $L : \mathcal{M} \to \mathbf{R}^2$ defined by $L(r_1, r_3) = (\log(\frac{r_1}{r_1 - 1}), \theta(r_1, r_3))$. Similarly, one can use the above parameters to find explicit models for the Teichmüller spaces of surfaces of type $(0, m)$, where $m \geq 5$.

5.2. Teichmüller Spaces of Riemann Surfaces of Type (1,1)

A Riemann surface of type $(1, 1)$ is topologically equivalent to a surface of genus one with a deleted point. If the boundary is a puncture, then we call it a parabolic Riemann surface of type $(1, 1)$.; otherwise, we call it a hyperbolic Riemann surface of type $(1, 1)$.

A hyperbolic Riemann surface S of type $(1,1)$ can be constructed from a pant P, where two of its boundaries are bounded by closed simple geodesics of the same length, by identifying these two boundaries. This can be seen as follows. Start with a Fuchsian group Γ uniformizing a surface S of type $(1,1)$. Let l be a non-dividing simple closed geodesic on S and denote by $R = S - l$. Fix a lifting T of the region R in the hyperbolic plane and consider G_1 the subgroup of G fixing T. The group G_1 uniformizes a pant P with two boundaries bounded by simple closed geodesics of the same length. We may assume up to conjugation that $G_1 =< A_1, A_2 >$ is normalized by $a(A_1) = \infty$, $a(A_2) = 0$ and $a(A_2 \circ A_1) = 1$, and the axis of A_1 and A_2 are projected to the two geodesics of the same length. If $r_1 = r(A_1)$, $r_2 = r(A_2)$ and $s = r(A_2 \circ A_1)$, then the axis of the transformations A_1, A_2 and $A_2 \circ A_1$, denoted as N_1, N_2 and $N_{2,1}$, respectively, are as shown in figure 6. A fundamental domain for G_1 is also shown in figure 6.

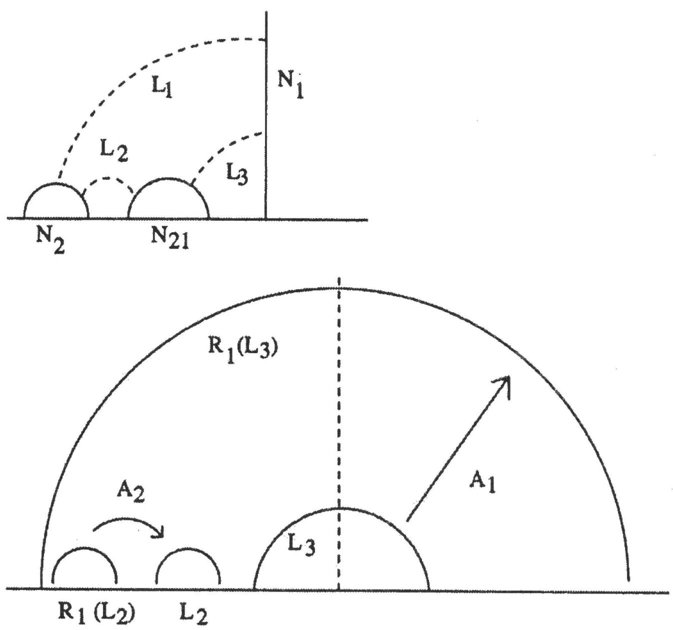

Figure 6.

