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WEIGHTED HOMOGENEOUS MAP GERMS OF CORANK ONE FROM C^3 TO C^3 AND POLAR MULTIPLICITIES

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Abstract

For quasi-homogeneous and finitely determined corank one map germs $f : (\mathbf{C}^3, 0) \to (\mathbf{C}^3, 0)$ we obtain formulae in function of the degree and weight of f for invariantes on the stable types of f, as polar multiplicities, number of Milnor, number of Lê. We minimize also the number of invariantes for 7, to resolve the problem that decides the Whitney equisingularity of families of such maps germs. To finalize use these formulae to increase the list of invariants of some normal forms of f.

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1. Introduction

Teissier introduced in some key papers in singularity theory ([16], [17]) the notions of polar varieties and polar multiplicities. His work was taken up by Gaffney in the study of equisingularity of map-germs ([3], [4]). Gaffney's theorem states that, for an important class of finitely determined map germs $\mathbf{C}^n, 0 \to \mathbf{C}^p, 0$, if all the polar multiplicities of the strata in the source and target of a well chosen stratification are constant along a family of such germs, then this family is Whitney equisingular. The results in [3] show that the number of invariants involved depends on the dimensions (n, p) and this number can be very big according to n and p are big.

Work is done for reducing this number in some particular cases; see [5] for $n = p = 2$, [3] for $n = 2, p = 3$, and [9] for $n = p = 3$, [10] $n = 3, p = 4$.

The work of Teissier and Gaffney show that the polar varieties and multiplicities are powerfull tools for solving some problems in singularity theory. However these invariants have not been used extensively. This could be due to the fact that they are difficult to compute in practice. In this paper we show that for corank 1, quasi-homogeneous and finitely determined germs $f : (\mathbf{C}^3, 0) \to (\mathbf{C}^3, 0)$ one can obtain formulae for the polar multiplicities defined on the stable types (§3). In the §4 we use the formulae to deduce more invariants for the list of the $\mathcal{A}-$simple germs in [14].

2. Notation and preliminaries

We shall denote by $\mathcal{O}(3, 3)$ the set of origin preserving germs of holomorphic mappings from $\mathbf{C}^3, 0$ to $\mathbf{C}^3, 0$. We let $\mathcal{R}$ denote origin preserving diffeomorphisms of the source, $\mathcal{L}$ the corresponding group of the target; $\mathcal{A} = \mathcal{R} \times \mathcal{L}$. There is a natural action of $\mathcal{A}$ on $\mathcal{O}(3, 3)$, given by $k \circ f \circ h^{-1}$, where $f \in \mathcal{O}(3, 3)$ and $(h, k) \in \mathcal{A}$. The critical set of f is denoted by $\Sigma(f)$, and its image, the discriminant of f, by $\Delta(f)$. We denote the determinant of the derivative of f by $J[f]$.

For a given germ f, we stratify the source and target as follows. In the source, we have the set of critical points $\Sigma(f)$, the cuspidal edge curve, denoted by $\Sigma^{1,1}(f)$, the set of double points $f(D^2_1(f|\Sigma(f)))$ (for corank 1 germs), and the rest of the neighborhood of the origin. We should note that, in general, the set of double points $D^2(f|\Sigma(f))$ lies in $\mathbf{C}^6$. However, for corank 1 germs we consider instead the $D^2_1(f|\Sigma(f))$ which is defined as follows. We can write $f = (x, y, g(x, y, z))$, and consider $D^2(f|\Sigma(f))$ (see

[6] for details) as the subset

$$\Big\{(x,y,z,z_1)\in\mathbf{C}^4 : \tfrac{g(x,y,z)-g(x,y,z_1)}{z-z_1} = \tfrac{z_1\frac{\partial g(x,y,z)}{\partial z}-z\frac{\partial g(x,y,z_1)}{\partial z}}{z-z_1} = \tfrac{\frac{\partial g(x,y,z)}{\partial z}-\frac{\partial g(x,y,z_1)}{\partial z}}{z-z_1}=0\Big\}.$$

