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ON THE COHOMOLOGY OF FOLIATED BUNDLES *

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Abstract

We prove a de Rham-like theorem for foliated bundles $F \to (M, \mathcal{F}) \xrightarrow{\pi} B$ *showing that the cohomology* $H^*(\mathcal{F})$ *is isomorphic to the equivariant cohomology* $H_\Gamma\left(\widetilde{B}, C^\infty(F)\right), \Gamma = \pi_1(B)$ *and* $\widetilde{B}$ *the universal covering of* B*. When* B *is an Eilenberg-Mac Lane space* $K(\Gamma, 1)$ *the cohomology* $H^*(\mathcal{F})$ *is the cohomology of the* Γ*-module* $C^\infty(F)$*. This gives algebraic models for* $H^*(\mathcal{F})$ *and geometrial models for the cohomology of the* Γ*-module* $C^\infty(F)$*. Using this isomorphism and a theorem of J. Palis and J.C. Yoccoz on the triviality of centralizers of diffeomorphisms, [14] and [15] we show that* $H^*(\mathcal{F})$ *is infinite dimensional for a large class of foliated bundles.*

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Introduction

The cohomology of foliated manifolds appears naturally in the study of locally free actions of Lie groups and characteristic classes of foliations, [3], [17] and [18]. In this article we study the cohomology of foliated bundles $F \to (M, \mathcal{F}) \xrightarrow{\pi} B$ suspension of actions $\varphi : \Gamma \to Diff\,(F)$, where B and F are connected C^∞ manifolds and $\Gamma = \pi_1(B)$. We show in Theorem 2.1 that the foliated cohomology $H^*(\mathcal{F})$ of a foliated bundle π is isomorphic to the equivariant cohomology $H^*_\Gamma(\tilde{B}, C^\infty(F))$ where $\tilde{B}$ is the universal covering of B and $C^\infty(F)$ has the Γ-module structure given by the action φ. This isomorphism is via a natural de Rham mapping. When B is an Eilenberg-Mac Lane space $K(\Gamma, 1)$, i.e. $\tilde{B}$ is contractible then $H^*(\mathcal{F})$ is isomorphic to the cohomology $H^*(\Gamma, C^\infty(F))$ of the Γ-module $C^\infty(F)$, the action on $C^\infty(F)$ being $\gamma \cdot h = h \circ \varphi_\gamma$, $\gamma \in \Gamma$ and $h \in C^\infty(F)$. In this way we have both algebraic models for the foliated cohomology $H^*(\mathcal{F})$ and geometrical models for the cohomology $H^*(\Gamma, C^\infty(F))$ of the Γ-module $C^\infty(F)$. Theorems 2.4 and 3.3 and J. Palis and J.C. Yoccoz results on the triviality of centralizers of diffeomorphisms, [14], [15] show that the cohomology $H^*(\mathcal{F})$ of a foliated bundle suspension of an action $\varphi : \Gamma \to Diff\,(F)$ is infinite dimensional for a large class of actions. In section 4 we discuss the case $B = T^p$ and show that the cohomology of groups gives a procedure for the computation of $H^*(\mathbf{Z}^p, C^\infty(F))$ which can be used to give an alternative simple way for computing the cohomology of linear foliations of T^n, [2]. We also state R.U. Luz, [9] computation of the cohomology of the actions of $\mathbf{Z}^p$ by affine transformations of T^q.

1. The Cohomology of a Foliated Bundle

The foliated cohomology introduced by B.L. Reinhart in [16] appears naturally in the study of locally free actions of Lie groups and characteristic classes of foliations, [3], [17] and [18].

Let $\mathcal{F}$ be a p-dimensional foliation of M and $\Lambda\,(M)$ the graded algebra of all C^∞ differential forms on M. If $I(\mathcal{F}) \subset \Lambda\,(M)$ is the annihilating ideal of $\mathcal{F}$, then $I(\mathcal{F})^{q+1} = 0$, $q = m - p$ being the codimension of $\mathcal{F}$. Thus $\Lambda(\mathcal{F}) = \dfrac{\Lambda(M)}{I(\mathcal{F})}$ is a graded algebra, called the algebra of *differential forms along* $\mathcal{F}$. The elements of $\Lambda^j(\mathcal{F})$ may be thought of as sections of the j-th exterior power of the dual bundle of the tangent bundle $T\mathcal{F}$ of $\mathcal{F}$. Since by Frobenius's theorem $dI(\mathcal{F}) \subset I(\mathcal{F})$ the differential $d : \Lambda(M) \to \Lambda(M)$ induces the *foliated differential* $d_f : \Lambda(\mathcal{F}) \to \Lambda(\mathcal{F})$. The kernel $\mathbf{Z}(\mathcal{F})$ of d_f is the set of d_f-closed forms and the image $B(\mathcal{F})$ of d_f is the set of d_f-exact forms. The cohomology $H^*(\mathcal{F})$ of the differential complex $(\Lambda(\mathcal{F}), d_f)$ is the *cohomology of the foliated manifold* $(M, \mathcal{F})$. This is a natural generalization to foliations of the de Rham cohomology. Let $\tau : \Lambda(\mathcal{M}) \to \Lambda(\mathcal{F})$ be the canonical projection. We say that $\xi \in \Lambda^j(M)$ is d_f-closed if $d\xi \in I(\mathcal{F})$ which is equivalent to $\tau(\xi) \in \mathbf{Z}(\mathcal{F})$ and we denote by $\overline{\xi}$ the cohomology class of $\tau(\xi)$ in $H^j(\mathcal{F})$.

In this article we study the cohomology of a foliated bundle. Let B and F be connected orientable smooth manifolds, p and q dimensional, respectively, and $\varphi : \pi_1(B) \to Diff\,(F)$ be a left action. The *foliated bundle* $F \to (M, \mathcal{F}) \xrightarrow{\pi} B$ *suspension* of φ is constructed as follows: let $p : \tilde{B} \to B$ be the universal covering of B and $x_o \in B$. Associated to φ there is an action

$$\Phi : \pi_1(B, x_o) \to Diff\left(\widetilde{B} \times F\right) \tag{1.1}$$

given by $\Phi_\gamma(\tilde{x}, y) = (\gamma \cdot \tilde{x}, \varphi_\gamma^{-1}(y))$ where $\gamma \cdot \tilde{x}$ denotes the image of $\tilde{x}$ by the deck transformation of $\tilde{B}$ corresponding to the homotopy class γ of $\pi_1(B, x_o)$. The orbit space M of Φ is a manifold and actually we have a fiber bundle $F \to M \xrightarrow{\pi} B$. Every object of $\tilde{B} \times F$ which is invariant under Φ induces a corresponding object on M. To the natural foliation $\tilde{\mathcal{F}}$ given by the projection $\tilde{B} \times F \to F$, which is invariant under Φ, corresponds a foliation $\mathcal{F}$ on M which is transverse to the fibers of π. We think of $\Lambda(\mathcal{F})$ as the set of differential forms ξ of $\Lambda\left(\widetilde{\mathcal{F}}\right)$ which are invariant under Φ i.e. $\Phi_\gamma^*(\xi) = \xi$. We observe that $\xi \in \Lambda\left(\widetilde{\mathcal{F}}\right)$ invariant under the action Φ is equivalent to

$$[\gamma]^*(\xi_y) = \xi_{\varphi_\gamma(y)} \tag{1.2}$$

where $\xi_y = j_y^*(\xi)$, $j_y : \tilde{B} \to \tilde{B} \times F$ being the inclusion $j_y(\tilde{x}) = (\tilde{x}, y)$ and $[\gamma]$ the automorphism of $\tilde{B} \times F$ given by $[\gamma](\tilde{x}, y) = (\gamma \cdot \tilde{x}, y)$.