As a consequence of Theorem 1, we have that the multiplier of the transformations A_1 and A_2 are given by $k_1^2 = \frac{r_1(r_1-r_2)}{(r_1-1)(r_1-s)}$ and $k_2^2 = \frac{r_1(1-r_2)(s-r_2)}{s(r_1-r_2)}$, respectively. Our assumption on the equality of the geodesics lengths implies that $s = \frac{r_2(1-r_1)}{1-r_2}$. Since $s > 1$ we have the inequality $r_2(2-r_1) < 1$. Observe that $r_2 < 0$ and $r_1 > 1$ imply that $\frac{r_2(1-r_1)}{1-r_2} < r_1$. On the other hand, both geodesics N_1 and N_2 project onto l on S. It implies that there is a transformation A_3 in Γ such that $A_3 \circ A_2 \circ A_3^{-1} = A_1^{-1}$. We denote by D_2 and D_1 the discs bounded by N_2 and N_1, respectively, where the boundary of D_1 contains $r_1 + 1$ and the boundary of D_2 contains $\frac{r_2}{2}$. We have that $A_3(D_2)$ is equal to the complement of $D_1 \cup N_1$. In this way the conditions of the Maskit's second combination theorem ([7]) are satisfied for G_1 and $G_2 =< A_3 >$. In particular, Γ is the HNN-extension of G_1 by A_3 (see figure 7 for a fundamental domain of Γ).

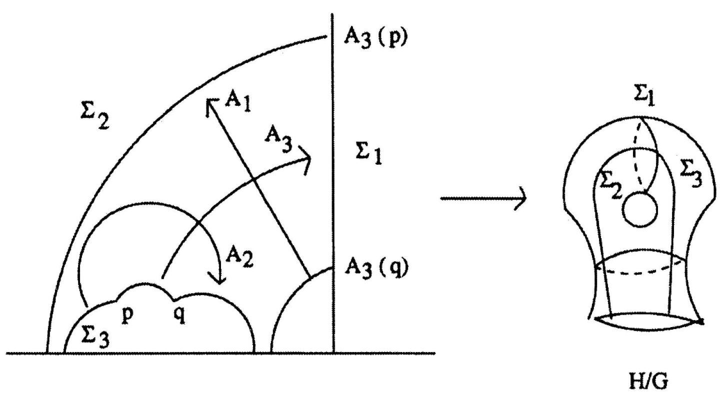

Figure 7.

Denote by a and r the attracting and repelling fixed points of A_3, respectively. In this case, $r_2 < r < 0$ and $r_1 < a$. If $k^2 > 1$ denotes the multiplier of A_3, then

$$A_3 = \begin{pmatrix} r - ak^2 & ar(k^2-1) \\ 1-k^2 & rk^2 - a \end{pmatrix}.$$

Since necessarily $A_3(0) = r_1$ and $A_3(r_2) = \infty$, we obtain $a = \frac{r_1 r_2}{r}$ and $k^2 = \frac{r_2(r_1 - r)}{r(r-r_2)}$. The inequality $r > r_2$ implies the inequality $r_1 < a$. In this way, we obtain that the only variables are given by r_1, r_2 and r satisfying the inequalities

(F1) $r_2(2 - r_1) < 1$; and

(F2) $r_2 < r < 0 < 1 < r_1$.

The Teichmüller space of Γ is identified with the set W of marked groups $(G, (B_1, B_2, B_3))$ satisfying:

(1) G is discrete subset of $PGL^+(2, \mathbf{R})$;

(2) $a(B_1) = \infty$, $a(B_2) = 0$ and $a(B_2 \circ B_1) = 1$; and

(3) There is a quasiconformal homeomorphism $F : \mathbf{H} \to \mathbf{H}$ such that $F \circ A_i \circ F^{-1} = B_i$, for $i = 1, 2, 3$.

In particular, for $(G, (B_1, B_2, B_3))$ in W, the only relation is given by $B_3 \circ B_2 \circ B_3^{-1} = B_1^{-1}$.

Let us consider the region $\mathcal{H}$ in $\mathbf{R}^3$ consisting of the points (r_1, r_2, r) satisfying the inequalities (F1) and (F2). As a consequence of theorem 1, we have a one-to-one real analytic map $\Theta : W \to \mathcal{H}$ defined by $\Theta((G, (B_1, B_2, B_3))) = (r(B_1), r(B_2), r(B_3))$.