Then we denoted by $D^2_1(f|\Sigma(f))$ the projection of $D^2(f|\Sigma(f))$ to the (x,y,z)-space. In the target we have the smooth part of the discriminant $\Delta(f)=f(\Sigma(f))$, the image of the cuspidal edge curve $f(\Sigma^{1,1}(f))$, the image of the double points curve $f(D^2_1(f|\Sigma(f)))$, the zero-dimensional stable types, and the rest of the neighborhood of the origin. The zero-dimensional stable types are the swallowtail points (A_3), normal crossing of a plane with a cuspidal edge (A_1A_2) and the set of triple points (A^3_1). To a $k-$dimensional variety are associated $k+1$ polar invariants. As $\Sigma(f)$ and $\Delta(f)$ are of dimension 2 and the dimension of $D^2_1(f|\Sigma(f))$, $\Sigma^{1,1}(f)$, $f(\Sigma^{1,1}(f))$ and $f(D^2_1(f|\Sigma(f)))$ is 1, we have 14 polar invariants defined on these sets. We also have 3 multiplicities of the zero-dimensional stable types. These are the number swallowtails $(\sharp A_3)$, the number of normal crossing of a plane with a cuspidal edge $(\sharp A_1A_2)$ and the number of triple points $(\sharp A^3_1)$. Therefore to apply Gaffney's result to germs in $\mathcal{O}(3,3)$ we needed the constancy of 18 invariants, including the degree of f. In [9], we established relations amongst the above invariants, and using the fact that these are upper semi-continuous, we reduced the number of necessary invarinants for Whitney equisingularity to 6 for corank 1 germs.

Using some of the relations in [9], we obtain here formulae for the polar multiplicities of the strata in the target for weighted homogeneous corank 1 map-germs. We observe that in the source the strata are analytic spaces that are I.C.I.S. so the the formulae for their polar multiplicities are obtained in [4].

We shall need the following definitions

Definition 2.1. *(1) An analytic map-germ* $f:(\mathbf{C}^n,0)\to(\mathbf{C}^p,0)$, $f=(f_1,...,f_p)$ *is said to be quasi-homogeneous, or weighted homogeneous, if there are positive integers* w_1, w_2, ..., w_n, *the weights, and positive integers* d_1, d_2, ..., d_p, *the degrees, such that,* $f_i(\lambda^{w_1}x_1,\lambda^{w_2}x_2,...,\lambda^{w_n}x_n)=\lambda^{d_i}f_i(x)$ *for all* $x\in\mathbf{C}^n$, $\lambda\in\mathbf{C}$, $i=1,...,p$.

(2) An analytic map-germ f *that defines a space with an isolated singularity complete intersection (I.C.I.S.) is said to be semi quasi-homogeneous,*

if

$$f = (f_1, f_2, \ldots, f_p) = (f_1^0 + f_1^1 + \ldots, f_2^0 + f_2^1 + \ldots, \ldots, f_p^0 + f_p^1 + \ldots)$$

were f_i^0 are quasi-homogeneous polynomials of the smallest degree in each f_i, and the map $f^0 = (f_1^0, f_2^0, \ldots, f_p^0)$ defines a quasi-homogeneous analytic space that is also I.C.I.S. with the same weights of f. The germ f^0 is called initial part of f.

Theorem 2.2. *[7] Let $f : (\mathbf{C}^n, 0) \to (\mathbf{C}^k, 0)$ be a semi quasi-homogeneous germ that defines a complete intersection. Then The space $f^{-1}(0)$ is an I.C.I.S. of dimension $(n-k)$, and $\mu(f) = \mu(f^0)$.*

Theorem 2.3. *[13] Let $f : (\mathbf{C}^n, 0) \to (\mathbf{C}^n, 0)$ be a corank 1 weighted-homogeneous $\mathcal{A}-$finite map-germ with weights and degrees as above. For any stabilization of f, and any partition $\mathcal{P}$ of n,*

$$\sharp A_{\mathcal{P}} = \frac{w_n^{n-1}}{N(\mathcal{P})w} \prod_{j=1}^{n+l-1} (\frac{d}{w_n} - j)$$

where l is the length of $\mathcal{P}$, $w_n = wt(f_n)$, $d = degree(f_n)$, $w = \prod_{i=1}^{n-1}$ and $N(\mathcal{P})$ define the order of the sub group of S_l which fixes $\mathcal{P}$. Here S_l acts on $\mathbf{R}^l$ by permuting the coordinates.

3. Formulae for polar multiplicities of the stable types of the target

In this section we calculate the polar multiplicities of the strata in the target associated to a finitely determined, quasi-homogeneous, corank 1 germ $f \in \mathcal{O}(3,3)$. We start with the regular part of the discriminant, denoted by $\Delta(f)$.