The image of $\pi^* : \Lambda(B) \to \Lambda(\mathcal{F})$ is called the space of *basic forms.* In [3] to each invariant Borel probability measure μ on F there was associated an epimorphism

$$P_\mu : \Lambda(\mathcal{F}) \to \Lambda(B)$$

of differential complexes i.e. P_μ is a continuous surjective linear mapping which commutes with the differentials: $P_\mu \circ d_f = d \circ P_\mu$. P_μ is defined by the properties:

(i) $P_\mu \circ \pi^* = id$

(ii) $P_\mu(h) = \int_F h(\tilde{x}, y) d\mu(y)$ if $h \in \Lambda^o(\mathcal{F})$ (i.e. $h \in \Lambda^o(\tilde{\mathcal{F}})$ is Φ-invariant).

The split short exact sequence

$$0 \to Ker \to \Lambda(\mathcal{F}) \xrightarrow{P_\mu} \Lambda(B) \to 0$$

gives the split short exact cohomology sequences

$$0 \to H^j(Ker) \to H^j(\mathcal{F}) \to H^j(B) \to 0, \qquad 0 \le j \le p.$$

Thus

$$H^j(\mathcal{F}) = H^j(B) \oplus H^j(Ker) \tag{1.3}$$

for $0 \le j \le p$.

2. The Cohomology of a Foliated Bundle as Equivariant Cohomology

Let $F \to (M, \mathcal{F}) \xrightarrow{\pi} B$ be the foliated bundle suspension of an action $\varphi : \pi_1(B, x_o) \to Diff(F)$. The fundamental group $\Gamma = \pi_1(B, x_o)$ acts freely on the universal covering $\tilde{B}$ of B by deck transformations $\gamma : \tilde{x} \to \gamma \cdot \tilde{x}$. A (left) $\mathbf{Z}\Gamma$-module, also called a Γ-module, consists of an abelian group A together with a homomorphism of the integral group ring $\mathbf{Z}\Gamma$ of Γ to the ring of endomorphisms of A, [4], [5] and [12].

Since Γ acts freely on $\tilde{B}$ the simplicial complex $S_*(\tilde{B})$ is a Γ-module: if $\sigma : \Delta^k \to \tilde{B}$ is a singular k-simplex of $\tilde{B}$ then $\gamma \cdot \sigma = \gamma \circ \sigma$ is again a k-simplex of $\tilde{B}$. Thus $S_*(\tilde{B})$ is a Γ-module. The ring $C^\infty(F)$

of all C^∞ functions $h : F \to R$ also has a Γ-module structure given by the action $\varphi : \Gamma \to Diff\,(F)$. Γ acts on $C^\infty(F)$ by $\gamma \cdot h = h \circ \varphi_\gamma$. We also write $C^\infty(F)$ for this Γ-module. Let $Hom_\Gamma(S_*(\tilde{B}), C^\infty(F))$ be the group of all Γ-homomorphisms of $S_*(\tilde{B})$ into $C^\infty(F)$. A Γ-homomorphism is a $\mathbf{Z}$-homomorphism $\ell : S_*(\tilde{B}) \to C^\infty(F)$ such that

$$\ell(\gamma \cdot \sigma) = \gamma \cdot \ell(\sigma) = \ell(\sigma) \circ \varphi_\gamma \tag{2.1}$$

for all σ in $S_*(\tilde{B})$ and γ in Γ.

Thus $Hom_\Gamma(S_*(\tilde{B}), C^\infty(F))$ is a differential complex. The boundary operator δ being, as usual, $\delta\ell(\sigma) = \ell(\partial\sigma)$, ∂ the boundary operator of $S_*(\tilde{B})$. The cohomology $H^*_\Gamma(\tilde{B}, C^\infty(F))$ of the above complex is known as the *equivariant cohomology* of $\tilde{B}$ with coefficients $C^\infty(F)$, [12].

In this section we prove a de Rham type theorem which says that the cohomology $H^*(\mathcal{F})$ of a foliated bundle π is isomorphic to the equivariant cohomology $H^*_\Gamma(\tilde{B}, C^\infty(F))$ via the natural de Rham mapping. Let $\tilde{\mathcal{F}}$ be the foliation on $\tilde{B} \times F$ given by the natural projection $\tilde{B} \times F \to F$. We have a natural de Rham mapping

$$k : \Lambda^j(\tilde{\mathcal{F}}) \to Hom\left(S_j\left(\tilde{B}\right), C^\infty\left(F\right)\right), \quad 0 \le j \le p$$

given by

$$k_\xi(\sigma)(y) = \int_\sigma \xi_y \tag{2.2}$$

for ξ in $\Lambda^j(\tilde{\mathcal{F}})$, $\sigma \in S_j(\tilde{B})$ and $y \in F$. This mapping retricts to the subcomplex $\Lambda\left(\mathcal{F}\right)$ of Φ-invariant forms of $\Lambda\left(\tilde{\mathcal{F}}\right)$ as a natural de Rham mapping

$$k : \Lambda\left(\mathcal{F}\right) \to Hom_\Gamma\left(S_*\left(\tilde{B}\right), C^\infty\left(F\right)\right) \tag{2.3}$$

To see this notice that from (1.2) and (2.2) a form ξ in $\Lambda^j(\tilde{\mathcal{F}})$ is Φ-invariant if and only if $k_\xi : S_j(\tilde{B}) \to C^\infty(F)$ is a Γ-homomorphism, $0 \le j \le p$.

2.1 Theorem (de Rham's Theorem for foliated bundles). *Let B and F be connected paracompact C^∞ manifolds and $F \to (M, \mathcal{F}) \xrightarrow{\pi} B$ be the suspension foliated bundle of an action φ*

$$\varphi : \Gamma = \pi_1(B, x_o) \to Diff\,(F)\,.$$

Then the de Rham mapping (2.3) induces an isomorphism

$$k^* : H^j(\mathcal{F}) \to H^j_\Gamma(\tilde{B}, C^\infty(F)), \quad 0 \leq j \leq p,$$

$p = dimB$.
The proof follows the same basic pattern of Massey's proof, [10] of the classical de Rham's theorem, based on Milnor's proof of the Poincaré duality theorem. For Massey's proof to work we have to show the existence of Mayer-Vietoris sequences for the equivariant cohomology. Mayer-Vietoris sequences for foliated cohomology, are well known, see [6].

Let $\mathcal{U} = \{U_j\}$ be an open covering of B such that $\pi^{-1}(U_j) \xrightarrow{\pi} U_j$ are trivial and U_j are contractible. Denote by $S_j(\tilde{B},\mathcal{U})$ the Γ-module generated by "$\mathcal{U}$-small" j-simplices i.e. simplices whose ranges lie in elements of $\mathcal{U}$ and let $H_\Gamma((\tilde{B},\mathcal{U}), C^\infty(F))$ be the cohomology of the complex $Hom_\Gamma(S_j(\tilde{B},\mathcal{U}), C^\infty(F))$. Let $j_\mathcal{U} : Hom_\Gamma(S_j(\tilde{B}), C^\infty(F)) \longrightarrow Hom_\Gamma(S_j(\tilde{B},\mathcal{U}), C^\infty(F))$ be the restriction homomorphism.

