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THE SPECTRUM OF THE LAPLACIAN MATRIX OF A BALANCED 2^p-ARY TREE

OSCAR ROJO *
Universidad Católica del Norte, Chile

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Abstract

Let $p > 1$ be an integer. We consider an unweighted balanced tree $\mathcal{B}_k^p$ of k levels with a root vertex of degree 2^p, vertices from the level 2 until the level $(k-1)$ of degree 2^p+1 and vertices in the level k of degree 1. The case $p = 1$ it was studied in [8, 9, 10]. We prove that the spectrum of the Laplacian matrix $L\left(\mathcal{B}_k^p\right)$ is $\sigma\left(L\left(\mathcal{B}_k^p\right)\right) = \cup_{j=1}^{k}\sigma\left(T_j^{(p)}\right)$ where, for $1 \le j \le k-1$, $T_j^{(p)}$ is the $j \times j$ principal submatrix of the tridiagonal $k \times k$ singular matrix $T_k^{(p)}$,

$$T_k^{(p)} = \begin{bmatrix} 1 & \sqrt{2^p} & 0 & \cdots & 0 \\ \sqrt{2^p} & 2^p+1 & \sqrt{2^p} & \ddots & \vdots \\ 0 & \sqrt{2^p} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2^p+1 & \sqrt{2^p} \\ 0 & \cdots & 0 & \sqrt{2^p} & 2^p \end{bmatrix}.$$

We derive that the multiplicity of each eigenvalue of T_j, as an eigenvalue of $L\left(\mathcal{B}_k^p\right)$, is at least $2^{(2^p-1)2^{(k-j-1)p}}$. Moreover, we show that the multiplicity of the eigenvalue $\lambda = 1$ of $L\left(\mathcal{B}_k^p\right)$ is exactly $2^{(2^p-1)2^{(k-2)p}}$. Finally, we prove that $3, 7 \in \sigma\left(L\left(\mathcal{B}_k^2\right)\right)$ if and only if k is a multiple of 3, that $5 \in \sigma\left(L\left(\mathcal{B}_k^2\right)\right)$ if and only if k is an even number, and that no others integer eigenvalues exist for $L\left(\mathcal{B}_k^2\right)$.

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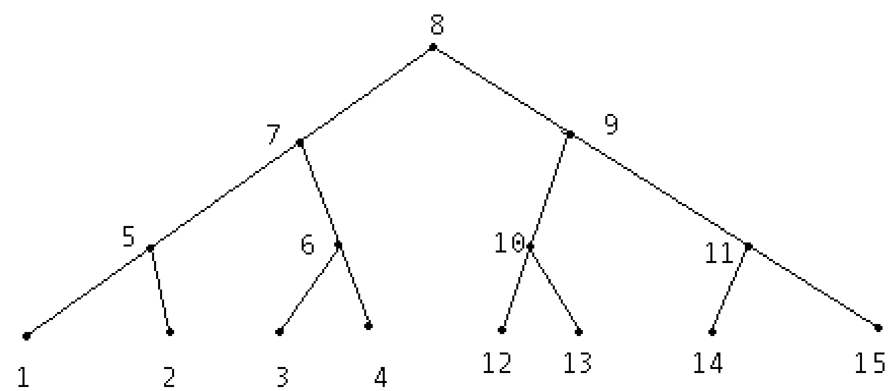
1. Introduction

Let G be a graph with vertices $1, 2, ..., n$. Let d_i be the degree of the vertex i. Let $A(G)$ be the adjacency matrix of G and let $D(G)$ be the diagonal matrix of vertex degrees. The Laplacian matrix of G is $L(G) = D(G) - A(G)$. Clearly, $L(G)$ is a real symmetric matrix. From this fact and Geršgorin's theorem, it follows that its eigenvalues are nonnegative real numbers. Moreover, since its rows sum to 0, 0 is the smallest eigenvalue of $L(G)$. In [7], some of the many results known for Laplacian matrices are given. Fiedler [3] proved that G is a connected graph if and only if the second smallest eigenvalue of $L(G)$ is positive. This eigenvalue is called the algebraic connectivity of G and it is denoted by $a(G)$. This concept has been studied by many authors. In section 3 of [7], some results concerning $a(G)$ and some of its many applications are presented.

Denote by $\mathcal{B}_k$ an unweighted balanced binary tree of k levels. The tree $\mathcal{B}_4$ and our labeling for its vertices are illustrated below



The number of vertices in $\mathcal{B}_k$ is

$$n = 1 + 2 + 2^2 + ... + 2^{k-1} = 2^k - 1.$$

The degree of the root vertex is 2 while the rest of the vertices have degree 3, except the vertices in the level k with degree 1, and there are two branches with the same number of vertices, say the left branch and right branch.

Using the labels $1, 2, 3, ..., n = 2^k - 1$, in this order, our labeling for the vertices of $\mathcal{B}_k$ [9] follows the steps:

1. Label the vertices on the left branch from the bottom to the root vertex and, in each level, from the left to the right.

2. Label the vertices on the right branch from the root vertex to the bottom and, in each level, from the left to the right.

With this labeling the Laplacian matrix $L(\mathcal{B}_k)$ becomes a symmetric persymmetric matrix, that is, a symmetric matrix with respect to the main diagonal as well as to the secondary diagonal. In [9] by using properties for this type of matrices, among other results, we characterized completely the spectrum of $L(\mathcal{B}_k)$: $\sigma(L(\mathcal{B}_k)) = \cup_{j=1}^{k} \sigma(T_j)$ where, for $1 \leq j \leq k-1$, T_j is the $j \times j$ principal submatrix of the tridiagonal $k \times k$ singular matrix T_k,

$$T_k = \begin{bmatrix} 1 & \sqrt{2} & 0 & \cdots & 0 \\ \sqrt{2} & 3 & \sqrt{2} & \ddots & \vdots \\ 0 & \sqrt{2} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 3 & \sqrt{2} \\ 0 & \cdots & 0 & \sqrt{2} & 2 \end{bmatrix}.$$

Other results concerning to $L(\mathcal{B}_k)$ can be found in [8] and [10]. In [8], quite tight lower and upper bounds for the algebraic connectivity of $\mathcal{B}_k$ are given; and, in [10], we find the integer eigenvalues of $L(\mathcal{B}_k)$.