In what follows we shall use the symbol $a \wedge b$ to represent the minimum integer number of a and b (where at least one of these is an integer), that is,

$$a \wedge b = \begin{cases} min(a,b) & \text{if} \quad a, b \in \mathbf{Z} \\ a & \text{if} \quad b \notin \mathbf{Z} \\ b & \text{if} \quad a \notin \mathbf{Z} \end{cases}$$

Theorem 3.1. *Let $f = (x, y, g(x, y, z)) \in \mathcal{O}(3,3)$ be a finitely determined, quasi-homogeneous, corank 1 map germ with weights w_1, w_2, w_3 and d the degree of g. Then,*

$$m_0(\Delta(f)) = \frac{(d-w_3)}{w_3},$$

$$m_1(\Delta(f)) = \frac{(d-w_3)(d-2w_3)}{w_1.w_3} \wedge \frac{(d-w_3)(d-2w_3)}{w_2.w_3},$$

$$m_2(\Delta(f)) = \frac{(d-w_3)(d-2w_3)}{w_1.w_3} \wedge \frac{(d-w_3)(d-2w_3)}{w_2.w_3} + \frac{(d-w_3)(d-2w_3)(d-w_1-w_2-w_3)}{w_1.w_2.w_3}.$$

Proof. Using the relation $m_0(\Delta(f)) = \delta(f) - 1$ in [9] (Theorem 3.3) we easily establish the formula for $m_0(\Delta(f))$, this formulae was obtained in [7]. For $m_1(\Delta(f))$, we showed in [9] (Theorem 3.4), that

$$\begin{aligned} m_1(\Delta(f)) &= dim_C \frac{\mathcal{O}_3}{(p_1 \circ f, J[f], J[p_2 \circ f, J[f]])} \\ &= \mu(p_1 \circ f, J[f], J[p_2 \circ f, J[f]]) + 1 \end{aligned}$$

where $p_1 : C^3 \to C$ and $p_2 : C^3 \to C^2$ are generic linear projections. The ideal $(p_1 \circ f, J[f], J[p_2 \circ f, J[f]])$ is not always quasi-homogeneous, but by Theorem 2.2 ([7]) the Milnor number of a semi quasi-homogeneous analytic space that is I.C.I.S. is the same as the Milnor number of the initial part of this space. Therefore we only need to verify that the ideal $(p_1 \circ f, J[f], J[p_2 \circ f, J[f]])$ is semi-quasi-homogeneous and computer the initial part. Let $p_2(x, y, z) = (a_1x + a_2y + a_3z, b_1x + b_2y + b_3z)$ be a generic linear projection. Then,

$$\begin{aligned} J[p_2 \circ f, J[f]] = (a_1b_2 - a_2b_1)g_{zz} + (a_3b_1 - a_1b_3)g_{zy}g_z + \\ (a_1b_3 - a_3b_1)g_y g_{zz} + (a_3b_2 - a_2b_3)g_x g_{zz} + \\ (a_2b_3 - a_3b_2)g_z g_{zx}. \end{aligned}$$

As p_2 is generic, $(a_1b_2 - a_2b_1) \neq 0$. Then the initial part of the ideal is given by

$$\begin{aligned} (x, J[f], (a_1b_2 - a_2b_1)g_{zz}) \quad &\text{if} \quad w_1 \leq w_2, \quad \text{or} \\ (y, J[f], (a_1b_2 - a_2b_1)g_{zz}) \quad &\text{if} \quad w_2 \leq w_1. \end{aligned}$$

These ideals define analytic spaces quasi-homogeneous that are I.C.I.S. and the degrees of their components are w_1, $(d - w_3)$, $(d - 2w_3)$ and w_2, $(d - w_3)$, $(d - 2w_3)$ respectively, then

$m_1 (\triangle (f)) = \mu (x, J [f], (a_1b_2 - a_2b_1) g_{zz}) + 1 = \dim_C \frac{\mathcal{O}_3}{(x, J[f], (a_1b_2 - a_2b_1)g_{zz})}$

or

$m_1 (\triangle (f)) = \mu (y, J [f], (a_1b_2 - a_2b_1) g_{zz}) + 1 = \dim_C \frac{\mathcal{O}_3}{(y, J[f], (a_1b_2 - a_2b_1)g_{zz})}$

The formula for $m_1(\Delta(f))$ now follows by applying Theorem 3.4 [7] or from Bezout's theorem applied to the ideals $(x, J[f], (a_1b_2 - a_2b_1)g_{zz})$ or $(x, J[f],\ (a_1b_2 - a_2b_1)g_{zz},$ ([12]). For $m_2\,(\triangle\,(f))$ we use the previous two formulae and the following relation in [9] (Theorem 3.4)

$$m_2(\Delta(f)) - m_1(\Delta(f)) + m_0(\Delta(f)) = \mu(\Sigma(f)) + 1.$$

From [12] we computer the Milnor number of $\Sigma(f)$ denoted by $\mu(\Sigma(f))$ for quasi-homogeneous germs and establish the formula for the polar multiplicity $m_2(\Delta(f))$. □

We now deal with the polar multiplicities of the image of the cuspidal edge curve $\Sigma^{1,1}(f)$.