2.2 Lemma. The restriction homomorphism $j_\mathcal{U}$ induces an isomorphism

$$j^*_\mathcal{U} : H^j_\Gamma(\tilde{B}, C^\infty(F)) \xrightarrow{\sim} H^j_\Gamma((\tilde{B},\mathcal{U}), C^\infty(F)), 0 \leq j \leq p.$$

Proof: The proof is essentially the same as for the classical isomorphism, [7]

$$H^j(B, C^\infty(F)) \simeq H^j((B,\mathcal{U}), C^\infty(F))$$

Let $Sd : S_j(\tilde{B}) \to S_j(\tilde{B})$ be the subdivision homomorphism and

$$R : S_j(\tilde{B}) \to S_{j+1}(\tilde{B})$$

be the corresponding homotopy operator i.e.

$$\left.\begin{array}{l} \partial \circ Sd = Sd \circ \partial \\ \text{and} \\ \partial \circ R + R \circ \partial = id - Sd \end{array}\right\} \quad (2.4)$$

Choose a $\mathbf{Z\Gamma}$-basis $\mathcal{B}$ of the Γ-module $S_*\left(\tilde{B}\right)$. Define Γ-homomorphisms sd and T by

$$sd\,(\sigma) = Sd\,(\sigma)$$

and

$$T(\sigma) = R(\sigma)$$

for every σ in B

$$sd(\gamma \cdot \sigma) = \gamma \cdot Sd(\sigma)$$
$$T(\gamma \cdot \sigma) = \gamma \cdot R(\sigma)$$

Thus

$$\left.\begin{array}{l} \partial \circ sd = sd \circ \partial \\ \text{and} \\ \partial \circ T + T \circ \partial = id - sd \end{array}\right\} \tag{2.5}$$

The proof follows now as in the classical case observing that if l is a Γ-cocycle then $\widetilde{l} = l \circ T$ is a Γ-homomorphism and $\delta\bar{l} = l - l \circ sd$. There exists an integer $n > 0$ such that $(sd)^n(\sigma)$ is a linear combination of $\mathcal{U}$-small singular simplices. Thus $l \circ (sd)^n$ is cohomologous to l, proving the lemma.

Proof of Theorem 2.1 We first notice that the theorem is true in degree zero since both $H^o(\mathcal{F})$ and $H^o_\Gamma(\tilde{B}, C^\infty(F))$ are isomorphic to the subspace $C^\infty(F)^\Gamma$ of all Γ-invariant C^∞ functions $h : F \to R$ i.e. $\gamma \cdot h = h \circ \varphi_\gamma = h$ for all $\gamma \in \Gamma$ and the de Rham mapping k^* is the identity in this case.

Case 1. U is an open contractible set of B and $\pi^{-1}(U) \xrightarrow{\pi} U$ is a trivial fiber bundle. Let $\mathcal{F}_U$ be the restriction of $\mathcal{F}$ to $\pi^{-1}(U)$. In this case we have
(i) $H^o(\mathcal{F}_U) = C^\infty(F)^\Gamma$ and $H^j(\mathcal{F}_U) = 0$ if $j \geq 1$ and
(ii) $H^o_\Gamma(p^{-1}(U), C^\infty(F)) = C^\infty(F)^\Gamma$ and $H^j_\Gamma(p^{-1}(U), C^\infty(F)) = 0$ if $j \geq 1$.

To prove (i) notice that $p^{-1}(U) = \bigcup\limits_{\gamma \in \Gamma} \gamma \cdot \tilde{U}$ and $p : \tilde{U} \to U$ is a diffeomorphism. Thus if $\xi \in \Lambda^j(\mathcal{F}_U)$ is d_f-closed then by the Poincaré lemma ξ is d_f-exact in $\tilde{U} \times F$ i.e. there exists a leafwise $(j-1)$-form w such that $d_f w = \xi$. Using the notation in (1.2) extend w to a Φ-invariant form on $p^{-1}(U) \times F$ by $[\gamma]^* w_y = w_{\varphi_\gamma(y)}$. Thus

$$d_f[\gamma]^* w_y = d_f w_{\varphi_\gamma(y)} = \xi_{\varphi_\gamma(y)} = [\gamma]^* \xi_y \tag{2.6}$$

and ξ is d_f-exact. Thus $H^j(\mathcal{F}_U) = 0$ for $j \geq 1$. Analogously, if $\ell : S_j(p^{-1}(U)) \to C^\infty(F)$ is a Γ-cocycle, then the restriction of ℓ

to $S_j(\tilde{U})$ is a **Z**-cocycle and as $\tilde{U}$ is contractible then there exists a **Z**-homomorphism $\tilde{\ell} : S_{j+1}(\tilde{U}) \to C^\infty(F)$ such that $\delta\tilde{\ell} = \ell$. We also denote by $\tilde{\ell}$ the extension of $\tilde{\ell}$ to a Γ-homomorphism $\tilde{\ell} : S_{j+1}(p^{-1}(U)) \to C^\infty(F)$. Thus $\delta\tilde{\ell} = \ell$ and $H^j_\Gamma(p^{-1}(U), C^\infty(F)) = 0$ if $j \geq 1$.

Now it is clear that

$$h^* : H^j(\mathcal{F}_U) \to H^j_\Gamma(p^{-1}(U), C^\infty(F))$$

is an isomorphism.

Case 2. B is the union of two open sets U and V and de Rham's theorem is assumed to hold for U, V and $U \cap V$. We show the theorem also holds for $U \cup V$. To prove the theorem in this case we use Mayer-Vietoris sequences. We denote by $\mathcal{F}_U$, $\mathcal{F}_V$ and $\mathcal{F}_{U\cap V}$ the restrictions of the foliation $\mathcal{F}$ to $\pi^{-1}(U)$, $\pi^{-1}(V)$ and $\pi^{-1}(U \cap V)$, respectively. Let $i : \pi^{-1}(U \cap V) \to \pi^{-1}(U)$, $j : \pi^{-1}(U \cap V) \to \pi^{-1}(V)$, $k : \pi^{-1}(U) \to \pi^{-1}(U \cup V)$ and $\ell : \pi^{-1}(V) \to \pi^{-1}(U \cup V)$ be the inclusions. We consider the mappings

$$a : \Lambda^*(\mathcal{F}) \to \Lambda^*(\mathcal{F}_U) \oplus \Lambda^*(\mathcal{F}_V)$$

and

$$\beta : \Lambda^*(\mathcal{F}_U) \oplus \Lambda^*(\mathcal{F}_V) \to \Lambda^*(\mathcal{F}_{U\cap V})$$

given by

$$a(w) = (k^*w, \ell^*w)$$

and

$$\beta(w_1, w_2) = i^*w_1 - j^*w_2$$

We have then the short exact sequence of differential complexes, [6]

$$(2.7) \qquad 0 \to \Lambda^*(\mathcal{F}) \xrightarrow{a} \Lambda^*(\mathcal{F}_U) \oplus \Lambda^*(\mathcal{F}_V) \xrightarrow{\beta} \Lambda^*(\mathcal{F}_{U\cap V}) \to 0$$