Here we consider an unweighted balanced tree of k levels with a root vertex of degree 2^p, vertices from the level 2 until the level $(k-1)$ of degree 2^p+1 and vertices in the level k of degree 1. Denote by $\mathcal{B}_k^p$ such a tree. In particular, $\mathcal{B}_k^1 = \mathcal{B}_k$. The number of vertices in $\mathcal{B}_k^p$ is

$$n = 1 + 2^p + 2^{2p} + .. + 2^{p(k-1)} = \frac{2^{pk}-1}{2^p-1}.$$

For $k=2$, $\mathcal{B}_2^p$ is the star graph $K_{1,2^p}$ and it is known that the eigenvalues of $L(K_{1,2^p})$ are $0, 2^p+1$ and 1 with multiplicity 2^p-1. Henceforth, we assume $k \geq 3$.

In this paper, we characterize completely the spectrum of $L(\mathcal{B}_k^p)$:

$$\sigma(L(\mathcal{B}_k^p)) = \cup_{j=1}^{k} \sigma\left(T_j^{(p)}\right)$$

where, for $1 \leq j \leq k-1$, $T_j^{(p)}$ is the $j \times j$ principal submatrix of the tridiagonal $k \times k$ singular matrix $T_k^{(p)}$,

$$T_k^{(p)} = \begin{bmatrix} 1 & \sqrt{2^p} & 0 & \cdots & 0 \\ \sqrt{2^p} & 2^p+1 & \sqrt{2^p} & \ddots & \vdots \\ 0 & \sqrt{2^p} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2^p+1 & \sqrt{2^p} \\ 0 & \cdots & 0 & \sqrt{2^p} & 2^p \end{bmatrix}.$$

2. The Laplacian matrix $L\left(\mathcal{B}_k^p\right)$ and its eigenvalues

We introduce the following notations:

If all the eigenvalues of an $n \times n$ matrix A are real numbers, we write

$$\lambda_n(A) \leq \lambda_{n-1}(A) \leq ..... \leq \lambda_2(A) \leq \lambda_1(A).$$

J is the reversal matrix, that is, the matrix with ones in the secondary diagonal and zeros elsewhere. Observe that J^2 is the identity matrix. The order of J will be clear from the context in which it is used.

I_m is the identity matrix of order $m \times m$.

$\mathbf{e}_m$ is the all ones column vector of dimension m.

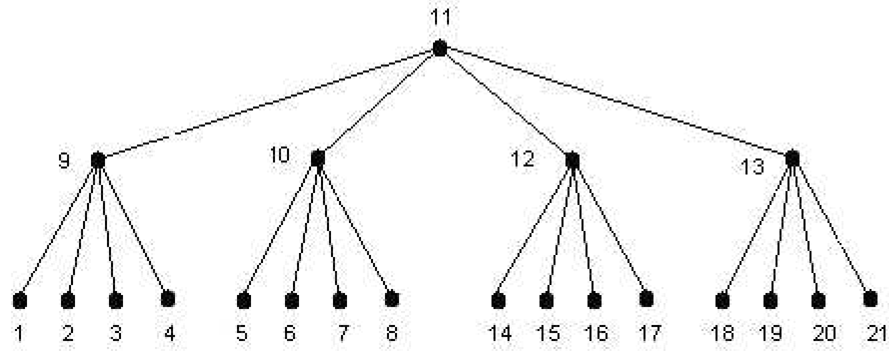
For $j = 1, 2, 3, ..., k-1$, $n_j = 2^{(k-j)p-1}$.

For $j = 1, 2, ..., k-3$, C_j is the block diagonal matrix defined by

$$C_j = \begin{bmatrix} \mathbf{e}_{2^p} & 0 & \cdots & 0 \\ \vdots & \mathbf{e}_{2^p} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \mathbf{e}_{2^p} \end{bmatrix}$$

with n_{j+1} diagonal blocks. The order of C_j is $n_j \times n_{j+1}$.

Since the root vertex degree of $\mathcal{B}_k^p$ is an even number, we can distinguish two parts in $\mathcal{B}_k^p$ with the same number of vertices, say the left part and the right part. The tree $\mathcal{B}_3^2$ and our labeling for its vertices are shown below



According to our labeling the vertices on the left part of $\mathcal{B}_3^2$ are 1, 2, 3, 4, 5, 6, 7, 8 on the level 3 and $9, 10$ on the level 2, 11 is the root vertex, and the vertices on the right part of $\mathcal{B}_3^2$ are $12, 13$ on the level 2 and 14, 15, 16, 17, 18, 19, 20, 21 on the level 3. The Laplacian matrix $L\left(\mathcal{B}_3^2\right)$ becomes

$$L\left(\mathcal{B}_3^2\right) = \begin{bmatrix} U & \mathbf{b} & 0 \\ \mathbf{b}^T & 4 & \mathbf{b}^T J \\ 0 & J\mathbf{b} & JUJ \end{bmatrix}$$

where

$$U = \begin{bmatrix} I_8 & -C_1 \\ -C_1 & 5I_2 \end{bmatrix},\ C_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ -1 \end{bmatrix}.$$

In general, using the labels $1, 2, 3, ..., n = \dfrac{2^{kp}-1}{2^p-1}$, in this order, our labeling for the vertices of $\mathcal{B}_k^p$ follows the steps:

1. Label the vertices on the left part of $\mathcal{B}_k^p$ from the bottom to the root vertex and, in each level, from the left to the right.

2. Label the vertices on the right part of $\mathcal{B}_k^p$ from the root vertex to the bottom and, in each level, from the left to the right.