Theorem 3.2. *Let $f = (x, y, g(x, y, z)) \in \mathcal{O}(3, 3)$ be a finitely determined, quasi-homogeneous, corank 1 germ with weights w_1, w_2, w_3 and d the degree of g. Then,*

$$\begin{aligned} m_0(f(\Sigma^{1,1}(f))) = &\ \tfrac{(d-w_3)(d-2w_3)}{w_1.w_3} \wedge \tfrac{(d-w_3)(d-2w_3)}{w_2.w_3}, \\ m_1(f(\Sigma^{1,1}(f))) = &\ \textstyle\sum_{j=1}^{2}\prod_{i=1}^{3}(\frac{d_j}{w_i} - 1)\prod_{k=1,k\neq j}^{2}(\frac{d_k}{d_j - d_k}) + \\ &\ \tfrac{(d-w_3)(d-2w_3)}{w_1.w_3} \wedge \tfrac{(d-w_3)(d-2w_3)}{w_2.w_3} - (\tfrac{w_1.w_2.w_3 + \prod_{j=1}^{3}(d-jw_3)}{w_1.w_2.w_3}) \end{aligned}$$

Proof. We showed in [9] (Theoren 3.3) the following relation

$$m_0(f(\Sigma^{1,1}(f))) = dim_C \frac{\mathcal{O}_3}{(I^{1,1}(f), p \circ f)}$$

where $I^{1,1}(f)$ is the ideal that defines the cuspidal curve $\Sigma^{1,1}(f)$ and p is a generic linear projection. As $I^{1,1}(f) = (\frac{\partial g}{\partial z}, \frac{\partial^2 g}{\partial z^2})$, we have

$$m_0(f(\Sigma^{1,1}(f))) = dim_C \frac{\mathcal{O}_3}{(\frac{\partial g}{\partial z}, \frac{\partial^2 g}{\partial z^2}, ax + by + cg(x, y, z))}.$$

If $w_1 < w_2$, the initial part of the ideal semi quasi-homogeneous $(\frac{\partial g}{\partial z}, \frac{\partial^2 g}{\partial z^2}, ax + by + cg(x, y, z))$, is given by $(\frac{\partial g}{\partial z}, \frac{\partial^2 g}{\partial z^2}, ax)$, for $a \neq 0$ and from Bezout's theorem applied to this ideal we obtain,

$$m_0(f(\Sigma^{1,1}(f))) = \frac{(d - w_3)(d - 2w_3)}{w_2.w_3}.$$

If $w_2 < w_1$, the initial part of the ideal semi quasi-homogeneous $(\frac{\partial g}{\partial z}, \frac{\partial^2 g}{\partial z^2}, ax + by + cg)$, is $(\frac{\partial g}{\partial z}, \frac{\partial^2 g}{\partial z^2}, by)$, for $b \neq 0$. Therefore, from Bezout's theorem

$$m_0(f(\Sigma^{1,1}(f))) = \frac{(d - w_3)(d - 2w_3)}{w_1.w_3}.$$

The formula in the theorem now follows from the above two cases.

For the first polar multiplicity $m_1(f(\Sigma^{1,1}(f)))$, we showed the following relation in [9] (Theorem 3.7)

$$m_1(f(\Sigma^{1,1}(f))) - m_0(f(\Sigma^{1,1}(f))) = \mu(\Sigma^{1,1}(f))) - \sharp A_3 - 1.$$

We apply the Corollary 3.10 in [7] and the Theorem 2.3 in [13] to compute $\mu(\Sigma^{1,1}(f)))$ and $\sharp A_3$. We get

$$\mu(\Sigma^{1,1}(f)) = \sum_{j=1}^{2} \prod_{i=1}^{3} (\frac{d_j}{w_i} - 1) \prod_{k=1, k \neq j}^{2} (\frac{d_k}{d_j - d_k}),$$

where $d_1 = d - w_3$ e $d_2 = d - 2w_3$, and

$$\sharp A_3 = \frac{w_3^2}{w_1.w_2} \prod_{j=1}^{3} (\frac{d}{w_3} - j).$$

The formula for $m_1(f(\Sigma^{1,1}(f)))$ now follows by substituting in the above relation $m_0(f(\Sigma^{1,1}(f)))$, $\mu(\Sigma^{1,1}(f)))$ and $\sharp A_3$ by their values. $\square$