Now we consider Mayer-Vietoris sequence for equivariant cohomology. Consider the cochain mappings

$$\begin{aligned} a' : Hom_\Gamma(S_r(\tilde{B}, U), C^\infty(F)) \to Hom_\Gamma(S_r(p^{-1}(U)), C^\infty(F)) \oplus \\ \otimes Hom_\Gamma(S_r(p^{-1}(V), C^\infty(F)) \end{aligned}$$

and

$$\beta' : Hom_\Gamma(S_r(p^{-1}(U)), C^\infty(F)) \oplus Hom_\Gamma(S_r(p^{-1}(V)), C^\infty(F)) \to \\ \to Hom_\Gamma(S_r(p^{-1}(U \cap V)), C^\infty(F))$$

where $U = \{U, V\}$ and $0 \leq r \leq p$. These mappings are defined in the usual way, [9]. Clearly a' is a monomorphism. It follows from the Mayer-Vietoris sequence for singular cohomology that β' is an epimorphism. For if $\tilde{\ell} : S_r(p^{-1}(U \cap V)) \to C^\infty(F)$ is a Γ-homomorphism consider the Z-homomorphism $\ell : S_r(U \cap V) \to C^\infty(F)$ given by $\ell(\sigma) = \tilde{\ell}(\tilde{\sigma})$ for every $\tilde{\sigma}$ in the $\mathbf{Z\Gamma}$-basis B fixed before and $\tilde{\sigma} \in S_r(p^{-1}(U \cap V))$. Thus there exist Z-homomorphisms $\ell_1 : S_r(U) \to C^\infty(F)$ and $\ell_2 : S_r(V) \to C^\infty(F)$ such that $\ell = i^*(\ell_1) - j^*(\ell_2)$ where we also denote by i and j the inclusions of $U \cap V$ in U and V, respectively. We define Γ-homomorphisms $\tilde{\ell}_1 : S_r(p^{-1}(U)) \to C^\infty(F)$ and $\tilde{\ell}_2 : S_r(p^{-1}(V)) \to C^\infty(F)$ by $\tilde{\ell}_1(\tilde{\sigma}) = \ell_1(p \circ \tilde{\sigma}) = \ell_1(\sigma)$ for every $\tilde{\sigma}$ in $B \cap S_r(p^{-1}(U))$ and analogously for $\tilde{\ell}_2$. Clearly $\beta'(\tilde{\ell}_1, \tilde{\ell}_2) = \tilde{\ell}$. Thus we have the short exact sequence of cochain complexes

$$0 \to Hom_\Gamma(S_*((\tilde{B}, U), C^\infty(F)) \xrightarrow{a'} Hom_\Gamma(S_*(p^{-1}(U), C^\infty(F)) \oplus$$

$$\oplus Hom_\Gamma(S_*(p^{-1}(V), C^\infty(F)) \xrightarrow{\beta'} Hom_\Gamma(S_*(p^{-1}(U \cap V)), C^\infty(F)) \to 0$$
(2.8)

Finally, we put the sequences (2.7) and (2.8) into a commutative diagram where the first vertical mapping is the composite of de Rham mapping k and the restriction homomorphism $j_{\mathcal{U}}$ and the last two mappings are de Rham mappings. Now taking the cohomology sequences associated to (2.7) and (2.8) the theorem is proved in this case using Lemma 2.2 and the five lemma, [12].

Case 3. $B = \bigcup_{i=1}^{\infty} U_i$, where $U_1 \subset U_2 \subset \ldots, \subset U_n \subset U_{n+1} \subset \ldots$ is a nested sequence of open sets with compact closures. It is assumed that de Rham's theorem holds for each U_i; we will show that it also holds for B. To carry out the proof in this case, we need to use inverse limits as in Massey's proof, [10] and [11]. The inclusions $U_i \subset B$ induce cochain mappings $\Lambda^*(\mathcal{F}) \to \Lambda^*(\mathcal{F}_i)$, $\mathcal{F}_i$ being the restriction of

$\mathcal{F}$ to $\pi^{-1}(U_i)$. The inverse sequence $\Lambda^*(\mathcal{F}_i)$ satisfies the Mittag-Leffler condition, thus

$$lim^1 \Lambda^j(\mathcal{F}_i) = 0 \quad \text{for all} \quad j$$

and there is the natural short exact sequence

$$(2.9) \qquad 0 \to lim^1 H^{j-1}(\mathcal{F}_i) \to H^j(\mathcal{F}_i) \to \lim inv H^j(\mathcal{F}_i) \to 0$$

Similarly for the cochain complexes $Hom_\Gamma(S_*(U_i), C^\infty(F))$ i.e. we also have the short exact sequences

$$(2.10) \qquad \begin{aligned} 0 \to \lim{}^1 H_\Gamma^{j-1}(\pi^{-1}(U_i), C^\infty(F)) \to H_\Gamma^j(\tilde{B}, C^\infty(F)) \to \\ \to \lim inv H_\Gamma^j(\pi^{-1}(U_i), C^\infty(F)) \to 0 \end{aligned}$$

Now apply the de Rham homomorphism from sequence (2.2) to the sequence (2.3) and one easily prove the theorem in this case.

Case 4. B is an open set of R^p. Thus B is a countable union of open sets U_i as in Case 1 with compact closures. By Cases 1 and 2, de Rham's theorem holds for finite unions $\bigcup_{i=1}^{n} U_i$, by induction on n. To complete the proof in this case one passes to the limit as $n \to \infty$ using Case 3.

Case 5. B is paracompact. In this case B is a countable union of open sets U_i diffeomorphic to R^p and with compact closures. Let $V_n = \bigcup_{i=1}^{n} U_i$. We can prove by induction on n using Cases 2 and 4 that de Rham's theorem holds true for each V_n and each V_n is compact. Then it follows from Case 3 that de Rham's theorem holds for B.

Let as before $\varphi : \Gamma \to Diff(F)$ be an action, $\Gamma = \pi_1(B)$ and $F \to (M, \mathcal{F}) \xrightarrow{\pi} B$ the suspension foliated bundle of φ. Let H be the submodule of the Γ-module $C^\infty(F)$ generated by the generated by the functions h-$\gamma \cdot h$, $\gamma \in \Gamma$. Let $C^\infty(F)^\Gamma$ denote the set of functions $h \in C^\infty(F)$ which are invariant under the action of Γ i. e. $\gamma \cdot h = h \circ \varphi_\gamma = h$; $C^\infty(F)^\Gamma$ is the largest submodule of $C^\infty(F)$ on which Γ acts trivially. The group of *co-invariants* of $C^\infty(F)$ is denoted by

$C^{\infty}(F)_{\Gamma}$ and is the quotient $\dfrac{C^{\infty}(F)}{\mathcal{H}}$. A simple computation shows that $H^{0}_{\Gamma}\left(\widetilde{B}, C^{\infty}(F)\right) = C^{\infty}(F)^{\Gamma}$. If the universal covering $\widetilde{B}$ of B is contractible (or equivalently $\pi_i(B) = 0$ for $i \succeq 2$) i. e. B is an Eilenberg - Mac Lane space $K(\Gamma, 1)$, then the *augmented cellular chain complex*

$$\cdots \to S_{j+1}\left(\widetilde{B}\right) \to S_j\left(\widetilde{B}\right) \to \cdots S_0\left(\widetilde{B}\right) \xrightarrow{\varepsilon} \mathbf{Z} \to 0 \tag{2.11}$$

is a free resolution of $\mathbf{Z}$ over $\mathbf{Z}\Gamma$, [5]. In this case the groups $H^{j}_{\Gamma}\left(\widetilde{B}, C^{\infty}(F)\right)$ are called the *cohomology groups of the* Γ-*module* $C^{\infty}(F)$ and denoted by $H^{j}(\Gamma, C^{\infty}(F)), 0 \leq j \leq p$. Theorem 2.1 gives a geometrical interpretation for the cohomology of the Γ-module $C^{\infty}(F)$ and an alternative way of understanding the cohomology of a foliated bundle.

