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A VARIATIONAL INEQUALITY RELATED TO AN ELLIPTIC OPERATOR

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Abstract

It is considered the non-linear operator

$$A(u) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right), \qquad p > 2,$$

and a variational inequality associated to the operator

$$A(u) + g(x, u)$$

with g satisfying some conditions.

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We will be considering the elliptic operator:

$$A(u) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\frac{\partial u}{\partial x_i}^{p-2} \frac{\partial u}{\partial x_i} \right), \qquad p > 2,$$

and a variational inequality associated to the operator

$$A(u) + g(x,u)$$

with g satisfying some conditions, on a not necessarily bounded domain $\Omega \subset \mathbf{R}^n$.

We will assume that $g(x,u)$ satisfies the following hypothesis:

(a) $g(x,r)$ is measurable, in x, on Ω, for a fixed $r \in \mathbf{R}$; it is continuous in r, for each x, fixed. For each $x \in \Omega$, $g(x,0) = 0$ and for all $r \in \mathbf{R}$, $x \in \Omega$, $g(x,r)r \geq 0$;

(b) $g(x,r)$ is a non-decreasing function in r, on $\mathbf{R}$. For each fixed r, $g_r(x) = g(x,r)$ is a $L^1(\Omega)$-function.

Let us remaind that, under (b), if

$$G(x,r) = \int_0^r g(x,s)\, ds,$$

G is continuous, convex, in r, for all x and r, with $G(x,0) = 0$.

Moreover,

$$G'(x,r) = g(x,r).$$

In what follows we will use the notation as in [3].

Our goal is to prove the following theorem, where $\Omega \subset \mathbf{R}^n$ is an open subset and $A(u)$ is the above described operator.

Theorem. *If $g(x,r)$ satisfies* (a) *and* (b) *and $G(x,r)$ is its primitive with respect to r then, if V is any closed subspace of $W_0^{1,p}(\Omega)$ and $K \subset V$ is a closed, convex subset of V with $0 \in K$ and $f \in V'$ then, there is a unique $u \in K$ such that $g(x,u)$ is in $L^1(\Omega)$, $g(x,u)u$ is in $L^1(\Omega)$ and $\int G(x,u)\, dx < \infty$. Moreover, u, satisfies both inequalities:*

(i) *for each $v \in K \cap L^\infty(\Omega)$,*

$(A(u) + g(x,u) - f, v - u) \geq 0$

(ii) *for each $v \in K$,*

$\int G(x,v)\, dx - \int G(x,u)\, dx + (A(u) - f, v - u) \geq 0.$

Proof: We know, from [3] that for each positive, integer n, there is a solution u_n, in K of the variational inequality:

$$(A(u_n) + g_n(x, u_n) - f, v - u_n) \geq 0, \quad (v \in K).$$

Since A is coercive and $0 \in K$,

$$(A(u_n) + g_n(x, u_n) - f, u_n) \leq 0.$$

Therefore,

$$\alpha||u_n||^p \leq (A(u_n), u_n) \leq (A(u_n) + g_n(x, u_n), u_n) \leq (f, u_n). \qquad *$$

with $\alpha \in \mathbf{R}$.

We will show that, if

$$u_n \rightharpoonup u, \quad \text{weakly in} \quad V,$$

u is a solution of the problem in the theorem and that

$$w = A(u).$$

From (*), we have that

$$\int_\Omega g_n(x, u_n) u_n \, dx$$

is uniformly bounded, for all n.

The sequence $\{g_n(x, u_n)\}\_n$ is equiuniformly integrable on Ω.

For each R, positive, integer,

$$R|g_n(x, u_n)| \leq u_n \, g_n(x, u_n) + R\left\{g(x, R) + |g(x, -R)|\right\}, \qquad **$$

since $g(x, \cdot)$ is non-decreasing.

Let $\varepsilon > 0$ and $B \subset \Omega$, measurable. We have

$$\int_B |g_n(x, u_n)| \, dx \leq \frac{1}{R} \int_B u_n \, g_n(x, u_n) \, dx + \int_B g(x, R) \, dx + |g(x, -R)|$$

and this may be taken less than ε for all n if $\mu(B)$ is sufficiently small, as far as $g(\cdot, r) \in L^1(\Omega)$.

From inequality (**), with $N \subset \Omega$,

$$\int_N |g_n(x,u_n)|\,dx \leq \frac{1}{R}\int_N u_n\,g_n(x,u_n)\,dx + \int_N (g(x,R)\,dx + |g(x,-R)|)\,dx.$$

Since $\int_N u_n\,g_n(x,u_n)\,dx \leq M_2$, independently of n, there exists $B_\varepsilon \subset \Omega$ measurable with $\mu(B_\varepsilon) < \infty$, such that

$$\int_{\Omega - B_\varepsilon} |g_n(x,u_n)|\,dx \leq \varepsilon, \quad \text{for all} \quad n \in \mathbf{N}.$$

Moreover, since $||u_n|| \leq C$ by the Sobolev immersion theorems, we may obtain (u_n) a subsequence of (u_n) such that

$$u_n \to u, \quad \text{a.e. in} \quad \Omega.$$

Therefore

$$g_n(x,u_n) \to g(x,u), \quad \text{a.e., in} \quad \Omega.$$

By the convergence theorem of Vitali, $g(x,u)$ is in $L^1(\Omega)$, and

$$g_n(x,u_n) \to g(x,u)$$

strongly in $L^1(\Omega)$. Using Fatou's lemma, $g(x,u)u$ is in $L^1(\Omega)$.