As we expect, for this labeling the Laplacian matrix $L\left(\mathcal{B}_k^p\right)$ becomes a symmetric persymmetric matrix. More precisely

$$L\left(\mathcal{B}_k^p\right) = \begin{bmatrix} U & \mathbf{b} & 0 \\ \mathbf{b}^T & 2^p & \mathbf{b}^T J \\ 0 & J\mathbf{b} & JUJ \end{bmatrix} \tag{2.1}$$

where

$$U = \begin{bmatrix} I_{n_1} & -C_1 & 0 & \cdots & 0 \\ -C_1^T & (2^p+1)\, I_{n_2} & -C_2 & \ddots & \vdots \\ 0 & -C_2^T & \ddots & \ddots & 0 \\ \vdots & & & (2^p+1)\, I_{n_{k-2}} & -C_{k-2} \\ 0 & \cdots & 0 & -C_{k-2}^T & (2^p+1)\, I_{n_{k-1}} \end{bmatrix} \tag{2.2}$$

and

$$\mathbf{b} = \begin{bmatrix} 0 & \cdots & \cdots & 0 & -\mathbf{e}_{2^{p-1}} \end{bmatrix}^T. \tag{2.3}$$

We recall a basic fact on symmetric persymmetric matrices of order $(2p+1)\times(2p+1)$ [2].

Lemma 1. *Let A be a complex symmetric persymmetric $(2p+1)\times(2p+1)$ matrix. Then, A has the form*

$$A = \begin{bmatrix} U & \mathbf{b} & VJ \\ \mathbf{b}^T & a & \mathbf{b}^T J \\ JV & R\mathbf{b} & JUJ \end{bmatrix}, \tag{2.4}$$

where U and V are complex symmetric matrices of order $p\times p$, $\mathbf{b}$ is a $p-$dimensional complex column vector and $a = a_{p+1,p+1}$ is a complex scalar. Moreover, if

$$F = \begin{bmatrix} U+V & \sqrt{2}\mathbf{b} \\ \sqrt{2}\mathbf{b}^T & a \end{bmatrix}$$

then

$$\sigma(A) = \sigma(F) \cup \sigma(U-V).$$

Theorem 2.

$$\sigma\left(L\left(\mathcal{B}_k^p\right)\right) = \sigma\left(F\right) \cup \sigma\left(U\right) \tag{2.5}$$

where

$$F = \begin{bmatrix} U & \sqrt{2}\mathbf{b} \\ \sqrt{2}\mathbf{b}^T & 2^p \end{bmatrix} \tag{2.6}$$

with U and $\mathbf{b}$ as in (2.2) and (2.3), respectively. Moreover, the smallest eigenvalue and the largest eigenvalue of U are, respectively, the algebraic connectivity of $\mathcal{B}_k^{(p)}$ and the second largest eigenvalue of $L\left(\mathcal{B}_k^{(p)}\right)$. Also, the largest eigenvalue of F is the largest eigenvalue of $L\left(\mathcal{B}_k^{(p)}\right)$.

Proof. We apply Lemma 1 to $L\left(\mathcal{B}_k^p\right)$. By comparing (2.4) and (2.1), we see that U is defined by (2.2), $V = 0$ and $\mathbf{b}$ is defined by (2.3). Thus, by Lemma 1, (2.5) follows immediately. The rest of the proof follows from (2.5) together with the fact that the eigenvalues of U interlace the eigenvalues of F. □

Now, we search for the eigenvalues of U and F. The following lemma is a slight variation of Lemma 3 in [9] and, because of this, its proof is omitted.

Lemma 3. *Let a, b and c be real numbers. Let*

$$\beta_1 = a,$$

$$\beta_j = b - \frac{2^p}{\beta_{j-1}},\ j = 2, 3, ..., k-1,\ \beta_{j-1} \neq 0,$$

and

$$\beta_k = c - \frac{2^p}{\beta_{k-1}}.$$

Let

$$M=\begin{bmatrix} aI_{n_1} & C_1 & 0 & \cdots & \cdots & \cdots & 0 \\ C_1^T & bI_{n_2} & C_2 & \ddots & & & \vdots \\ 0 & C_2^T & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & bI_{n_{k-2}} & C_{k-2} \\ 0 & \cdots & \cdots & \cdots & 0 & C_{k-2} & bI_{n_{k-1}} \end{bmatrix}$$

and

$$N=\begin{bmatrix} aI_{n_1} & C_1 & 0 & \cdots & \cdots & \cdots & 0 \\ C_1^T & bI_{n_2} & C_{n_2} & \ddots & & & \vdots \\ 0 & C_{n_2} & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & bI_{n_{k-2}} & C_{n_{k-2}} & 0 \\ \vdots & & & \ddots & \mathbf{C}_2^T & bI_{n_{k-1}} & \sqrt{2}\mathbf{e}_{2^{p-1}} \\ 0 & \cdots & \cdots & \cdots & 0 & \sqrt{2}\mathbf{e}_{2^{p-1}}^T & c \end{bmatrix}.$$

Then, (a) *If* $\beta_j \neq 0$ *for* $j=1,2,....,k-2$,

$$\det M=\beta_1^{2^{(k-1)p-1}}\beta_2^{2^{(k-2)p-1}}.....\beta_{k-3}^{2^{3p-1}}\beta_{k-2}^{2^{2p-1}}\beta_{k-1}^{2^{p-1}}. \tag{2.7}$$

(b) $\det M\neq 0$ *if and only if* $\beta_j\neq 0$ *for* $j=1,2,....,k-1$.
(c) *If* $\beta_j\neq 0$ *for* $j=1,2,....,k-1$,

$$\det N=\beta_1^{2^{(k-1)p-1}}\beta_2^{2^{(k-2)p-1}}.....\beta_{k-3}^{2^{3p-1}}\beta_{k-2}^{2^{2p-1}}\beta_{k-1}^{2^{p-1}}\beta_k. \tag{2.8}$$

(d) *If* $\beta_j\neq 0$ *for* $j=1,2,....,k$ *then* $\det N\neq 0$. *If* $\beta_j=0$ *for some* $j, 1\leq j\leq k-2$, *then* $\det N=0$. *Also, if* $\beta_{k-1}\neq 0$ *and* $\beta_k=0$ *then* $\det N=0$.