Associated to the action φ we also have a short exact sequence

$$\begin{aligned} 0 \to Hom_{\Gamma}(S_*(\tilde{B}), \mathcal{H}) &\xrightarrow{i} Hom_{\Gamma}(S_*(\tilde{B}), C^{\infty}(F)) \xrightarrow{\rho} \\ &\to Hom\left(S_*(B), C^{\infty}(F)_{\Gamma}\right) \to 0 \end{aligned} \tag{2.12}$$

The mapping ρ is defined as follows: let Σ_j be the submodule of $S_j(\tilde{B})$ generated by the elements $\sigma - \gamma \cdot \sigma$ for $\sigma \in S_j(\tilde{B})$ and $\gamma \in \Gamma$. Notice that $S_j(B) = \dfrac{S_j(\tilde{B})}{\Sigma_j}$ and if $\ell : S_j(\tilde{B}) \to C^{\infty}(F)$ is a Γ-homomorphism, then $\ell(\Sigma_j) \subset \mathcal{H}$ and ℓ induces a $\mathbf{Z}$-homomorphism $\hat{\ell} : S_j(B) \to C^{\infty}(F)_{\Gamma}$. The mapping ρ is defined by $\rho(\ell) = \hat{\ell}$. Clearly the kernel of ρ is $Hom_{\Gamma}(S_*(\tilde{B}), \mathcal{H})$. To show ρ is an epimorphism, choose a $Z\Gamma$-basis B for $S_j(\tilde{B})$. If $\hat{\ell} : S_j(B) \to C^{\infty}(F)_{\Gamma}$ is a homomorphism, choose a function h_{σ} in each coset $\hat{\ell}(p_*\sigma)$, $\sigma \in B$ and let $\ell(\tilde{\sigma}) = h_{\sigma}$. This defines a Γ-homomorphism $\ell : S_j(\tilde{B}) \to C^{\infty}(F)$ and clearly $\rho(\ell) = \hat{\ell}$. We notice that the sequence (2.12) is in general not split. Associated to (2.12) we have the long exact cohomology sequence

$$\begin{aligned} 0 \to H^{o}_{\Gamma}(\tilde{B}, \mathcal{H}) \to H^{o}_{\Gamma}(\tilde{B}, C^{\infty}(F)) \to H^{0}\left(B, C^{\infty}(F)_{\Gamma}\right) &\xrightarrow{\Delta} H^{1}_{\Gamma}\left(\tilde{B}, \mathcal{H}\right) \to \\ \to H^{1}_{\Gamma}(\tilde{B}, C^{\infty}(F)) \to H^{1}\left(B, C^{\infty}(F)_{\Gamma}\right) &\xrightarrow{\Delta} \ldots \end{aligned} \tag{2.13}$$

Let G be a connected Lie group and Γ be a discrete cocompact subgroup of G i.e. the right coset space $B = \dfrac{G}{\Gamma}$ is a compact manifold.

Each right translation $R_g : G \to G$ induces a diffeomorphism R_g^o on B and the right action induces a right action of Gon B. Let $\mathcal{G}$ be the Lie algebra of G. If $E \in \mathcal{G}$ is a left invariant vector field on G and $a : R \times G \to G$ its flow, then $a_t(g) = R_{a_t(e)}(g)$ i.e. the restriction of the right action $G \times G \xrightarrow{R} G$ to the 1-parameter subgroup $a_t(e)$is the flow of E. Thus the restriction of R^o to $a_t(e)$gives a flow E_o on B and we have an injective homomorphism of $\mathcal{G}$ into the Lie algebra $\chi(B)$ of all C^∞ vector fields on B. Let $\mathcal{G}_o$be the image of this homomorphism. Associated to a left action φ of Γ on a q-manifold F there is a left action $\Phi : \Gamma \to Diff\,(G \times F)$ given by $\Phi_\gamma(g, y) = (\gamma g, _{\gamma^{-1}}(y))$. The orbit space $M = \dfrac{G \times F}{\Gamma}$ is a manifold and we have the foliated bundle $F \to (M, \mathcal{F}) \xrightarrow{\pi} B$ suspension of φ. We notice that the mappings $R_g \times id$ on $G \times F$ induce automorphisms H_g of the foliated bundle π i.e. H_g preserves the leaves of $\mathcal{F}$ and $\pi \circ H_g = R_g^o \circ \pi$. Thus we have an action

$$H : G \to Aut\,(\pi) \tag{2.14}$$

of G as automorphisms of π, called the *canonical action* of π. The restriction of H to Γ gives an action $H^*$of Γ as automomorphisms of the cohomology $H^*(\mathcal{F})$, [17]. Also the right translations R_γ of G induce automorphisms R_γ^* of the equivariant cohomology $H_\Gamma^*(G, C^\infty(F))$.

2.3 Proposition. The action R^* of Γ as automorphisms of $H_\Gamma^*(G, C^\infty(F))$ is trivial i.e. $R_\gamma^* = id$ for all $\gamma \in \Gamma$.

Proof: Given a Γ-cocycle $\ell : S_j(G) \to C^\infty(F)$, $0 \leq j \leq p$, $p = dimG$ we show $R_\gamma^*(\ell) - \ell$ is a Γ-coboundary. Choose any smooth path $c : I \to G$, $c(0) = e$ and $c(1) = \gamma$. We have a homotopy $h : G \times I \to G$, $h(x, t) = x\,c(t)$and let $h_* : S_*(G \times I) \to S_*(G)$be the induced chain mapping. Consider the prism operators, [7]

$$P : S_j(G) \to S_{j+1}(G \times I), \quad 0 \leq j \leq p.$$

Since

$$\partial P + P\partial = j_*^1 - j_*^o \tag{2.15}$$

where $j^t(x) = (x,t)$, $t = 0,1$, the homotopy h and (2.8) give

$$\partial h_* P + h_* P \partial = (R_\gamma)_* - id.$$

Since ℓ is a Γ-cocycle we get

$$R_\gamma^*(\ell) - \ell = \delta\widehat{\ell}$$

where $\hat{\ell} = \ell \circ h_* P$. To show that $\hat{\ell}$ is a Γ-homomorphism we show that $h_* P : S_j(G) \to S_j(G)$ is a Γ-homomorphism. In fact, for each $\gamma \in \Gamma$ we have, [7]

$$(L_\gamma \times id)_* P = P(L_\gamma)_*. \tag{2.16}$$

Now observing that

$$h \circ (L_\gamma \times id) = L_\gamma \circ h \tag{2.17}$$

we see that $(L_\gamma)_*$ commutes with $h_* P$, i.e. $h_* P$ is a Γ-homomorphism.

2.4 Theorem. Let G be a connected Lie group, Γ a discrete cocompact subgroup of G and $F \to (M, \mathcal{F}) \xrightarrow{\pi} B$ be the suspension foliated bundle of an action φ of Γ on a q-manifold F. Assume ∴ is generated by the iterates of a diffeomorphism φ_γ where $\gamma \in \Gamma$ is in the center of G. If the only φ-invariant functions are the constants, then there is a natural isomorphism

$$H_\Gamma^j(G, C^\infty(F)) \simeq Hom\left(H_j\left(B\right), C^\infty\left(F\right)_\Gamma\right)$$

for $0 \le j \le p$, $p = dimG$.