For each $n \in \mathbf{N}$, let us define

$$G_n(x,r) = \int_0^r g_n(x,s)\,ds.$$

For each r and s,

$$\begin{aligned} G_n(x,r) - G_n(x,s) = G_n'(x,\xi)(r-s) &= g_n(x,\xi)(r-s) \\ &\geq g_n(x,s)(r-s), r \leq \xi \leq s. \end{aligned}$$

Let $v \in K$ be arbitrary. We have:

$$G_n(x,v) - G_n(x,u_n) \geq g_n(x,u_n)(v-u_n).$$

Integrating over Ω, we obtain:

$$\int G_n(x,v) - G_n(x,u_n) \geq \int g_n(x,u_n)(v-u_n) \geq (f - A(u_n), v - u_n).$$

If v is such that $\int G(x,v) < \infty$ then

$$|G_n(x,v)| \leq |G(x,v)|$$

what implies that

$$\int G_n(x,v) \to \int G(x,v).$$

We also have

$$G_n(x,u_n) \to G(x,u) \quad \text{a.e. in} \quad \Omega.$$

Moreover,

$$G(x,u(x)) = \int_0^{u(x)} g(x,s)\,ds \leq g(x,u(x))u(x),$$

and since $g(x,u)u \in L^1(\Omega)$,

$$\int G(x,u)\,dx < \infty.$$

We obtain,

$$\int G(x,v) - \int G(x,u) \geq \limsup(A(u_n) - f, u_n - V),$$

for each $v \in K$, such that

$$\int G(x,v)\,dx < \infty.$$

Letting, $v = u$, we have

$$0 \geq \limsup(A(u_n) - f, u_n - u) = \limsup(A(u_n), u_n - u).$$

Since A is pseudo-monotonic from V to V', $w = A(u)$ that is,

$$A(u_n) \quad \text{converges weakly to} \quad A(u)$$

in V', and

$$(A(u_n), u_n) \to (A(u), u).$$

Therefore, for each $v \in K$ with

$$\int G(x,v)\,dx < \infty$$

we have:

$$\int G(x,v) - \int G(x,u) \geq (A(u) - f, u - v),$$

which is part our theorem.

Let now, $v \in K \cap L^\infty(\Omega)$.

We have,

$$\int g_n(x,u_n)(v - u_n) \geq (A(u_n) - f, u_n - v).$$

By the lemma of Fatou, we have, since $v \in L^\infty(\Omega) \cap K$:

$$\liminf g_n(x,u_n)(v-u_n) \geq \liminf (A(u_n)-f, u_n-v) = (A(u)-f, u-v).$$

Therefore

$$\int g(x,u)(v - u) \geq (A(u) - f, u - v)$$

or

$$(A(u) + g(x,u) - f, v - u) \geq 0$$

what is other part of our theorem.

Unicity

Let u_1 and u_2 be two solutions of our problem, for a given $f \in V'$.

Then,

$$\int G(x,v) - \int G(x,u_1) \geq (A(u_1) - f, u_1 - v)$$

and

$$\int G(x,v) - \int G(x,u_2) \geq (A(u_2) - f, u_2 - v).$$

$G(x,r)$ is convex in r. Hence if we put

$$v = \frac{1}{2}\,(u_1 + u_2)$$

v is a permissible element, and

$$u_1 - v = \frac{1}{2}(u_1 - u_2) = -(u_2 - v).$$

Hence,

$$\begin{aligned} &\textstyle\int G(x,v) - \int G(x,u_1) \geq \frac{1}{2}(Au_1 - f, u_1 - u_2) \\ &\textstyle\int G(x,v) - \int G(x,u_2) \geq \frac{1}{2}(Au_2 - f, u_2 - u_1). \end{aligned}$$

Adding the inequalities, we obtain:

$$\frac{1}{2}(A(u_1) - A(u_2), u_1 - u_2) + \int G(x,u_1) + \int G(x,u_2) - 2\int G(x,v) \leq 0.$$

Therefore

$$0 \leq (Au_1 - Au_2, u_1 - u_2) + 2\left[2\int \tfrac{G(x,u_1)+G(x,u_2)}{2} - G\left(x, \tfrac{u_1+u_2}{2}\right)\right]$$
$$\leq 0.$$

G is convex and therefore the second term is zero. Hence,

$$(Au_1 - Au_2, u_1 - u_2) = 0$$

$$\sum_{i=1}^{n} \int_\Omega \left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u_1}{\partial x_i} - \left|\frac{\partial u_2}{\partial x_i}\right|^{p-2} \frac{\partial u_2}{\partial x_i} \right) \left(\frac{\partial u_1}{\partial x_i} - \frac{\partial u_2}{\partial x_i} \right) dx = 0.$$

The function

$$\lambda \to |\lambda|^{p-2}\lambda$$

is monotone. Therefore, for each i,

$$\left(\left|\frac{\partial u_1}{\partial x_i}\right|^{p-2} \frac{\partial u_1}{\partial x_i} - \left|\frac{\partial u_2}{\partial x_i}\right|^{p-2} \frac{\partial u_2}{\partial x_i} \right) \left(\frac{\partial u_1}{\partial x_i} - \frac{\partial u_2}{\partial x_i} \right) = 0.$$

for almost all $x \in \Omega$.

By the same reason, $\dfrac{\partial u_1}{\partial x_i} = \dfrac{\partial u_2}{\partial x_i}$, for each i.

But $u_1 - u_2 = 0$, on Γ, since $u_1 - u_2 \in W_0^{1,p}(\Omega)$. Therefore,

$$u_1 = u_2 \,.$$

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