Theorem 4. *Let*

$$P_0(\lambda)=1,\ P_1(\lambda)=\lambda-1, \tag{2.9}$$

$$P_j(\lambda)=\left(\lambda-2^p-1\right)P_{j-1}(\lambda)-2^pP_{j-2}(\lambda) \quad \text{for } j=2,3,....,k-1 \tag{2.10}$$

and

$$P_k(\lambda) = (\lambda - 2^p) P_{k-1}(\lambda) - 2^p P_{k-2}(\lambda). \tag{2.11}$$

Hence,

(a) *If* $\lambda \in \mathbf{R}$ *is such that* $P_j(\lambda) \neq 0$ *for* $j = 1, 2, ..., k-2$, *then*

$$\det(\lambda I - U) \tag{2.12}$$

$$= P_1^{(2^p-1)2^{(k-2)p-1}}(\lambda) P_2^{(2^p-1)2^{(k-3)p-1}}(\lambda) ....$$

$$.......P_{k-3}^{(2^p-1)2^{2p-1}}(\lambda) P_{k-2}^{(2^p-1)2^{p-1}}(\lambda) P_{k-1}^{2^{p-1}}(\lambda)$$

(b) $\det(\lambda I - U) \neq 0$ *if and only if* $\lambda \in \mathbf{R}$ *is such that* $P_j(\lambda) \neq 0$ *for* $j = 1, 2, ..., k-1$.

(c) *If* $\lambda \in \mathbf{R}$ *is such that* $P_j(\lambda) \neq 0$ *for* $j = 1, 2, ..., k-1$, *then*

$$\det(\lambda I - F) \tag{2.13}$$

$$= P_1^{(2^p-1)2^{(k-2)p-1}}(\lambda) P_2^{(2^p-1)2^{(k-3)p-1}}(\lambda) ....$$

$$........P_{k-3}^{(2^p-1)2^{2p-1}}(\lambda) P_{k-2}^{(2^p-1)2^{p-1}}(\lambda) P_{k-1}^{2^{p-1}-1}(\lambda) P_k(\lambda).$$

(d) *If* $\lambda \in \mathbf{R}$ *is such that* $P_j(\lambda) \neq 0$, *for* $j = 1, 2, ..., k$ *then* $\det(\lambda I - F) \neq 0$. *If* $\lambda \in \mathbf{R}$ *is such that* $P_j(\lambda) = 0$, *for some* j, $1 \leq j \leq k-2$, *or if* $\lambda \in \mathbf{R}$ *is such that* $P_k(\lambda) = 0$ *then* $\det(\lambda I - F) = 0$.

Proof. Let $\lambda \in \mathbf{R}$. We apply Lemma 3 to the matrices $M = \lambda I - U$ and $N = \lambda I - F$. Then, $a = \lambda - 1$, $b = \lambda - 2^p - 1$ and $c = \lambda - 2^p$.

(a) Suppose that $P_j(\lambda) \neq 0$ for $j = 1, 2, ..., k-2$. For brevity, we write $P_j(\lambda) = P_j$. Then,

$$\beta_1 = \lambda - 1 = \frac{P_1}{P_0} \neq 0 \text{ and } \beta_j = (\lambda - 2^p - 1) - \frac{2^p P_{j-2}}{P_{j-1}} = \frac{P_j}{P_{j-1}} \neq 0.$$

From (2.7),

$$\begin{aligned}
&\det(\lambda I - U) \\
&= P_1^{2^{(k-1)p-1}} \frac{P_2^{2^{(k-2)p-1}}}{P_1^{2^{(k-2)p-1}}} \cdots\cdots \frac{P_{k-3}^{2^{3p-1}}}{P_{k-4}^{2^{3p-1}}} \frac{P_{k-2}^{2^{2p-1}}}{P_{k-3}^{2^{2p-1}}} \frac{P_{k-1}^{2^{p-1}}}{P_{k-2}^{2^{p-1}}} \\
&= P_1^{(2^p-1)2^{(k-2)p-1}} P_2^{(2^p-1)2^{(k-3)p-1}} \cdots\cdot P_{k-3}^{(2^p-1)2^{2p-1}} P_{k-2}^{(2^p-1)2^{p-1}} P_{k-1}^{2^{p-1}}.
\end{aligned}$$

Thus, (2.12) is proved.

(b) Suppose that $P_j(\lambda) \neq 0$ for $j = 1, 2, ..., k-1$. Then, $\beta_j = \frac{P_j(\lambda)}{P_{j-1}(\lambda)} \neq 0$, for $j = 1, 2, ..., k-1$. From Lemma 3, part (b), it follows that $\det(\lambda I - U) \neq 0$. Conversely, suppose that $\det(\lambda I - U) \neq 0$ and $P_j(\lambda) = 0$ for some $j, 1 \leq j \leq k-1$. Since $P_0(\lambda) = 1 \neq 0$, we may assume $P_{j-1}(\lambda) \neq 0$ and $P_j(\lambda) = 0$. Then $\beta_j = \frac{P_j(\lambda)}{P_{j-1}(\lambda)} = 0$ and thus $\det(\lambda I - U) = 0$, which is a contradiction.

(c) Suppose that $P_j(\lambda) \neq 0$ for $j = 1, 2, ..., k-1$. Then, as in part (a), $\beta_j = \frac{P_j(\lambda)}{P_{j-1}(\lambda)} \neq 0$ for $j = 1, 2, ..., k-1$. Moreover,

$$\beta_k = (\lambda - 2^p) - \frac{2^p}{\beta_{k-1}} = (\lambda - 2^p) - \frac{2^p P_{k-2}(\lambda)}{P_{k-1}(\lambda)} = \frac{P_k(\lambda)}{P_{k-1}(\lambda)}.$$

From (2.7),(2.8) and (2.12),

$$\begin{aligned}
&\det(\lambda I - F) \\
&= P_1^{(2^p-1)2^{(k-2)p-1}} P_2^{(2^p-1)2^{(k-3)p-1}} \cdots\cdot P_{k-3}^{(2^p-1)2^{2p-1}} P_{k-2}^{(2^p-1)2^{p-1}} P_{k-1}^{2^{p-1}} \frac{P_k}{P_{k-1}} \\
&= P_1^{(2^p-1)2^{(k-2)p-1}} P_2^{(2^p-1)2^{(k-3)p-1}} \cdots\cdot P_{k-3}^{(2^p-1)2^{2p-1}} P_{k-2}^{(2^p-1)2^{p-1}} P_{k-1}^{2^{p-1}-1} P_k.
\end{aligned}$$

Thus, (2.15) is proved.