Corollary 3. *$\Theta(W) = \mathcal{H}$.*

Proof. Start with a point (r_1, r_2, r) in $\mathcal{H}$. Set $s = \frac{r_2(1-r_1)}{1-r_2}$ and $a = \frac{r_1 r_2}{r}$. Inequality (F1) ensure that $1 < s < r_1$, and (F2) ensure that $r_1 < a$. Using the values r_1, r_2 and s we can construct a pant group $G_1 =< B_1, B_2 >$ where $a(B_1) = \infty$, $r(B_1) = r_1$, $a(B_2) = 0$, $r(B_2) = r_2$, $a(B_2 \circ B_1) = 1$. Direct computations shows that $r(B_2 \circ B_1) = s$. We also construct B_3 satisfying $r(B_3) = r$, $a(B_3) = a$ and $k^2 = \frac{a-r_2}{r-r_2}$. Inequality (F2) ensures that B_3 maps the disc D_2 (bounded by the geodesic joining r_2 and 0 and containing $\frac{r_2}{2}$ in the boundary) onto the disc D_1' (bounded by the geodesic joining ∞ and r_1 and containing r_2 in the boundary). One can see that the conditions of the second combination theorem of Maskit holds for G_1 and $G_2 =< B_3 >$. As a consequence of the same theorem, we have that the group G generated by G_1 and G_2 has the presentation $G =< G_1, G_2 >=< B_1, B_2, B_3; B_3 \circ B_2 \circ B_3^{-1} = B_1^{-1} >$, and $(G, (B_1, B_2, B_3))$ belongs to $\mathcal{H}$. $\square$

As a consequence of Theorem 4, we have that $\mathcal{H}$ is an explicit model for the Teichmüller space of hyperbolic surfaces of type $(1, 1)$. This model only uses the fixed points of some elements. Compare to Maskit's model in [5] in which one of the parameters is a multiplier.

In the boundary of $\mathcal{H}$ (given by $r_2(2-r_1) = 1$) we have an explicit model for the Teichmüller space of parabolic Riemann surfaces of type $(1, 1)$. In this case, the model is given by $\mathcal{N} = \{(r_1, r); \frac{1}{2-r_1} < r < 0 < 1 < r_1\}$. Figure 8 shows a fundamental domain for a marked group $(G, (B_1, B_2, B_3))$ obtained from a point (r_1, r) in $\mathcal{N}$.

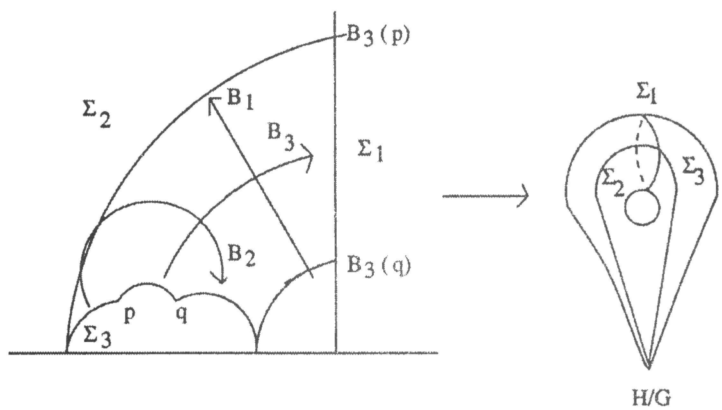

Figure 8.

5.3. Teichmüller Spaces of Closed Riemann Surfaces of Genus Two

Let S be a closed Riemann surface of genus two. On S we consider a set of oriented simple loops (through a point z) γ, α_1, α_2, β_1 and β_2 as shown in figure 9.

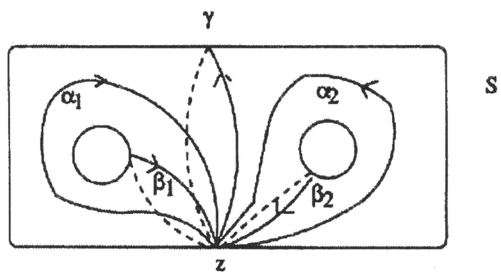

Figure 9.

