Proyecciones
Vol. 23, N^o 2, pp. 123-129, August 2004.
Universidad Católica del Norte
Antofagasta - Chile
DOI: 10.4067/S0716-09172004000200005

SOLVABILITY OF COMMUTATIVE POWER-ASSOCIATIVE NILALGEBRAS OF NILINDEX 4 AND DIMENSION

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Received : October 2003. Accepted : May 2004

Abstract

Let A be a commutative power-associative nilalgebra. In this paper we prove that when A (of characteristic $\neq 2$) is of dimension ≤ 8 and $x^4 = 0$ for all $x \in A$, then $((A^2)^2)^2 = 0$. That is, A is solvable. We conclude that if A is of dimension ≤ 7 over a field of characteristic $\neq 2$, 3 and 5, then A is solvable.

1. PRELIMINARIES

Let A be a commutative algebra over a field K. If x is an element of A , we define $x^1 = x$ and $x^{k+1} = x^k x$ for all $k \geq 1$. A is called power-associative, if the subalgebra of A generated by any element $x \in A$ is associative. An element $x \in A$ is called nilpotent, if there is an integer $r \geq 1$ such that $x^r = 0$. If any element in A is nilpotent, then A is called a nilalgebra. Now A is called a nilalgebra of nilindex $n \geq 2$, if $y^n = 0$ for all $y \in A$ and there is $x \in A$ such that $x^{n-1} \neq 0$.

If B, D are subspaces of A, then BD is the subspace of A spanned by all products bd with b in B, d in D. Also we define $B^1 = B$ and $B^{k+1} = B^k B$ for all $k \geq 1$. If there exists an integer $n \geq 2$ such that $B^n = 0$ and $B^{n-1} \neq 0$, then B is nilpotent of index n.

A is called solvable in case $A^{(k)} = 0$ for some integer k , where $A^{(1)} = A$ and $A^{(n+1)} = (A^{(n)})^2$ for all $n \geq 1$.

A is a Jordan algebra, if it satisfies the Jordan identity $x^2(yx) = (x^2y)x$ for all x, y in A , and A is a commutative algebra. It is known that any Jordan algebra (of characteristic $\neq 2$) is power-associative, and also that any finite-dimensional Jordan nilalgebra is nilpotent (see, [7]).

It is known that if A is a commutative algebra such that the identity $x^3 = 0$ is valid in A, then A is a Jordan algebra (see [9] , page 114) . Therefore, if A is a finite dimensional algebra then A is nilpotente, and hence solvable.

We will denote by $< a_1, ..., a_j > K$ the subspace of A generated over K by the elements $a_1, ..., a_j \in A$. In the following a greek letter indicates an element of the field K.

In [8] , D. Suttles constructs (as a counterexample to a conjecture due to A. A. Albert) a commutative power-associative nilalgebra of nilindex 4 and dimension 5, which is solvable and is not nilpotent. In [3] (Theorem 3.3), we prove that this algebra is the unique commutative power-associative nilalgebra of nilindex 4 and dimension 5, which is not Jordan algebra. At present there exists the following conjecture: Any finite-dimensional commutative power-associative algebra is solvable. The solvability of these algebras for dimension 4 , 5 and 6 , are proved in [5] , [3] and [2] respectively.

From Theorem 2 of [4] and [6] we obtain the following result:

Theorem 1.1 : If A is a commutative power-associative nilalgebra of nilindex n with dimension $\leq n + 2$ and the characteristic is zero or $\geq n$, then A is solvable.

The following result is proved in [3] :

Lemma 1.2 : If A is a commutative power-associative nilalgebra of nilindex 4, dimension 5 and is not a Jordan algebra, then $\dim_K(A^2) = 3$.