(d) Suppose $P_j(\lambda) \neq 0$, for $j = 1, 2, ..., k$. Then, $\beta_j = \frac{P_j(\lambda)}{P_{j-1}(\lambda)} \neq 0$ for $j = 1, 2, ..., k-1$ and $\beta_k = \frac{P_k(\lambda)}{P_{k-1}(\lambda)} \neq 0$. From Lemma 3, part (d), $\det(\lambda I - F) \neq 0$. Suppose $P_j(\lambda) = 0$ for some j, $1 \leq j \leq k-2$. Since $P_0(\lambda) = 1 \neq 0$, we may suppose $P_{j-1}(\lambda) \neq 0$ and $P_j(\lambda) = 0$. Thus, $\beta_j = \frac{P_j(\lambda)}{P_{j-1}(\lambda)} = 0$ and therefore $\det(\lambda I - F) = 0$. Finally, if $P_{k-1}(\lambda) \neq 0$ and $P_k(\lambda) = 0$ then $\beta_k = \frac{P_k(\lambda)}{P_{k-1}(\lambda)} = 0$ and hence $\det(\lambda I - F) = 0$. $\square$

An immediate consequence of theorem 2 and theorem 4 is

Corollary 5.
$$\sigma(U) = \cup_{j=1}^{k-1} \{\lambda \in \mathbf{R} : P_j(\lambda) = 0\},$$

$$\sigma(F) = \cup_{j=1}^{k} \{\lambda \in \mathbf{R} : P_j(\lambda) = 0\}$$

and
$$\begin{aligned} &\det\left(\lambda I - L\left(\mathcal{B}_k^p\right)\right) \\ &= P_1^{(2^p-1)2^{(k-2)p}}(\lambda)\,..P_2^{(2^p-1)2^{(k-3)p}}(\lambda)\,.. \\ &\quad P_{k-3}^{(2^p-1)2^{2p}}(\lambda)\,P_{k-2}^{(2^p-1)2^{p}}(\lambda)\,P_{k-1}^{2^p-1}(\lambda)\,P_k(\lambda) \end{aligned} \tag{2.14}$$

Lemma 6. *For $j = 1, 2, 3, ..., k-1$, let T_j be the $j \times j$ principal submatrix of the tridiagonal $k \times k$ matrix T_k*

$$T_k = \begin{bmatrix} 1 & \sqrt{2^p} & 0 & \cdots & \cdots & 0 \\ \sqrt{2^p} & 2^p+1 & \sqrt{2^p} & \ddots & \ddots & \vdots \\ 0 & \sqrt{2^p} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 2^p+1 & \sqrt{2^p} \\ 0 & \cdots & \cdots & 0 & \sqrt{2^p} & 2^p \end{bmatrix}. \tag{2.15}$$

Then,
$$\det(\lambda I - T_j) = P_j(\lambda),\ j = 1, 2, ..., k.$$

Proof. It is well known (see for instance [1, page 229]) that the characteristic polynomials p_j of the $j \times j$ principal submatrix of the symmetric tridiagonal $k \times k$ matrix T,

$$T = \begin{bmatrix} a_1 & b_1 & 0 & \cdots & \cdots & 0 \\ b_1 & a_2 & b_2 & \ddots & \ddots & \vdots \\ 0 & b_2 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_{k-1} & b_{k-1} \\ 0 & \cdots & \cdots & 0 & b_{k-1} & a_k \end{bmatrix},$$

satisfy the three-term recursion formula

$$p_j(\lambda) = (\lambda - a_j)\, p_{j-1}(\lambda) - b_{j-1}^2 p_{j-2}(\lambda)$$

with

$$p_0(\lambda) = 1 \text{ and } p_1(\lambda) = \lambda - a_1.$$

In our case, $a_1 = 1$, $a_j = 2^p + 1$ for $j = 2, 3, ..., k-1$, $a_k = 2^p$ and $b_j = \sqrt{2^p}$ for $j = 1, 2, ..., k-1$. For these values, the above recursion formula gives the polynomials P_j, $j = 0, 1, 2, ..., k$. $\square$

Theorem 7. *Let $p \geq 2$. Let T_j, $j = 1, 2, ..., k$ defined in lemma 6. Then,*

(a) $\sigma(U) = \cup_{j=1}^{k-1} \sigma(T_j)$

(b) $\sigma(F) = \cup_{j=1}^{k} \sigma(T_j)$.

(c) $\sigma\left(L\left(\mathcal{B}_k^p\right)\right) = \cup_{j=1}^{k} \sigma(T_j)$.

(d) *For $j = 1, 2, ..., k-1$, the multiplicity of each eigenvalue of the matrix T_j, as an eigenvalue of $\mathbf{L}\left(\mathcal{B}_k^p\right)$, is at least $2^{(2^p-1)2^{(k-j-1)p}}$.*

Proof. (a), (b) and (c) are immediate consequences of Corollary 5 and Lemma 6. Moreover, since

$$\begin{aligned} & \det\left(\lambda I - L\left(\mathcal{B}_k^{(p)}\right)\right) \\ = \; & P_1^{(2^p-1)2^{(k-2)p}}(\lambda)\, P_2^{(2^p-1)2^{(k-3)p}}(\lambda) \ldots\ldots \\ & \ldots\ldots\ldots P_{k-3}^{(2^p-1)2^{2p}}(\lambda)\, P_{k-2}^{(2^p-1)2^{p}}(\lambda)\, P_{k-1}^{(2^p-1)}(\lambda)\, P_k(\lambda) \end{aligned}$$

and

$$\det(\lambda I - T_j) = P_j(\lambda),\; j = 1, 2, ..., k-1,$$

we have that the multiplicity of each eigenvalue of the matrix T_j, as an eigenvalue of $L(\mathcal{B}_k)$, is at least $2^{(2^p-1)2^{(k-j-1)p}}$. $\square$