Proof: The Theorem follows from the cohomology sequence (2.6) if we show that $H_\Gamma^j(G, \mathcal{H}) = 0$ for $0 \le j \le p$. To show that $\mathcal{H}$ is acyclic choose a probability measure μ on F which is invariant under φ_γ and let $C_o^\infty(F)$ be the subspace of functions h in $C^\infty(F)$ with μ-measure zero i.e. $\int_F h\, d\mu = 0$. Thus $\mathcal{H} \subset C_o^\infty(F)$ and each function h in $\mathcal{H}$ can be written uniquely as $h = \chi - \chi \circ \varphi_\gamma, \chi \in C_o^\infty(F)$. It follows that each Γ-cocycle $\ell : S_j(G) \to \mathcal{H}$ can be written uniquely as $\ell = \hat{\ell} - \gamma \cdot \hat{\ell}$ where $\hat{\ell} : S_j(G) \to C_o^\infty(F)$ is a Γ-cocycle i.e.

$$\ell(\sigma) = \hat{\ell}(\sigma) - \hat{\ell}(\sigma) \circ \varphi_\gamma \tag{2.18}$$

for $\sigma \in S_j(G)$, $0 \leq j \leq p$. Since γ is in the center of G then $\ell = \hat{\ell} - R^*_\gamma(\hat{\ell})$ and by Proposition 2.3 ℓ is a Γ-coboundary, proving the theorem.

3. On the Group of Co-invariants of and Action $\varphi : \Gamma \to Diff\,(F)$.

Let Γ be a finitely generated group. The main result of this section is that for a large set of actions φ of Γ on a closed manifold F the space $C^\infty(F)_\Gamma = \dfrac{C^\infty(F)}{\mathcal{H}}$ of co-invariants of $C^\infty(F)$ is infinite dimensional.

A *distribution* on F is a continuous linear functional $\lambda : C^\infty(F) \to R$. A distribution λ is invariant under an action $\varphi : \Gamma \to Diff(F)$ if $\lambda(h \circ \varphi_\gamma) = \lambda(h)$ for all $\gamma \in \Gamma$ and $h \in C^\infty(F)_\Gamma$. So it defines a continuous linear functional on the co-invariants $C^\infty(F)_\Gamma$. We denote by D_φ the space of φ-invariant distributions.

A *signed measure* is a continuous linear functional μ on the space $C^0(F)$ of all continuous functions on F. We denote by M_φ the space of φ-invariant measures. Clearly M_φ is a linear subespace of D_φ. We remark that $C^\infty(F)_\Gamma$ is infinite dimensional if D_φ is infinite dimensional. If φ has subexponential growth, [13] then there exists a φ-invariant probability measure $\mu : C^0(F) \to R$. If $dimD_\varphi = 1$, then $M_\varphi = D_\varphi$ and the C^∞ closure of $\mathcal{H}$ is $C^\infty_0(F) = ker\mu \cap C^\infty(F)$. This follows from the Hahn-Banach theorem.

3.1 Example. Let $R_a : S^1 \to S^1$ be a rotation $R_a(z) = e^{2\pi i a}z$, $z \in S^1$, a irrational and $\varphi : \mathbf{Z} \to Diff(S^1)$ be the action generated by R_a. Here $dimD_\varphi = 1$. If a is a diophantine number then $\mathcal{H}''$ is closed and $\mathcal{H} = C^\infty_o(F)$ and the group of co-invariants has dimension one. If α is Liouville then $\mathcal{H}$ is not closed and the group of co-invariants is infinite dimensional. A proof of this facts uses standard argument on Fourier series, [2].

A set $\mathcal{M} \subset F$ if *minimal* for an action $\varphi : \Gamma \to Diff(F)$ if it is closed and invariant under φ and has no proper closed invariant subset. If F is compact then every action has a minimal set.

3.2 Proposition. Let $\varphi : \Gamma \to Diff\,(F)$ be an action of a countable generated group Γ on a closed manifold F. If φ has subexponential growth and infinitely many minimal sets, then $C^\infty\,(F)_\Gamma$ is an infinite dimensional vector space.

Proof: We show that the space M_φ of invariant measures is infinite dimensional. In fact, given any positive integer $\underline{n}$ choose n distinct minimal sets $\mathcal{M}_1, \ldots, \mathcal{M}_n$. Since φ has subexponential growth then by [13] there exist invariant probability measures $\mu_1, \ldots, \mu_n$ such that $\sup p\mu_j \subset \mathcal{M}_j$, $1 \le j \le n$. These measures are linearly independent. For, let $f_1, \ldots, f_n$ be smooth functions with disjoint support such that $f_j^{-1}(1) = \mathcal{M}_j$, $1 \le j \le n$. If $c_1\mu_2 + \cdots + c_n\mu_n = 0$, then $0 = c_j\mu_j(f_j) = c_j$, $1 \le j \le n$, finishing the proof. □

The *centralizer group* of a diffeomorphism $\varphi : F \to F$ is the set of elements in $Diff(F)$ which commute with φ and is denoted by $C(\varphi)$. We denote by $\mathbf{Z}(\varphi)$ the cyclic group generated by φ. The group $Diff(F)$ of all C^∞ diffeomorphisms of F is endowed with the C^∞ topology. We say that φ has trivial centralizer if $Z(\varphi) = C(\varphi)$. A question posed by S. Smale, [19] is whether there exists an open dense set of diffeomorphisms in $Diff(F)$ having trivial centralizer. The question was answered affirmatively in the case of the circle by N. Kopell in [8]. J. Palis and J.C. Yoccoz gave an affirmative answer for a large set in $Diff(F)$, [14]. Smale's question can be related with the question whether for a large set of foliated bundles the foliated cohomology is infinite dimensional. In fact, it is believed that the set of foliations with finite dimensional cohomology is a very small set.

Suppose the group Γ is generated by $\gamma_1, \ldots, \gamma_p$. We say that an action $\varphi : \Gamma \to Diff(F)$ is generated by the iterates of a diffeomorphism $\psi \in Diff(F)$ if $\varphi_{\gamma_j} = \psi^{k_j}$, $k_j \in \mathbf{Z}$, $1 \le j \le p$. In this case every function h in $\mathcal{H}$ can be written as $\chi - \chi \circ \varphi = h$ for $\chi \in C^\infty(F)$. We remark that if Γ has a non-trivial center then Palis-Yoccoz results on the centralizer of diffeomorphisms say that the actions of Γ on F which are generated by the iterates of some diffeomorphism $\psi \in Diff(F)$ form a large class. Next we extend Theorem 2.1 of [1] to these actions.

The *orbit* of an action φ through a point $x \in F$ is the set $\mathcal{O}(x)$ of all points $\varphi_\gamma(x)$, $\gamma \in \Gamma$. The closure $\overline{\mathcal{O}\,(x)}$ of $\mathcal{O}(x)$ will be refferred to as an *orbit closure* of φ.

3.3 Theorem. Let φ be an action of a finitely generated group Γ on a closed manifold F. Suppose φ is generated by the iterates of a diffeomorphism ψ with infinitely many orbit closures. Then the space of co-invariants $C^\infty(F)_\Gamma$ of φ is infinite dimensional.