2. SOLVABILITY

In this section, A is a commutative power-associative algebra over a field K with characteristic $\neq 2$, such that the identity $x^4 = 0$ is valid in A. Linearizing the identities $(x^2)^2 = 0$ and $x^4 = 0$, we obtain that for all y, x, z, v in A :

$$(2.1) \qquad (yx)x^2 = 0, \ 2(xy)^2 + x^2y^2 = 0$$

$$(2.2) \qquad x^2(yz) + 2(xy)(xz) = 0, \ (x^2y)(x^2v) = 0$$

$$(2.3) \qquad (xy)(zv) + (xz)(yv) + (xv)(yz) = 0$$

$$(2.4) \qquad 2((yx)x)x + (yx^2)x + yx^3 = 0$$

We see that replacing y by yx in (4) and using (1), we obtain that $2(((yx)x)x)x + (yx)x^3 = 0$. Replacing z by x^2 in (2), we get $x^2(yx^2) + 2(yx)x^3 = 0$. Therefore for all y, x in A we have that:

$$(2.5) \qquad 4(((yx)x)x)x = x^2(yx^2) = -2(yx)x^3$$

If now we replace y by yx in (5) and using (1), we obtain that:

$$(2.6) \qquad ((((yx)x)x)x)x = 0$$

We observe that using the identity (6), it is easy to prove the following result:

Lemma 2.1 : If y, x are elements in A such that $y \neq 0$ and $xy = \alpha y$, then $\alpha = 0$.

Lemma 2.2 : If A is of dimension ≤ 8, then the following identities are valid in A :

$$(2.7) \qquad (((yx)x)x)x = x^2(yx^2) = (yx)x^3 = 0$$

$$(2.8) \qquad (yx)^3 = 0$$

Proof. We will prove first that $(((yx)x)x)x = 0$ for all y, x in A. Suppose that there exist elements y, x in A such that $(((yx)x)x)x \neq 0$. By (5) we have that $x^2(yx^2) \neq 0$ and $x^3 \neq 0$. Let $X =< x, x^2, x^3 >_K$. The elements $x^2(yx^2)$ and x^3 are linearly independent. In fact: If $x^2(yx^2) = \sigma x^3$, then by (5) we get that $(yx)x^3 = -\frac{1}{2}\sigma x^3$. Using the Lemma 2.1, we obtain that $\sigma = 0$, which is a contradiction. We note that we can have that either $(((yx)x)x)x \in X$ or $(((yx)x)x)x \notin X$. Suppose that $(((yx)x)x)x \notin X$. In this case we will prove that y, yx, $(yx)x$, $((yx)x)x$, $(((yx)x)x)x$, x, x^2, x^3, yx^2 are linearly independent. Let $\alpha y + \beta yx + \gamma(yx)x + \delta((yx)x)x + \varepsilon(((yx)x)x)x + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \lambda yx^2 = 0$. Multiplying by x^2 and using (1), we get that $\alpha yx^2 + \beta_1 x^3 + \lambda(yx^2)x^2 = 0$. Now using (5) and (1) : $0 = x^2(\alpha yx^2 + \beta_1 x^3 + \lambda(yx^2)x^2) = \alpha x^2(yx^2)$ implies $\alpha = 0$, and since x^3, $(yx^2)x^2$ are linearly independent, we get $\beta_1 = \lambda = 0$. Hence we have that $\beta yx+\gamma(yx)x+\delta((yx)x)x+\varepsilon(((yx)x)x)x+\beta_2 x^2+\beta_3 x^3 = 0$. Multiplying two times by x and using (6), we get that $\beta = \gamma = 0$. Multiplying by x we get that $0 = \delta(((yx)x)x)x+\beta_2 x^3 = \frac{1}{4}\delta x^2(yx^2)+\beta_2 x^3$, which implies $\delta = \beta_2 = 0$. Now $0 = \varepsilon(((yx)x)x)x+\beta_3 x^3 = \frac{1}{4}\varepsilon x^2(yx^2)+\beta_3 x^3$ implies $\varepsilon = \beta_3 = 0$. Thus we conclude that $\dim_K(A) \geq 9$, which is a contradiction. Therefore we must have $(((yx)x)x)x \in X$, and so $(((yx)x)x)x = \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$. Now $0 = ((((yx)x)x)x)x = \alpha_1 x^2 + \alpha_2 x^3$ implies $\alpha_1 = \alpha_2 = 0$, and relation (5) together with Lemma 2.1 imply $\alpha_3 = 0$. That is, $(((yx)x)x)x = 0$, a contradiction. Hence we prove that the identity $(((yx)x)x)x = 0$ is valid in A, and thus by (5) we obtain (7).