We recall the following interlacing property [4]:

Let T be a symmetric tridiagonal matrix with nonzero codiagonal entries and $\lambda_i^{(j)}$ be the $i-$th smallest eigenvalue of its $j \times j$ principal submatrix. Then,

$$\begin{aligned} \lambda_{j+1}^{(j+1)} < \lambda_j^{(j)} < \lambda_j^{(j+1)} < \ldots < \lambda_{i+1}^{(j+1)} < \lambda_i^{(j)} < \lambda_i^{(j+1)} < \\ \ldots < \lambda_2^{(j+1)} < \lambda_1^{(j)} < \lambda_1^{(j+1)}. \end{aligned}$$

From this interlacing property and theorem 7, we have

Theorem 8. *(a)* $\sigma(T_{j-1}) \cap \sigma(T_j) = \phi$ *for* $j = 2, 3, ..., k$.

(b) The largest eigenvalue of T_k *is the largest eigenvalue of* $L(\mathcal{B}_k^p)$.

(c) The smallest eigenvalue of T_{k-1} *is the algebraic connectivity of* $\mathcal{B}_k^p$.

(d) The largest eigenvalue of T_{k-1} *is the second largest eigenvalue of* $L(\mathcal{B}_k^p)$.

Example 1. *Let* $p = 2$ *and* $k = 6$. *Then,* $n = 1365$. *The eigenvalues of* $\mathbf{L}(\mathcal{B}_6^2)$, *rounded to four decimal places, are given in the following table:*

T_1 :	1					
T_2 :	0.1716	5.8284				
T_3 :	0.0376	3.6222	7.3402			
T_4 :	0.0090	2.5606	5.4394	7.9910		
T_5 :	0.0022	2.0082	4.1501	6.5147	8.3248	
S_6 :	0	1.5359	3	5	7	8.4641

3. The integer eigenvalues of $L(\mathcal{B}_k^p)$

The purpose of this section is to search for the integer eigenvalues of $L(\mathcal{B}_k^p)$. The case $p = 1$ was studied in [10]. Then, we assume $p \geq 2$. Since $\mathcal{B}_k^p$ is connected, 0 is a simple eigenvalue of $\mathcal{B}_k^p$. From theorem 7, Geršgorin's theorem and the fact that $L(\mathcal{B}_k^p)$ is a positive semidefinite matrix, it follows that if λ is an eigenvalue of $L(\mathcal{B}_k^p)$ then

$$0 \leq \lambda \leq 2^p + 1 + 2\sqrt{2^p}.$$

Since $P_1(1) = 0$, it follows that 1 is an eigenvalue of $L(\mathcal{B}_k^p)$. Next, we prove that $P_j(1) \neq 0$ for $j = 1, 2, ..., k$. Keep in mind that we are assuming $p \geq 2$. From (2.10), we obtain the second order homogeneous linear difference equation

$$P_j(1) + 2^p P_{j-1}(1) + 2^p P_{j-2}(1) = 0 \tag{3.1}$$

for $j = 1, 2, ..., k-1$, with the initial conditions $P_0(1) = 1$ and $P_1(1) = 0$. The characteristic equation of (3.1) is

$$u^2 + 2^p u + 2^p = 0. \tag{3.2}$$

We have $2^{2p-2} - 2^p = 2^p\left(2^{p-2} - 1\right) \geq 0$ because $p \geq 2$. Thus, the roots of (3.2) are

$$\begin{aligned} \alpha &= -2^{p-1} + \sqrt{2^{2p-2} - 2^p} \\ \beta &= -2^{p-1} - \sqrt{2^{2p-2} - 2^p}. \end{aligned}$$

Let $p = 2$. Then $\alpha = \beta = -2$ and the general solution of (3.1) is [6]:

$$P_j(1) = c_1(-2)^j + c_2 j(-2)^j.$$

From the initial conditions, we get $c_1 = 1$ and $c_2 = -1$. Hence

$$P_j(1) = (-2)^j(1-j),\ j = 1, 2, ..., k-1.$$

Moreover

$$\begin{aligned} P_k(1) &= -3P_{k-1}(1) - 4P_{k-2}(1) \\ &= -3(-2)^{k-1}(-k) - 4(-2)^{k-2}(3-k) \\ &= -3(-2)^{k-1}(-k) - 2(-2)^{k-1}(3-k) \\ &= (-2)^{k-1}(5k-6) \neq 0. \end{aligned}$$

Therefore, if $p = 2$ then $P_j(1) = 0$ if and only if $j = 1$. Suppose now $p > 2$. In this case, $\beta < \alpha < 0$ and the general solution of (3.1) is [6]:

$$P_j(1) = c_1\alpha^j + c_2\beta^j.$$

From the initial conditions, $c_1 + c_2 = 1$ and $c_1\alpha + c_2\beta = 0$, we get $c_1 = \dfrac{\beta}{\beta - \alpha}$ and $c_2 = -\dfrac{\alpha}{\beta-\alpha}$. Hence

$$\begin{aligned} P_j(1) &= \tfrac{\beta}{\beta-\alpha}\alpha^j - \tfrac{\alpha}{\beta-\alpha}\beta^j \\ &= \tfrac{1}{\beta-\alpha}\left(\alpha^j\beta - \alpha\beta^j\right). \end{aligned}$$

Moreover

$$\begin{aligned} P_k(1) &= (1-2^p)P_{k-1}(1) - 2^p P_{k-2}(1) \\ &= P_{k-1}(1) - 2^p\left(P_{k-1}(1) + P_{k-2}(1)\right) \neq 0. \end{aligned}$$

Hence, if $p > 1$ then $P_j(1) = 0$ if and only if $j = 1$.