The fundamental group of S, with base point at z, has a presentation of the form

$$\Pi_1(S,z) =< \gamma, \alpha_1, \alpha_2, \beta_1, \beta_2; \alpha_1^{-1}\beta_1\alpha_1 = \gamma\beta_1, \alpha_2^{-1}\beta_2\alpha_2 = \gamma\beta_2 >.$$

Let F be a Fuchsian group (acting on the hyperbolic plane $\mathbf{H} = \{z \in \mathbf{C}; lm(z) > 0\}$) uniformizing the surface S, that is, there is a holomorphic covering $\pi : \mathbf{H} \to S$ with F as covering group. Choose a point x in $\mathbf{H}$ such that $\pi(x) = z$. We have a natural isomorphism $\lambda : \Pi_1(S,z) \to F$ as follows. For a class $[\eta] \in \Pi_1(S,z)$ we consider a representative η. Now we lift η under π at the point x. The end point of such a lifting is of the form $f_\eta(x)$ for a unique element $f_\eta \in F$. Basic covering theory asserts that if ρ is another representative of $[\eta]$, then $f_\rho = f_\eta$. We define $\lambda([\eta]) = f_\eta$. We set $A_1 = \lambda([\gamma])$, $A_2 = \lambda([\beta_1])$, $A_3 = \lambda([\beta_2])$, $F_1 = \lambda([\alpha_1])$ and $F_2 = \lambda([\alpha_2])$. In particular, a presentation of F is given by

$$F =< A_1, A_2, A_3, F_1, F_2; F_1 \circ A_2 \circ F_1^{-1} = A_2 \circ A_1, F_2 \circ A_3 \circ F_2^{-1} = A_3 \circ A_1 >.$$

Denote by $\tilde{\gamma}$, $\tilde{\alpha_1}$, $\tilde{\alpha_2}$, $\tilde{\beta_1}$ and $\tilde{\beta_2}$ the projections on S under π of the axis of the transformations A_1, F_1, F_2, A_2 and A_3, respectively, as shown in figure 10.

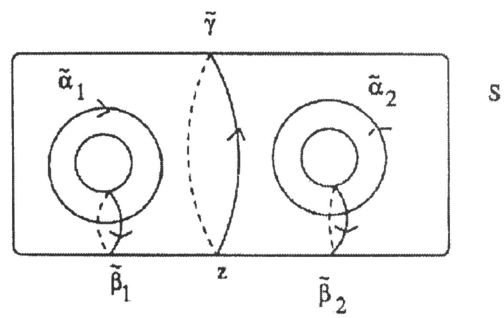

Figure 10.

We have oriented the axe $Ax(H)$ of a (hyperbolic) transformation H in such a way that the attracting fixed point of H is the end point. The orientations of the projections of the above axis carry the natural orientation induced from the one given to the axis. We can normalize F in $PGL(2, \mathbf{R})$ such that $a(A_1) = \infty$, $a(A_2) = 0$ and $a(A_2 \circ A_1) = 1$. The choice made for the transformations A_1, A_2, A_3, F_1 and F_2 imply that the axis of these transformations are as shown in figure 11.

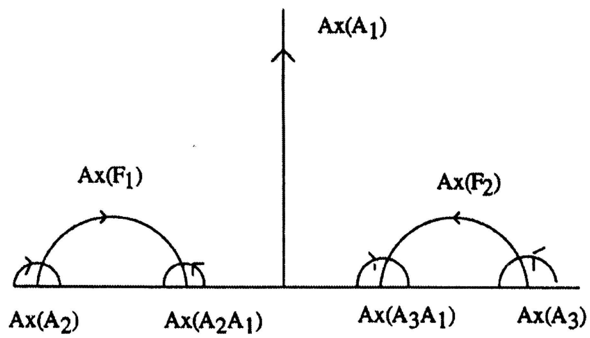

Figure 11.