Now we will prove the identity (8). We know that $x^2(yx^2) = 0$ for all y, x in A. By linearization we get that the following identity is valid in A :

$$x^2((xz)y) + (xz)(yx^2) = 0 \tag{2.9}$$

Replacing z by x^3 in (2) we get $x^2(yx^3) = 0$, and replacing z by x^2 in (9) we obtain $x^3(yx^2) = -x^2(yx^3) = 0$. Finally, replacing z by y, y by y^2 in (9) and using (1), we obtain that $0 = x^2((xy)y^2) = -(xy)(y^2x^2) = 2(yx)^3 = 0$. □

Remark 2.3 : Lemma 2.2 is not valid when A is of dimension 9. In fact : Let B be a commutative algebra of dimension 9 with basis $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$ and nonzero multiplication given by : $v_1v_6 = v_2$, $v_1v_7 = v_9$, $v_1v_8 = -(2+\beta)v_4 - \gamma v_5$, $v_2v_6 = v_3$, $v_2v_8 = -2v_5$, $v_3v_6 = v_4$, $v_4v_6 = v_5$, $v_6^2 = v_7$, $v_6v_7 = v_8$, $v_6v_9 = \beta v_4 + \gamma v_5$, $v_7v_9 = 4v_5$. We obtain that B is a commutative power-associative nilalgebra of nilindex 4, such that : $(((v_1v_6)v_6)v_6)v_6 = v_5$, $(v_1v_6^2)v_6^2 = 4v_5$ and $(v_1v_6)v_6^3 = -2v_5$.

Theorem 2.4 : If $A$is of dimension ≤ 8,then $((A^2)^2)^2 = 0$.

Proof. Suppose that $((A^2)^2)^2 \neq 0$. Then there exist elements y, x, u, v in A such that $(x^2y^2)(u^2v^2) \neq 0$. From (1) we obtain that $(x^2y^2)(u^2v^2) = 4(xy)^2(uv)^2 = -8((xy)(uv))^2 \neq 0$. We will prove that x^2, y^2, xy, x^2y^2, u^2, v^2, uv, u^2v^2 are linearly independent. The elements x^2, y^2, xy, x^2y^2 are linearly independent. In fact: if $\alpha x^2 + \beta y^2 + \gamma xy + \delta x^2y^2 = 0$, then multiplying by x^2 and using the identities (1) and (7) we obtain that $\beta x^2y^2 = 0$, which implies $\beta = 0$. Similarly we obtain that $\alpha = 0$. Now using (8), $0 = \gamma yx + \delta x^2y^2 = \gamma yx - 2\delta(xy)^2$ implies $\gamma = \delta = 0$. Let $\alpha x^2 + \beta y^2 + \gamma xy + \delta x^2y^2 + \alpha_0 u^2 + \beta_0 v^2 + \gamma_0 uv + \delta_0 u^2v^2 = 0$. Multiplying by v^2, afterwards by $x^2y^2 = -2(xy)^2$ and using (1), (2) and (7) we obtain $\alpha_0(x^2y^2)(u^2v^2) = 0$, which implies $\alpha_0 = 0$. Similarly, $\beta_0 = 0$. Now we have that $\alpha x^2 + \beta y^2 + \gamma xy + \delta x^2y^2 = -(\gamma_0 uv + \delta_0 u^2v^2)$. Using (1), (7) and (8) : $(\alpha x^2 + \beta y^2 + \gamma xy + \delta x^2y^2)^2 = (2\alpha\beta - \frac{1}{2}\gamma^2)x^2y^2$ and $(-(\gamma_0 uv + \delta_0 u^2v^2))^2 = -\frac{1}{2}\gamma_0^2 u^2v^2$. Hence $(2\alpha\beta - \frac{1}{2}\gamma^2)x^2y^2 = -\frac{1}{2}\gamma_0^2 u^2v^2$, which implies that $-\frac{1}{2}\gamma_0^2(u^2v^2)(x^2y^2) = 0$, and so $\gamma_0 = 0$. Now we have that $\alpha x^2 + \beta y^2 + \gamma xy + \delta x^2y^2 + \delta_0 u^2v^2 = 0$. Multiplying by x^2y^2 we get $\delta_0 = 0$. Since x^2, y^2, xy, x^2y^2 are linearly independent, we conclude that $\{x^2, y^2, xy, x^2y^2, u^2, v^2, uv, u^2v^2\}$ is a basis of A, and hence $A^2 = A$. Now using the above identities we get that $A = A^2 =< x^2y^2, u^2v^2, x^2u^2, x^2v^2, x^2(uv), x^2(u^2v^2), y^2u^2, y^2v^2, y^2(uv), y^2(u^2v^2), (xy)u^2, (xy)v^2, (xy)(uv), (xy)(u^2v^2), (x^2y^2)u^2, (x^2y^2)v^2, (x^2y^2)(uv), (x^2y^2)(u^2v^2) >_K$. Let $J =< (x^2y^2)(u^2v^2) >_K$. We observe that $(x^2y^2)A = J$ and $(u^2v^2)A = J$. Now we will show that $((x^2y^2)(u^2v^2))A = 0$. Since $A^2 = A$, it is sufficient to prove that $((x^2y^2)(u^2v^2))(z_1z_2) = 0$ for all z_1, $z_2 \in A$. Now using (3), we obtain that $((x^2y^2)(u^2v^2))(z_1z_2) = -((x^2y^2)z_1)((u^2v^2)z_2) - ((x^2y^2)z_2)((u^2v^2)z_1) \in J^2 = 0$, and therefore $((x^2y^2)(u^2v^2))A = 0$. It is easy to prove that x^2y^2, u^2v^2, $(x^2y^2)(u^2v^2)$ are linearly independent, and moreover clearly the subspace $I =< x^2y^2, u^2v^2, (x^2y^2)(u^2v^2) >_K$ is an ideal of A. Now $\overline{A} = A/I$ is a commutative power-associative nilalgebra of dimension 5. If $\overline{A}$ is a Jordan algebra, then is nilpotent and so $\dim_K(\overline{A}^2) < 5$. Now if $\overline{A}$ is not a Jordan algebra, then by Lemma 1.2 we get that $\dim_K(\overline{A}^2) < 5$. Finally we conclude that $\overline{A}^2 = A^2/I = A/I = \overline{A}$, which is a contradiction. Therefore $((A^2)^2)^2 = 0$, as desired. $\square$