Theorem 9. *If $p \geq 2$ then the multiplicity of $\lambda = 1$ as an eigenvalue of $L\left(\mathcal{B}_k^p\right)$ is $(2^p - 1)\,2^{(k-2)p}$.*

Proof. It is an immediate consequence of (2.14) and the fact that $P_j(1) \neq 0$ for all $j \geq 2$ if $p \geq 2$. $\square$

We recall an important fact concerning an integer eigenvalue of a tree [5]: *If $\lambda > 1$ is an integer eigenvalue of the Laplacian matrix of a tree with n vértices then λ is a simple eigenvalue and exactly divides n.*

Since, in addition, $n = \dfrac{2^{kp}-1}{2^p-1}$ is odd for any k and p, the only possible integer eigenvalues of $L\left(\mathcal{B}_k^p\right)$ greater than 1 are restricted to the odd positive integers not exceeding $2^p + 1 + 2\sqrt{2^p}$.

Lemma 10. Let $p \geq 2$. If $j = 1, 2, ..., k-1$ then $P_j(\lambda) \neq 0$ for all integer $\lambda > 1$.

Proof. We already know that the multiplicity of each eigenvalue of the matrix T_j, as an eigenvalue of $\mathbf{L}(\mathcal{B}_k)$, is at least $2^{(2^p-1)2^{(k-j-1)p}}$ and that $\det(\lambda I - T_j) = P_j(\lambda)$. Moreover, for $j = 1, 2, ..., k-1$, $2^{(2^p-1)2^{(k-j-1)p}} > 1$ because $p \geq 2$. Hence, if $\lambda > 1$ is an integer number such that $P_j(\lambda) = 0$ then λ is an eigenvalue of T_j with multiplicity greater than 1. This is a contradiction because the integer eigenvalues greater than 1 of a tree are simple. $\square$

Therefore in searching for the integer eigenvalues of $L\left(\mathcal{B}_k^p\right)$ greater than 1, we have to look for the integer roots of the equation $P_k(\lambda) = 0$.

From now on, we consider $p = 2$. In this case, we may restrict the values of λ to the interval $(1, 9]$. From the recursion formula

$$P_j(\lambda) = (\lambda - 5) P_{k-1}(\lambda) - 4P_{k-2}(\lambda), \; j = 2, ..., k-1$$

we obtain the second order homogeneous linear difference equation

$$P_j(\lambda) - (\lambda - 5) P_{k-1}(\lambda) + 4P_{k-2}(\lambda) = 0 \tag{3.3}$$

with $P_0(\lambda) = 1$ and $P_1(\lambda) = \lambda - 1$. The corresponding characteristic equation is

$$u^2 - (\lambda - 5) u + 4 = 0. \tag{3.4}$$

Let $\lambda = 9$. For this value the equation (3.4) becomes

$$(u-2)^2 = 0$$

and the solution of (3.3) is

$$P_j(9) = 2^j + 3j2^j = 2^j(1+3j).$$

Hence

$$\begin{aligned} P_k(9) &= 4P_{k-1}(9) - 4P_{k-2}(9) \\ &= 2^{k+1}(3k-2) - 2^k(3k-5) \\ &= 2^k(3k+1) \neq 0. \end{aligned}$$

Thus, we may consider $\lambda \in (0,9)$. Then, $\lambda^2 - 10\lambda + 9 < 0$ and the roots of (3.4) is a pair of conjugate complex root :

$$u_1 = \tfrac{\lambda-5}{2} + \tfrac{\sqrt{10\lambda-\lambda^2-9}}{2}i$$
$$\text{and } u_2 = \tfrac{\lambda-5}{2} - \tfrac{\sqrt{10\lambda-\lambda^2-9}}{2}i$$

where $i = \sqrt{-1}$. Therefore, the general solution of (3.3) is [6, p. 29] :

$$P_j(\lambda) = \rho^j\left(c_1 \cos\phi(\lambda)\, j + c_2 \sin\phi(\lambda)\, j\right)$$

where

$$\rho = \left[\left(\frac{\lambda-5}{2}\right)^2 + \left(\frac{\sqrt{10\lambda-\lambda^2-9}}{2}\right)^2\right]^{\frac{1}{2}} = 2$$

and

$$\cos\phi(\lambda) = \frac{\lambda-5}{4} \text{ and } \sin\phi(\lambda) = \frac{\sqrt{10\lambda-\lambda^2-9}}{4}. \tag{3.5}$$

From the initial conditions, we obtain $c_1 = 1$ and

$$2\left(\cos\phi(\lambda) + c_2 \sin\phi(\lambda)\right) = \lambda - 1.$$

Hence, for $j = 2, 3, ..., k-1$,

$$P_j(\lambda) = 2^j\left(\cos\phi(\lambda)\, j + \frac{\lambda+3}{\sqrt{10\lambda-\lambda^2-9}} \sin\phi(\lambda)\, j\right).$$

We already know that the possible integer eigenvalues greater than 1 of $L\left(\mathcal{B}_k^2\right)$ are $3, 5$ and 7.

Theorem 11. (a) $3 \in L\left(\mathcal{B}_k^2\right)$ if and only if k is a multiple of 3.
(b) $7 \in L\left(\mathcal{B}_k^2\right)$ if and only if k is a multiple of 3.
(b) $5 \in L\left(\mathcal{B}_k^2\right)$ if and only if k is even.

Proof. (a) Let $\lambda = 3$. Then, $\cos\phi(3) = -\frac{1}{2}$ and $\sin\phi(3) = \frac{\sqrt{3}}{2}$. Hence, $\phi(3) = \frac{2\pi}{3}$. Moreover