The case of the image of the double point curve $f(D_1^2(f|\Sigma(f)))$ is more complicated. As we use two different projections (one to define $D_1^2(f|\Sigma(f))$ as in §2 and the other to define the polar varieties), we are unable to obtain independent formulae for m_0 and m_1 as is the case of the previous strata. We shall give a relation between these two. Before doing so we have the following result.

Proposition 3.3. *Let* $f \in \mathcal{O}(3, 3)$ *be a corank* 1 *germ. Then*

$$2m_1(X) - 2m_0(X) = -3\sharp A_3 + 6\sharp A_1^3 + \mu(D^2(f|\Sigma(f))) - 1,$$

where $X = f(D_1^2(f|\Sigma(f)))$.

Proof. The proof follows by putting together the relation

$$2m_0(X) - 2m_1(X) + \mu(D_1^2(f|\Sigma(f))) = 2\sharp A_1 A_2 + 3\sharp A_3 + 1$$

from [9] (Theorem 3.8) and the following relation

$$\mu(D^2_1(f|\Sigma(f))) = \mu(D^2(f|\Sigma(f))) + 2\sharp A_1A_2 + 6\sharp A_1^3$$

in [8]. □

With the relation in the above proposition, the formulae of Theorem 2.3, and the following result from [8]

$$\mu(D^2(f|\Sigma(f))) = 1 + \frac{(d-w_3)(d-2w_3)(d-3w_3)}{w_3w_1w_2}(3d-(8w_3+w_1+w_2))$$

we deduce the following relation

$$2m_1(X) - 2m_0(X) = \frac{\prod_{i=1}^3(d-iw_3)}{w_1.w_2.w_3^3}(w_3(w_1+w_2-9w_3)-d(d-6w_3)),$$

where $X = f(D^2(f|\Sigma(f)))$.

Remark 3.4. *(i) We shall consider now the the stratum $f^{-1}(\Delta(f)) - \Sigma(f)$, which is a singular hypersurface, $dim_{\mathbf{C}}(\Sigma(X_0(f))) = 1$. We denote this set by*

$$X(f) = \overline{f^{-1}(\Delta(f)) - \Sigma(f) - \{0\}}$$

and we denoted by $X_0(f)$ the ideal that define $X(f)$. To verify the equisingularity of the hypersurface $X(f_t) \subset \mathbf{C} \times \mathbf{C}^3$, we consider the following Theorem of Teissier [17]: The pair $(X(f_t) - \Sigma(X_0(f)), T)$ is Whitney equisingular iff the relative polar multiplicities $m_0(X(f_t))$, $m_1(X(f_t))$, $m_2(X(f_t))$ are constant on $T = \{0\} \times \mathbf{C}$. Therefore we have 3 invariants, since our purppse is to minimize the number of invariants we use the Lê numbers, see [1]. These numbers are the generalization of the Milnor numbers and also characterise the Whitney equisingularity, see [2] , pag. 701 or [15], pag. 95. We have that the pair $(X(f_t) - \Sigma(X_0(f)), T)$ is Whitmey equisingular iff the Lê numbers of $X(f_t)$ and of all the Lê number of the generic planar sections of $X(f_t)$ are constant on T, i.e., $\lambda^i(X_0(f_t))$ and $\lambda^i(X_0(f_t)/H^k)$ are constant on T, for all $i = 0, ..., k-1$, $k = 1, 2$, where H^k is a generic plane of $\{t\} \times \mathbf{C}^3$. Using the relations of Gafffney and Gassler in [2]:

$$\lambda^0(X_0(f_t)/H^2) = \lambda^1(X_0(f_t)) + m_1(X_0(f_t)),$$
$$\lambda^1(X_0(f_t)/H^2) = \lambda^1(X_0(f_t))$$

and as $\lambda^1(X_0(f_t))$ is lexicographically upper semi continuous and $m_1(X_0(f_t))$ is also upper semi continuous, we obtain that, $(X(f_t) -$

$\Sigma(X_0(f)), T)$ is Whitney equisingular iff $\lambda^0(X_0(f_t)/H^2)$ and $\lambda^0(X_0(f_t))$ are constants on T. We remark that $\lambda^2(X_0(f_t)) = 0$ and $\lambda^1(X_0(f_t)) = 0$ since $dim_C(\Sigma(X_0(f))) = 1$, see [1], pag. 18.