For simplicity, we denote $a(A_3) = a_3$, $r(A_i) = r_i$ $(i = 1, 2, 3)$, $a(A_3 \circ A_1) = u$, $r(A_j \circ A_1) = s_j$ $(j = 2, 3)$, $r(F_k) = x_k$, $a(F_k) = y_k$ $(k = 1, 2)$. In particular, we have that these fixed points satisfy the following inequalities:

$$r_2 < x_1 < 0 < 1 < y_1 < s_2 < r_1 < s_3 < y_2 < u < a_3 < x_2 < r_3.$$

From the above, we observe that the group $G_1 = \langle A_1, A_2, A_3 \rangle$ uniformizes a hyperbolic surface of type $(0, 4)$ with two pairs of holes of the same length. These two pairs of holes are the ones bounded by the loops $\tilde{\beta}_1$ and $\pi(Ax(A_2 \circ A_1))$, and the loops $\tilde{\beta}_2$ and $\pi(Ax(A_3 \circ A_1))$, respectively. We denote by D_2, D_2', D_3 and D_3' the disjoint discs bounded by the axis of the transformations A_2, $A_2 \circ A_1$, A_3 and $A_3 \circ A_1$, respectively. Since $F_1 \circ A_2 \circ F_1^{-1} = A_2 \circ A_1$ and $F_2 \circ A_3 \circ F_2^{-1} = A_3 \circ A_1$, we have that $F_1(\bar{D}_2) \cap D_2' = \emptyset$ and $F_1(\bar{D}_3) \cap D_3' = \emptyset$, where the bar represents the Euclidean closure. One can apply the second combination theorem of Maskit [7] to the pair of groups G_1 and $G_2 =< F_1 >$. The resulting group $G_3 =< G_1, G_2 >= G_1 *_{F_1}$ uniformizes a surface of genus one with two boundary components of the same length. Now we apply again the second combination theorem of Maskit to the pair G_3 and $G_4 =< F_2 >$ to obtain the group $F =< G_3, G_4 >= G_3 *_{F_2}$ uniformizing the surface S.

For the group F in we have some equalities. We denote by $k^2(H) > 1$ the multiplier of the (hyperbolic) transformation H. Then we have the following:

(1) $k^2(A_2) = k^2(A_2 \circ A_1)$;

(2) $k^2(A_3) = k^2(A_3 \circ A_1)$;

(3) $F_1(0) = 1$, $F_1(r_2) = s_2$;

(4) $F_2(a_3) = u$, $F_2(r_3) = s_3$;

(5) $k^2(A_1) = \frac{r_1(r_1-r_2)}{(1-r_1)(s_2-r_1)}$;

(6) $k^2(A_2) = \frac{r_1(r_2-1)(r_2-s_2)}{s_2(r_1-r_2)}$;

(7) $k^2(A_3) = \frac{(r_3-s_3)(a_3-k^2(A_1)s_3-r_1(1-k^2(A_1)))}{(a_3-s_3)(r_3-k^2(A_1)s_3-r_1(1-k^2(A_1)))}$.

Direct computations imply that $u = \frac{r_1^2(1-k^2(A_1))+r_1(k^2(A_1)s_3-a_3-r_3)+a_3r_3}{k^2(A_1)(s_3-r_1)}$. Equality (1) implies $s_2 = \frac{r_1(r_2-1)}{r_2-r_1}$. Equality (2) and the fact that $r_1 < s_3$ imply $s_3 = \frac{a_3(1-r_1)+r_1(r_2-1)}{r_2-r_1}$. As a consequence, we obtain that $u = \frac{r_1(1-r_2+r_3)-r_3}{r_1-r_2}$.