Finally, by Theorems 1.1 and 2.4 we get:

Corollary 2.5 : Let Abe a commutative power-associative nilalgebra over a field of characteristic $\neq$ 2,3and 5.If $A$is of dimension ≤ 7,then $A$is solvable.

The authors thank the referee for many useful remarks.

3. REFERENCES

[1] Correa, I.; Suazo, A. On a class of commutative power-associative nilalgebras. Journal of Algebra, 215, pp. 412-417, (1999).

[2] Correa, I.; Hentzel I. R.; Peresi, L. A. On the solvability of the commutative power-associative nilalgebras of dimension 6. Linear Alg. Appl., 369, pp. 185-192, (2003).

[3] Elgueta, L.; Suazo, A. Jordan nilalgebras of nilindex n and dimension $n+1$. Communications in Algebra, 30, pp. 5547-5561, (2002).

[4] Elgueta, L.; Gutierrez Fernandez, J. C.; Suazo, A. Nilpotence of a class of commutative power-associative nilalgebras. Submitted.

[5] Gerstenhaber, M.; Myung, H.C. On commutative power-associative nilalgebras of low dimension. Proc. Amer. Math. Soc., 48, pp. 29-32, (1975).

[6] Gutierrez Fernandez J.C. On commutative power-associative nilalgebras. Communications in Algebra, 32(6), pp. 2243-2250, (2004).

[7] Schafer, R.D. An Introduction to Nonassociative Algebras; Academic Press: New York/London, (1966).

[8] Suttles, D.A. Counterexample to a conjeture of Albert. Notices Amer. Math. Soc. , 19, A-566, (1972).

[9] Zhevlakov, K.A.; Slin'ko, A.M.; Shestakov, I.P.; Shirshov, A.I. Rings That Are Nearly Associative; Academic Press: New York/London, (1992).

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