(*ii*) Let $(X, 0) \subset (C^n, 0)$ be a germ of an analytic space of dimension d that is an I.C.I.S. Then there are formulae for the polar multiplicities of X in terms of $C-$codimension of an algebra associated to X (see [4], Lemma 3.1). Therefore when the stable types are I.C.I.S. unless of the stratum $X(f)$, which is the case for the stable types in the source for corank 1 germs. The formulae for the polar multiplicities are as given in [4].

We conclude this section given the following equisingularity result for quasi-homogeneous corank 1 germs. The proof is a consequence of a more general result for corank 1 germs in [9] (Theorem 3.12) and the relations obtained in this sections. We observe that, from Remark 3.4 (ii) or [4], for quasi-homogeneous of corank 1 germs $m_1(\Sigma(f)) = m_0(\Sigma^{1,1}(f))$. Therefore we have the Theorem 3.12 in [9] for case quasi-homogeneous of corank 1 germs.

Theorem 3.5. *Let $f = (x, y, g(x, y, z)) \in \mathcal{O}(3, 3)$ be a finitely determined quasi-homogeneous of corank 1 germs. Let $F = (t, f_t)$ be a good 1-parameter unfolding of f. Then F is Whitney equisingular along the parameter t if and only if $m_1(\Delta(f_t))$, $\mu(\Sigma(f_t))$, $m_1(\Sigma^{1,1}(f_t))$, $m_0(f_t(D^2_1(f_t|\Sigma(f_t))))$, $m_1(D^2_1(f_t|\Sigma(f_t)))$, $\lambda^0(X_0(f_t)/H^2)$ and $\lambda^0(X_0(f_t))$ are constant for all t close to zero.*

4. Polar invariants for simple germs

A classification of the $\mathcal{A}$-simple germs $C^3, 0 \to C^3, 0$ is given in [14]. Also, in that paper, are computed a list of invariants associated to these germs. We increase this list here by computing the polar multiplicities of the discriminant and of the image of the cuspidal edge curve. Before that we treat in details the case of the swallowtail .

Example 4.1. The normal form of the swallowtail singularity is $f(x, y, z) = (x, y, z^4 + yz^2 + xz)$. The critical set $\Sigma(f)$ is a smooth surface given by $4z^3 + 2yz + x = 0$, and the discriminant can be parametrised by $\phi(y, z) = (-4z^3 - 2zy, y, -3z^4 + z^2y)$. As $f|_{\Sigma(f)}$ bimeoromorfic, therefore, for computing the polar variety $P_1(\Delta(f))$ it is sufficient to obtain the set of critical points of $p\circ\phi$, $\Sigma(p\circ\phi)$, where $p:\mathbf{C}^3\to\mathbf{C}^2$ is a generic linear projection. Write $p(x,y,z)=(a_1x+a_2y+a_3z, b_1x+b_2y+b_3z)$, then

$$\begin{aligned}\Sigma(p\circ\phi) = \;& 2(a_2b_1-b_2a_1)(-6z^6-y)+(a_2b_3-b_2a_3)(-12z^3+2zy)+\\ & (b_3a_1-b_1a_3)(24z^4-4z^2y)+(a_3b_1-b_3a_1)(-12z^5-2z^3).\end{aligned}$$

As p is generic, we can suppose $a_2b_1-b_2a_1\neq 0$. The set of critical points of $p\circ\phi$ is then locally a regular curve in $\Sigma(f)$. The image of this curve in $\Delta(f)$ is given by $\tilde{\phi}(z)=(-2z^3,6z^2,-3z^4)$. This is a complete intersection defined by the ideal $I=(y^2-z,x^2-yz)$. It defines the absolute polar varieties $P_1(\Delta(f))=V(y^2-z,x^2-yz)$ in $\Delta(f)$, see Figure 1. The polar variety $P_0(\Delta(f))$ is $\Delta(f)$ by definition, and $P_2(\Delta(f))$ is empty. We can compute now the polar multiplicities of these varieties:

$$\begin{aligned} m_0(P_0(\Delta(f)) &= m_0(\Delta(f)) &&= 3\\ m_0(P_1(\Delta(f)) &= m_1(\Delta(f)) &&= dim_{\mathbf{C}}\frac{\mathcal{O}_3}{(y^2-z,x^2-yz,y)} = 2\\ m_0(P_2(\Delta(f)) &= m_2(\Delta(f)) &&= 0\end{aligned}$$