Proof: If ψ has infinitely many minimals, the result follows from Proposition 3.2. Assume ψ has a finite number of minimals. Let $C = \{\overline{\mathcal{O}(x_i)}, i = 0, 1, 2, \ldots\}$ be a countable family of distinct orbit closures. By Lemma 2.2 of [1] we may assume that $A = \bigcap\limits_{j=0}^{\infty} a(x_i)$ and $W = \bigcap\limits_{j=0}^{\infty} w(x_i)$ are both non void (here $a(x_i)$ and $\omega(x_i)$ denote the a-limit and ω-limit sets of x_i, respectively). The inclusion gives a partial ordering on $\mathcal{C}$. So there are two possibilities.
Case 1. C has an infinite totally ordered subset C'.
Case 2. Any totally ordered subset C' of C is finite.

Case 1. Let $n > 0$ be any integer. By assumption there exist $n+1$ distinct orbit closures in C', say

$$\overline{\mathcal{O}(x_0)} \subset \overline{\mathcal{O}(x_1)} \subset \ldots \subset \overline{\mathcal{O}(x_n)}.$$

Choose $\underline{n}$ non-negative C^∞ functions $f_i : F \to R$ such that

$$\begin{array}{c} f_i^{-1}(1) = x_i, supp\ f_i \text{are disjoint from } \overline{\mathcal{O}(x_0)} \\ \text{and if } supp f_i \cap \overline{\mathcal{O}(x_j)} \neq \emptyset \text{ then } i \leq j. \end{array} \tag{3.1}$$

We claim that $f_1, ..., f_n$ are linearly independent in $C^\infty(F)_\Gamma$ i. e. if $c_1 f_1 + ... + c_n f_n = h$, $h \in \mathcal{H}$, and $c_j \in R$, $1 \leq j \leq n$, then $c_1 = ... = c_n = 0$. In fact, since $supp f_i$ are disjoint from $\overline{\mathcal{O}(x_0)}$, if $h \in \mathcal{H}$ then $\chi - \chi \circ \psi = h$ for some $\chi \in C^\infty(F)$ and from this we see that $\chi(x_o) - \chi(\varphi^k(x_o)) = \sum\limits_{j=0}^{k-1} h(\varphi^j(x_o)) = 0$ for each integer k. Thus χ is constant on $\overline{\mathcal{O}(x_0)}$ and we may as well assume that χ vanishes on

$\overline{\mathcal{O}\,(x_0)}$. Thus χ vanishes on A and W. Now

$$\left.\begin{array}{l} \chi(x_1)-\chi\left(\varphi^k\,(x_1)\right)=c_1\sum_{j=0}^{k-1}f_1(\varphi^j(x_1)) \\ \text{and} \\ \chi(x_1)-\chi\left(\varphi^k\,(x_1)\right)=-c_1\sum_{j=1}^{k}f_1(\varphi^{-j}(x_1)) \end{array}\right\} \tag{3.2}$$

for each integer $k>0$.
Since χ vanishes on A and W, (3.2) gives

$$\left.\begin{array}{l} \chi(x_1)=c_1\lim_{j\to\infty}\sum_{j=0}^{k_j-1}f_1(\varphi^j(x_1)) \\ \text{and} \\ \chi(x_1)=-c_1\lim_{j\to\infty}\sum_{j=0}^{l_j}f_1(\varphi^{-j}(x_1)) \end{array}\right\} \tag{3.3}$$

for some subsequences (k_j) and (l_j). Since the function f_i are non negative and take the value 1 on x_i, then the above limits are positive. Thus (3.3) gives $c_1=0$. Similarly we show that $c_2=...=c_n=0$. Thus $dimC^\infty\,(F)_\Gamma=\infty$.

Case 2. In this case given any positive integer $\underline{n}$ there exist $n+1$orbit closures $\overline{\mathcal{O}\,(x_0)},\ldots,\overline{\mathcal{O}\,(x_0)}$ such that $\mathcal{O}(x_i)$ is disjoint from $\mathcal{O}(x_j)$ if $i\neq j$. Thus we may choose open neighborhoods V_i of each x_i, $1\leq i\leq n$ such that $V_i\cap V_j=\phi$ if $i\neq j$. Now choose non-negative C^∞ functions $f_i:F\to R$ such that $f_i^{-1}(1)=x_i$ and $suppf_i\subset V_i$, $1\leq i\leq n$. Now as in Case 1 we show that these functions are linearly independent in $C^\infty(F)_\Gamma$, proving the theorem.

4. Foliated Bundles over the Torus T^p

Since the torus T^P is a $K\,(Z^p,1)$ space then the cohomology of a foliated bundle $F\to(M,\mathcal{F})\xrightarrow{\pi}T^p$ is, by Theorem 2.1, isomorphic to the cohomology of the Z^p-module $C^\infty(F)$In this section we give a procedure for the computation of this cohomology.

Let $\Lambda(R^p)^*$ be the exterior algebra over $\mathbf{Z}$ generated by the canonical 1-forms $dx_1,\ldots,dx_p$ with trivial $\mathbf{Z}^p$-action. Consider the graded $\mathbf{Z}^p$-module

$$\Lambda(R^p)^*\otimes C^\infty(F)=C^\infty(F)+\sum_{j=1}^{p}\Lambda^j(R^p)^*\otimes C^\infty(F) \tag{4.1}$$

We think of the elements of $\Lambda^j(R^p)^* \otimes C^\infty(F)$ as "differential forms" $\xi = \sum\limits_I h_I dx_I$ where $dx_I = dx_{i_1} \wedge \ldots \wedge dx_{i_j}$, $I = (i_1, \ldots, i_j)$, $1 \leq i_1 < \cdots < i_j \leq p$, $h_I \in C^\infty(F)$. We define differential operators:
(i) $d : C^\infty(F) \to \Lambda^1(R^p)^* \otimes C^\infty(F)$,

$$dh = \sum_{j=1}^{p} \partial_j\, h\, dx_j \text{ where } \partial_j\, h = h - e_j\, h \text{ and} \tag{4.2}$$

(ii) $d : \Lambda^j(R^p)^* \otimes C^\infty(F) \to \Lambda^{j+1}(R^p)^* \otimes C^\infty(F)$
is given on the generators by $d(h\, dx_I) = d\,h \wedge dx_I$. The cohomology of the differential complex $(\Lambda(R^p)^* \otimes C^\infty(F), d)$ is the cohomology of the Z^p-module $C^\infty(F)$. For a proof of this fact, see [12, Chap. VI.6].