The transformation F_k is uniquely determined by x_k, y_k and $k^2(F_k)$. Equality (3) implies

$y_1 = \frac{r_1(1-x_1-r_2)+x_1r_2}{x_1(1-2r_1+r_2)+r_1-r_2}$, and $k^2(F_1) = \frac{(x_1-1)(r_1(x_1-1+r_2)-x_1r_2)}{x_1(r_1-1)(r_2-x_1)}$.

Equality (4) implies $y_2 = \frac{P}{Q}$ and $k^2(F_2) = \frac{(-a_3+r_1+a_3r_1-r_1x_2+r_2x_2-r_1r_2)(r_1-r_1x_2+r_2x_2-r_1r_2-r_3+r_1r_3)}{(a_3-x_2)(r_1-1)(r_1-r_2)(x_2-r_3)}$, where $P = a_3r_1 - a_3r_1x_2 - r_1^2 - a_3r_1^2 + a_3r_1^2x_2 + a_3r_2x_2 - a_3r_1r_2 - r_1r_2x_2 - a_3r_1r_2x_2 + 2r_1^2r_2 + a_3r_1^2r_2 - r_1^2r_2x_2 + r_1r_2^2x_2 - r_1^2r_2^2 - a_3r_3 + r_1r_3 + 3a_3r_1r_3 - r_1r_3x_2 - r_1^2r_3 - 2a_3r_1^3r_3 + r_1^2r_3x_2 - a_3r_2r_3 + r_2r_3x_2 - r_1r_2r_3 + a_3r_1r_2r_3 - r_1r_2r_3x_2 + r_1^2r_2r_3$, and

$Q = (r_2 - r_1)(-a_3 + x_2 + r_1 + a_3r_1 - 2r_1x_2 + r_2x_2 - r_1r_2 - r_3 + r_1r_3)$.

We observe from the above that the group F is uniquely determined by the fixed points r_1, r_2, r_3, a_3, x_1 and x_2.

We consider the open (connected) region (in $\mathbf{R}^6$) $\mathcal{F}$ defined as:

$$\mathcal{F} = \{(r_1, r_2, r_3, a_3, x_1, x_2); r_2 < x_1 < 0 < 1 < y_1 < s_2 < r_1 < s_3 < y_2 < u < a_3 < x_2 < r_3\},$$

where the values y_1, s_2, s_3, y_2 and u are given by the formulae above.

The Teichmüller space $\mathcal{T}(F)$ can be identified with the space (subspace of the suitable deformation space) consisting of marked groups $(G, (B_1, B_2, B_3, H_1, H_2))$ satisfying the following properties:

(1) G is a discrete subgroup of $PGL^+(2, \mathbf{R})$;

(2) $a(B_1) = \infty$, $a(B_2) = 0$, $a(B_2 \circ B_1) = 1$;

(3) There exists a quasiconformal homeomorphism $K : \mathbf{H} \to \mathbf{H}$ such that $K \circ A_i \circ K^{-1} = B_i$, for $i = 1, 2, 3$, and $K \circ F_j \circ K^{-1} = H_j$, for $j = 1, 2$.

In particular, for $(G, (B_1, B_2, B_3))$ in $\mathcal{T}(F)$, the only relations are given by $H_1 \circ B_2 \circ H_1^{-1} = B_2 \circ B_1$ and $H_2 \circ B_3 \circ H_2^{-1} = B_3 \circ B_1 >$.

We have, as a consequence of the previous sections, an one-to-one real analytic map $\phi : \mathcal{T}(F) \to \mathcal{F}$, defined as

$$\phi((G, (B_1, B_2, B_3, H_1, H_2))) = (r(B_1), r(B_2), r(B_3), a(B_3), r(H_1), r(H_2)).$$