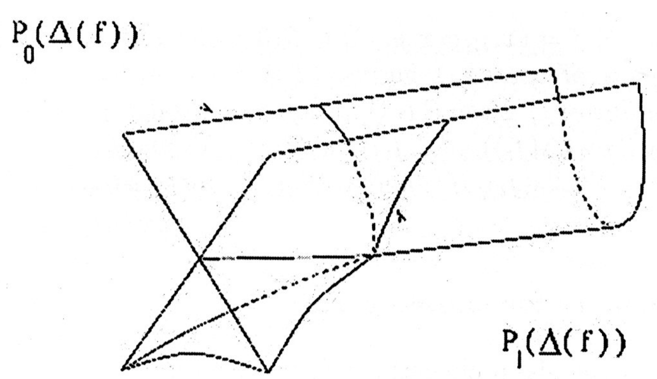

Figure 1 : Absolutes Polar Varietes of the swallowtail

Using the formulae obtained in the previous section and the relations in [9], we compute the polar multiplicities of the discriminants of the simple germs in [14].

Proposition 4.2. *The polar multiplicities of* $\Delta(f)$, *where* f *is one of the* $\mathcal{A}$*-simple germ in [14], are as follows:*

Normal Form	$\delta(f)$	$m_0(\Delta(f))$	$m_1(\Delta(f))$	$m_2(\Delta(f))$
(x, y, z^2)	2	1	0	0
$(x, y, z^3 + (x^2 + y^{k+1})z), k \geq 0$	3	2	2	$k + 1$
$(x, y, z^3 + (x^2 y + y^{k-1})z), k \geq 4$	3	2	$k - 1$	$2k - 2$
$(x, y, z^3 + (x^3 + y^4)z)$	3	2	3	8
$(x, y, z^3 + (x^3 + xy^3)z)$	3	2	3	9
$(x, y, z^3 + (x^3 + y^5)z)$	3	2	3	10
$(x, y, z^4 + xz + y^k z^2), k \geq 1$	4	3	2	0
$(x, y, z^4 + (y^2 + x^k)z + xz^2), k \geq 2$	4	3	3	2
$(x, y, z^5 + xz + yz^2)$	5	4	3	0
$(x, y, z^5 + xz + y^2 z^2 + yz^3)$	5	4	3	0
$(x, y, z^5 + xz + yz^3)$	5	4	3	0

Proof. For the quasi-homogeneous germs we use the formulae in the previous section. For the germ $f(x, y, z) = (x, y, z^4 + (y^2 + x^k)z + xz^2)$ that is not quasi-homogeneous,we proceed as follows. The set of critical points is given by

$f(x, y, z) = (x, y, z^4 + (y^2 + x^k)z + xz^2)$, so the polar multiplicity $m_1(\Delta(f))$ of the discriminant is equal to

$$dim_C \frac{\mathcal{O}_3}{(p_1 \circ f, J[f], J[p_2 \circ f, J[f]])}$$

where $p_2 : C^3 \to C^2$ and $p_1 : C^2 \longrightarrow C$ are generic projections. Choose $p_1 = y$ and $p_2 = (x, y)$ (which are generic) so that

$$m_1(\Delta(f)) = dim_C \frac{\mathcal{O}_3}{(y, 4z^3 + (y^2 + x^k) + 2xz, 12z^2 + 2x)} = 3 \text{ if } k \geq 2.$$

As $m_0(\Delta(f)) = \delta(f) - 1$ (see [9]), we get $m_0(\Delta(f)) = dim_C \frac{\mathcal{O}_3}{(x,y,z^4+(y^2+x^k)z+xz^2)} - 1 = 3$. Since $\mu(\Sigma(f)) = 1$ and $m_2(\Delta(f)) - m_1(\Delta(f)) + m_0(\Delta(f)) = \mu(\Sigma(f)) + 1$ (Theorem 3.4 in [9]) we deduce that $m_2(\Delta(f)) = 2$. In the same way we obtain the polar multiplicities associated to $f(x, y, z) = (x, y, z^5 + xz + y^2 z^2 + yz^3)$. The set of critical points is given by $5z^4 + x + 2y^2 z + 3yz^2$ and the polar multiplicity $m_1(\Delta(f))$ of the discriminant is equal to

$$dim_C \frac{\mathcal{O}_3}{(p_1 \circ f, J[f], J[p_2 \circ f, J[f]])}$$

where $p_2 : C^3 \to C^2$ e $p_1 : C^2 \to C$ are generic projections. Choose $p_1 = y$ and $p_2 = (x, y)$ (which are generic) so that

$$m_1(\Delta(f)) = dim_C \frac{\mathcal{O}_3}{(y, 5z^4 + x + 2y^2z + 3xyz^2, 20z^3 + 2y^2 + 6yz)} = 3.$$