$$\begin{aligned}
P_k(3) &= -P_{k-1}(3) - 4P_{k-2}(3) \\
&= -2^{k-1}\left(\cos(k-1)\phi(3) + \sqrt{3}\sin(k-1)\phi(3)\right) \\
&\quad -2^{k}\left(\cos(k-2)\phi(3) + \sqrt{3}\sin(k-2)\phi(3)\right) \\
&= -2^{k-1}\begin{pmatrix} \cos(k-1)\phi(3) + \sqrt{3}\sin(k-1)\phi(3) \\ +2\cos(k-2)\phi(3) + 2\sqrt{3}\sin(k-2)\phi(3) \end{pmatrix} \\
&= -2^{k-1}\begin{pmatrix} \cos k\phi(3)\cos\phi(3) + \sin k\phi(3)\sin\phi(3) \\ +\sqrt{3}\sin k\phi(3)\cos\phi(3) - \sqrt{3}\cos k\phi(3)\sin\phi(3) \\ +2\cos k\phi(3)\cos 2\phi(3) + 2\sin k\phi(3)\sin 2\phi(3) \\ +2\sqrt{3}\sin k\phi(3)\cos 2\phi(3) - 2\sqrt{3}\cos k\phi(3)\sin 2\phi(3) \end{pmatrix} \\
&= -2^{k-1}\begin{pmatrix} -\frac{1}{2}\cos k\phi(3) + \frac{1}{2}\sqrt{3}\sin k\phi(3) - \frac{\sqrt{3}}{2}\sin k\phi(3) \\ -\frac{3}{2}\cos k\phi(3) - \cos k\phi(3) - \sqrt{3}\sin k\phi(3) - \\ \sqrt{3}\sin k\phi(3) + 3\cos k\phi(3) \end{pmatrix} \\
&= 2^{k}\sqrt{3}\sin k\phi(3).
\end{aligned}$$

Consequently, $P_k(3) = 0$ if and only if $k = \frac{2\pi l}{\phi(3)} = 3l$ some positive integer l.

(b) Let $\lambda = 7$. Then, $\cos\phi(7) = \frac{1}{2}$ and $\sin\phi(7) = \frac{\sqrt{3}}{2}$. Hence, $\phi(7) = \frac{\pi}{3}$. Moreover

$$\begin{aligned}
P_k(7) &= 3P_{k-1}(7) - 4P_{k-2}(7) \\
&= 2^{k-1}\left(3\cos(k-1)\phi(7) + \frac{5}{\sqrt{3}}\sin(k-1)\phi(7)\right) \\
&\quad -2^{k}\left(\cos(k-2)\phi(7) + \frac{5}{\sqrt{3}}\sin(k-2)\phi(7)\right) \\
&= 2^{k-1}\begin{pmatrix} 3\cos(k-1)\phi(7) + 5\sqrt{3}\sin(k-1)\phi(7) \\ -2\cos(k-2)\phi(7) - \frac{10}{\sqrt{3}}\sin(k-2)\phi(7) \end{pmatrix} \\
&= 2^{k-1}\begin{pmatrix} 3\cos k\phi(7)\cos\phi(7) + 3\sin k\phi(7)\sin\phi(7) \\ +5\sqrt{3}\sin k\phi(7)\cos\phi(7) - 5\sqrt{3}\cos k\phi(7)\sin\phi(7) \\ -2\cos k\phi(7)\cos 2\phi(7) - 2\sin k\phi(7)\sin 2\phi(7) \\ -\frac{10}{\sqrt{3}}\sin k\phi(7)\cos 2\phi(7) + \frac{10}{\sqrt{3}}\cos k\phi(7)\sin 2\phi(7) \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= 2^{k-1}\left(\begin{array}{c} \frac{3}{2}\cos k\phi\left(7\right)+\frac{3}{2}\sqrt{3}\sin k\phi\left(7\right)+\frac{5\sqrt{3}}{2}\sin k\phi\left(7\right)- \\ \frac{15}{2}\cos k\phi\left(7\right)+\cos k\phi\left(7\right)-\sqrt{3}\sin k\phi\left(7\right) \\ +\frac{5}{\sqrt{3}}\sin k\phi\left(7\right)+5\cos k\phi\left(7\right) \end{array}\right) \\
&= 2^{k-1}\frac{14}{3}\sqrt{3}\sin k\phi\left(7\right).
\end{aligned}
$$

It follows, $P_k\left(7\right)=0$ if and only if $k=\frac{\pi l}{\phi(7)}=3l$ some for positive integer l.

(c) Let $\lambda=5$. Then, $\cos\phi\left(5\right)=0$ and $\sin\phi\left(5\right)=1$. Hence, $\phi\left(5\right)=\frac{\pi}{2}$. Moreover

$$
\begin{aligned}
P_k\left(5\right) &= P_{k-1}\left(5\right)-4P_{k-2}\left(5\right) \\
&= 2^{k-1}\left(\cos\left(k-1\right)\phi\left(5\right)+2\sin\left(k-1\right)\phi\left(5\right)\right) \\
&\quad -2^{k}\left(\cos\left(k-2\right)\phi\left(5\right)+2\sin\left(k-2\right)\phi\left(5\right)\right) \\
&= 2^{k-1}\left(\begin{array}{c} \cos\left(k-1\right)\phi\left(5\right)+2\sin\left(k-1\right)\phi\left(5\right) \\ -2\cos\left(k-2\right)\phi\left(5\right)-4\sin\left(k-2\right)\phi\left(5\right) \end{array}\right) \\
&= 2^{k-1}\left(\begin{array}{c} \cos k\phi\left(5\right)\cos\phi\left(5\right)+\sin k\phi\left(5\right)\sin\phi\left(5\right) \\ +2\sin k\phi\left(5\right)\cos\phi\left(5\right)-2\cos k\phi\left(5\right)\sin\phi\left(5\right) \\ -2\cos k\phi\left(5\right)\cos 2\phi\left(5\right)-2\sin k\phi\left(5\right)\sin 2\phi\left(5\right) \\ -4\sin k\phi\left(5\right)\cos 2\phi\left(5\right)+4\cos k\phi\left(5\right)\sin 2\phi\left(5\right) \end{array}\right) \\
&= 2^{k-1}\left(\sin k\phi\left(5\right)-2\cos k\phi\left(5\right)+2\cos k\phi\left(5\right)+4\sin k\phi\left(5\right)\right) \\
&= 2^{k-1}5\sin k\phi\left(5\right).
\end{aligned}
$$

It follows, $P_k\left(5\right)=0$ if and only if $k=\frac{\pi l}{\phi(5)}=2l$ for some positive integer l. □

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Oscar Rojo
Departamento de Matemáticas
Universidad Católica del Norte
Casilla 1280
Antofagasta
Chile
e-mail : orojo@ucn.cl