Corollary 4. *$\phi(\mathcal{T}(F)) = \mathcal{F}$.*

Proof. The map ϕ is a surjective map. This is a consequence of the combination theorems of Maskit in [7]. We sketch the idea of the proof of this assertion. Given a point $p = (r_1, r_2, r_3, a_3, x_1, x_2)$ in $\mathcal{F}$, we can construct the values s_2, s_3, u, y_1 and y_2. We can also construct values $k^2(H_1)$ and $k^2(H_2)$. Using these values we obtain unique transformations B_1, B_2, B_3, H_1 and H_2. The inequalities defining the set $\mathcal{F}$ asserts that the above transformations are all hyperbolic. The configuration of the axis of the transformations B_1, B_2, B_3, $B_2 \circ B_1$, $B_3 \circ B_1$, H_1 and H_2 is the same as for the axis of the transformations A_1, A_2, A_3, $A_2 \circ A_1$, $A_3 \circ A_1$, F_1 and F_2 as shown in figure 11. The group

$T_1 =< B_1, B_2 >$ is a free group of rank two uniformizing a pant. The group $T_2 =< B_1, B_3 >$ also uniformizes a pant. One can check that the conditions of the first combination theorem of Maskit are satisfied and one obtain that $T_3 =< T_1, T_2 >= T_1 *_{<B_1>} T_2$ uniformizes a surface of type $(0, 4)$. It is easy to check that $H_k \circ B_{k+1} \circ H_k^{-1} = B_{k+1} \circ B_1$, for $k = 1, 2$. The conditions of the second combination theorem holds for the group T_3 and $T_4 =< T_1 >$. The group $T_5 =< T_3, T_4 >= T_3 *_{H_1}$ uniformizes a surface of genus one with two boundary components. Again apply the second combination theorem to the groups T_5 and $T_6 =< H_2 >$ to obtain that the marked group $T_7 =< T_5, T_6 >= T_5 *_{H_2}$ belongs to $\mathcal{T}(F)$. $\square$

Remarks.

(1) The main difference between the above parameters and the ones in [9] is the fact that we only use fixed points. The parameters given by Maskit also only contain fixed points and are more easy than the ones obtained here.

(2) Maskit's model uses the fact that any Riemann surface of genus two is constructed from two isometric pants. Min model uses the fact that any Riemann surface of genus two is constructed from two surfaces of type $(1, 1)$. Our construction uses the fact that any Riemann surface of genus two is constructed from a surface of type $(0, 4)$ with two pairs of boundaries of the same length.

(3) We can relate our fixed point parameters to other parameters of Teichmüller space, for instance to Fenchel-Nielsen Parameters in the same way as done for the case of signature $(0, 4)$. The way to do this is the following. At each each axis $Ax(A_1)$, $Ax(A_2)$ and $Ax(A_3)$ we have associated the multiplier of the transformations A_1, A_2 and A_3, respectively. These multipliers are in function of the fixed point parameters and, in particular, the hyperbolic lengths of the geodesics $\widetilde{\alpha}_1$, $\widetilde{\alpha}_2$ and $\widetilde{\gamma}$ are in function of these fixed points parameters. To obtain the angle at $\widetilde{\gamma}$, we look at the common orthogonal geodesic L_1 (respectively, L_2) to both $Ax(A_1)$ and $Ax(A_2)$ (respectively, $Ax(A_1)$ and $Ax(A_3)$).

These two geodesics determine an arc in $Ax(A_1)$, whose hyperbolic length determine the angle (also in function of the fixed point parameters). To determine an angle at $\widetilde{\alpha}_1$ we look the common orthogonal geodesic L_3 of $Ax(A_1)$ and $Ax(A_2A_1)$. We proceed to see the hyperbolic arc in $Ax(A_2A_1)$ determined by the intersection point of L_3 with $Ax(A_2A_1)$ and the image by F_1 of the intersection point of L_3 with $Ax(A_2)$. Similarly for looking at the angle at $\widetilde{\alpha}_2$. The explicit computations are similar to the case $(0,4)$ and are left to the interested reader.

(4) We can apply Theorem 1 to construct explicit models for the Teichmüller space of arbitrary topologically finite Riemann surfaces. This will be done elsewhere.

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