As $m_0(\Delta(f)) = \delta(f) - 1$, we get

$$m_0(\Delta(f)) = dim_C \frac{\mathcal{O}_3}{(x, y, z^5 + xz + y^2z^2 + yz^3)} - 1 = 4.$$

Since $\mu(\Sigma(f)) = 0$ and $m_2(\Delta(f)) - m_1(\Delta(f)) + m_0(\Delta(f)) = \mu(\Sigma(f)) + 1$ we deduce that $m_2(\Delta(f)) = 0$. □

We compute in the same way the polar multiplicities of the cuspidal edge curve $f(\Sigma^{1,1}(f))$ of the germs in the above table.

Proposition 4.3. *The polar multiplicities of $f(\Sigma^{1,1}(f))$, where f is one of the $\mathcal{A}$-simple germ in [14], are as follows:*

Normal Form	$\mu(\Sigma^{1,1}(f))$	$m_0(f(\Sigma^{1,1}(f)))$	$m_1(f(\Sigma^{1,1}(f)))$
(x, y, z^2)	$-$	$-$	$-$
$(x, y, z^3 + (x^2 + y^{k+1})z)$	k	2	$k+1$
$(x, y, z^3 + (x^2y + y^{k-1})z)$	k	$k-1$	$2k-2$
$(x, y, z^3 + (x^3 + y^4)z)$	6	3	8
$(x, y, z^3 + (x^3 + xy^3)z)$	7	3	9
$(x, y, z^3 + (x^3 + y^5)z)$	8	3	10
$(x, y, z^4 + xz + y^kz^2)$	$k-1$	2	0
$(x, y, z^4 + (y^2 + x^k)z + xz^2)$	2	3	2
$(x, y, z^5 + xz + yz^2)$	0	3	0
$(x, y, z^5 + xz + y^2z^2 + yz^3)$	1	3	0
$(x, y, z^5 + xz + yz^3)$	1	3	0

Proof. The invariants for the quasi-homogeneous germs are obtained using the formulae of the previous section. For the germ $f(x, y, z) = (x, y, z^4 + (y^2 + x^k)z + xz^2)$ that is not quasi-homogeneous we calculate the multiplicities $m_0(f(\Sigma^{1,1}(f)))$ and $m_1(f(\Sigma^{1,1}(f)))$ as follows. The set $\Sigma^{1,1}(f)$ is defined by the ideal $(4z^3 + (y^2 + x^k) + 2xz, 12z^2 + 2x)$, so $\mu(\Sigma^{1,1}(f)) = 2$ (using the formula in [11]). We have

$$m_0(f(\Sigma^{1,1}(f))) = dim_C \frac{\mathcal{O}_3}{(4z^3 + (y^2 + x^k) + 2xz, 12z^2 + 2x, y)} = 3.$$

Therefore using the relation

$$\mu(\Sigma^{1,1}(f))) + m_0(f(\Sigma^{1,1}(f))) - 1 = m_1(f(\Sigma^{1,1}(f))) + \sharp A_3,$$

(Theorem 3.7 in [9]) and the fact that $\sharp A_3 = 2$, it follows that $m_1(f(\Sigma^{1,1}(f))) = 2$. For the germ $f(x,y,z) = (x,y,z^5+xz+y^2z^2+yz^3)$ we proceed as above. The set $\Sigma^{1,1}(f)$ is defined by the ideal $(5z^4+x+2y^2z+3yz^2, 20z^3+2y^2+3yz)$, so $\mu(\Sigma^{1,1}(f)) = 1$. We have

$$m_0(f(\Sigma^{1,1}(f))) = dim_C \frac{\mathcal{O}_3}{(5z^4+x+2y^2z+3yz^2, 20z^3+2y^2+3yz, y)} = 3.$$

Therefore using the relation

$$\mu(\Sigma^{1,1}(f))) + m_0(f(\Sigma^{1,1}(f))) - 1 = m_1(f(\Sigma^{1,1}(f))) + \sharp A_3,$$

it follows that $m_1(f(\Sigma^{1,1}(f))) = 0$. $\square$

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