4.1 Example. We describe the cocycles and coboundaries in the particular case $p = 3$. Notice that $dim\Lambda^j(R^p)^* = \binom{p}{j} = \dfrac{p!}{j!(p-j)!}$. Recall that $H^o(\Gamma, A) = A^\Gamma$ is group of invariants for any Γ-module A.
(i) A 1-cochain ξ is determined by its value on the generators dx_1, dx_2 and dx_3 of $\Lambda^1(R^3)^*$. So a 1-cochain is the same as a "1-form" $\xi = f_1\, dx_1 + f_2\, dx_2 + f_3\, dx_3$ where $f_i \in C^\infty(F)$, $i = 1, 2, 3$. ξ is a 1-cocycle if $d\xi = (\partial_1 f_2 - \partial_2 f_1)dx_1dx_2 + (\partial_1 f_3 - \partial_3 f_1)dx_1dx_3 + (\partial_2 f_3 - \partial_3 f_2)dx_2dx_3 = 0$. So ξ is a 1-cocycle $\Leftrightarrow \partial_i f_j = \partial_j f_i$, $1 \leq i < j \leq 3$. From this we have the equations

$$\left.\begin{array}{l} f_1 - e_2 \cdot f_1 = f_2 - e_1 \cdot f_2, \quad f_1 - e_3 \cdot f_1 = f_3 - e_1 \cdot f_3 \\ \text{and} \\ f_2 - e_3 \cdot f_2 = f_3 - e_2 \cdot f_3 \end{array}\right\} \tag{4.3}$$

ξ is a 1-coboundary if there is $h \in C^\infty(F)$ such that

$$h - e_j h = f_j, \quad 1 \leq j \leq 3 \tag{4.4}$$

(ii) A 2-cochain $\xi = f_{12}\, dx_1dx_2 + f_{13}\, dx_1dx_3 + f_{23}\, dx_2dx_3$ is 2-cocycle if

$$d\xi = (\partial_1 f_{23} - \partial_2 f_{13} + \partial_3 f_{12})dx_1dx_2dx_3 = 0$$

i.e. if the coefficients satisfy the equation

$$(f_{12} - e_3 \cdot f_{12}) - (f_{13} - e_2 \cdot f_{13}) + (f_{12} - e_3 \cdot f_{13}) = 0 \tag{4.5}$$

Notice that a 2-cocycle is a coboundary if there exists a 1-cochain $\lambda = h_1 dx_1 + h_2 dx_2 + h_3 dx_3$ such that $\xi = d\lambda$ and from this we get the equations $f_{ij} = \partial_i h_j - \partial_j h_i$ for $1 \leq i < j \leq 3$. Thus we have the system of equations

$$\begin{aligned}(h_2 - e_1 \cdot h_2) - (h_1 - e_2 \cdot h_1) &= f_{12} \\ (h_3 - e_1 \cdot h_3) - (h_1 - e_3 \cdot h_1) &= f_{13} \\ (h_3 - e_2 \cdot h_3) - (h_2 - e_3 \cdot h_2) &= f_{23}\end{aligned} \tag{4.6}$$

(iii) A 3-cocycle $\xi = h\, dx_1 dx_2 dx_3$ is a coboundary if there is a 2-cochain $\lambda = h_{12} dx_1 dx_2 + h_{13} dx_1 dx_3 + h_{23} dx_2 dx_3$ such that $d\lambda = \xi$ and from this we derive the equation $h = \partial_3 h_{12} - \partial_2 h_{13} + \partial_1 h_{23}$ i.e.

$$h = (h_{12} - e_3 \cdot h_{12}) - (h_{13} - e_2 \cdot h_{13}) + (h_{23} - e_1 \cdot h_{23}) \tag{4.7}$$

5. The Cohomolgy of Actions of Z^p on the Affine Group of T^q

Let $\text{Affin}(T^q)$ be the group of affine transformations of the torus T^q and $\varphi : \mathbf{Z}^p \to \text{Affin}(T^q)$ be an action (homomorphism). Let φ^* be the induced action of $\mathbf{Z}^p$ on the ring $C^\infty(T^q)$, $\varphi^*(f) = f \circ \varphi$ and $C^\infty_\varphi(T^q)$ the corresponding $\mathbf{Z}^p$-module.

By the *cohomology of the action* φ we mean the cohomology of the Z^p-module $C^\infty_\varphi(T^q)$. This cohomology was computed by J. L. Arraut and N. M. dos Santos for actions for $\mathbf{Z}^p$ by translations of T^q in [2] and, more generally, for actions of $\mathbf{Z}^p$ by affine transformations of T^q by R. U. Luz in [9].

We now discuss briefly these results. The derivative $A = D\varphi$ gives an action of $\mathbf{Z}^p$ on $S\ell(q, Z)$. The cohomology of the action φ depends on both the arithmetic nature of φ and the algebraic properties of A. The set $\sigma(A)$ of all eigenvalues of all $A(\ell)$, $\ell \in \mathbf{Z}^p$ is referred to as the *spectrum* of A. If φ is minimal (i.e. every orbit is dense) then $\sigma(A) = \{1\}$, [9]. The isotropy groups $I(k)$ of the action ${}^tA : Z^p \times Z^q \to Z^q$, ${}^tA(\ell, k) = {}^tA(\ell)k$ play an important role on the cohomology of φ. Let G be a non-trivial isotropy group of ${}^tA$ and $\{\ell_1, \ldots, \ell_r\}$ be a basis of G. Consider the matrix M whose columns are $\varphi(\ell_j)(0)$, $1 \leq j \leq r$. We say that φ satisfies a *Diophantine condition* for G if there exist β,

$c > 0$ such that

$$\|kM\| \geq \frac{c}{|k|^{1+\beta}} \quad \text{for all} \quad k \in \mathbf{Z}^q - \{0\}, \quad I(k) = G \tag{5.1}$$

where $\|x\| = \inf\{|x - \ell|\,,\ \ell \in Z^r\}$.

Definition. An action $\varphi : \mathbf{Z}^p \to \mathrm{Affin}(T^q)$ is *Diophantine* if 1. ${}^tA$ has only non-trivial isotropy groups and there exist only a finite number of them. 2. φ satisfies a Diophantine condition for each isotropy group of the action ${}^tA$. The cohomoly of Diophantine actions is given by

5.2 Theorem. (R.U. Luz, [9]). Let $\varphi : \mathbf{Z}^p \to \mathrm{Affin}(T^q)$ be a Diophantine action. Then

$$H^j(\mathbf{Z}^p, C^\infty_\varphi(T^q)) = H^j_{DR}(T^q), \quad 0 \leq j \leq p.$$

If φ acts by translations, we have

5.3 Theorem (J.L. Arraut and N.M. dos Santos, [2]). Let $\varphi : \mathbf{Z}^p \to \mathrm{Trans}(T^q)$ be an action of $\mathbf{Z}^p$ on the group $\mathrm{Trans}(T^q)$ of translations of T^q. Then

$$H^j(\mathbf{Z}^p, C^\infty_\varphi(T^q)) = H^j_{DR}(T^q), \quad 0 \leq j \leq p$$

if and only if φ is Diophantine.

5.4 Example. Let $\varphi : Z^2 \to \mathrm{Affin}(T^2)$ generated on the covering space R^2 by

$$\varphi(e_1) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \text{ and } \varphi\,(e_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 \\ \beta \end{pmatrix}$$

where $\underline{\alpha}$is a Diophantine number and $\beta \notin Q$. Then
(i) If β is Diophantine then φ is Diophantine and

$$H^1(\mathbf{Z}^2, C^\infty_\varphi(T^2)) = R^2,\ H^2(\mathbf{Z}^2, C^\infty_\varphi(T^2)) = R$$

by Theorem 5.2.

ii) If β is Liouville then

$$H^1(\mathbf{Z}^2, C^\infty_\varphi(T^2)) = R^2 \quad \text{and} \quad H^2(\mathbf{Z}^2, C^\infty_\varphi(T^2))$$

is a non-Hausdorff infinite dimensional space. We finish the section with the following oustanding problem.

5.5 Problem. Compute the cohomology group $H^1(Z, C^\infty_\varphi(S^1))$ where the action of $\mathbf{Z}$ is generated by a diffeomorphism $\varphi: S^1 \to S^1$ with irrational rotation number.

If φ is C^∞-conjugate to a rotation, then the problem reduces to actions of $\mathbf{Z} \to Transl(S^1)$, where the answer is known.

6. References

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