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A SIMPLE PROOF OF A THEOREM ON $(2n)$-WEAK AMENABILITY

MOHAMMAD SAL MOSLEHIAN
FATEMEH NEGAHBAN
Ferdowsi University, Irán

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Abstract

A simple proof of $(2n)$*-weak amenability of the triangular Banach algebra* $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{A} \\ 0 & \mathcal{A} \end{bmatrix}$ *is given where* $\mathcal{A}$ *is a unital* C^**-algebra.*

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1. Introduction

The topological cohomology groups provide us some significant information about Banach algebras such as their amenability, contractibility, stability, and singular extensions.[4]

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are unital Banach algebras and $\mathcal{M}$ is a unital Banach $\mathcal{A}-\mathcal{B}-$module that is simultaneously a Banach space, a left $\mathcal{A}$-module and a right $\mathcal{B}$-module satisfying $a(mb) = (am)b$, $1_{\mathcal{A}}m = m1_{\mathcal{B}}$ and $\| axb \| \leq \| a \| \| x \| \| b \|$. Then $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ 0 & \mathcal{B} \end{bmatrix} = \{ \begin{bmatrix} a & m \\ 0 & b \end{bmatrix}; a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B}\}$ equipped with the usual 2×2 matrix addition and formal multiplication and the norm $\| \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \| = \| a \| + \| m \| + \| b \|$ is said to be a triangular Banach algebra.

Note that the dual $\mathcal{M}^*$ of $\mathcal{M}$ together with the actions $(\phi a)(x) = \phi(ax)$ and $(b\phi)(x) = \phi(xb)$ is a Banach $\mathcal{B}-\mathcal{A}-$module. Similarly the $(2n)$-th dual $\mathcal{M}^{(2n)}$ of $\mathcal{M}$ is a Banach $\mathcal{A}-\mathcal{B}-$module and the $(2n-1)$-th dual $\mathcal{M}^{(2n-1)}$ of $\mathcal{M}$ is a Banach $\mathcal{B}-\mathcal{A}-$module. In particular, $\mathcal{A}^{(n)}$ is a Banach A-bimodule when $\mathcal{A}$ is regarded as an $\mathcal{A}-$bimodule in the natural way.

The notion of n-weak amenability was introduced by Dales, Ghahramani and Gronbæck [2]. Let $\mathcal{M}$ be a Banach A-bimodule, $Z^1(A, \mathcal{M}) = \{\delta : \mathcal{A} \longrightarrow \mathcal{M}; \delta$ is bounded and linear, and $\delta(ab) = a\delta(b) + \delta(a)b\}$ and $B^1(A, \mathcal{M}) = \{\delta_x : \mathcal{A} \longrightarrow \mathcal{M}; \delta_x(a) = ax - xa, a \in \mathcal{A}, x \in \mathcal{M}\}$. Then the first topological cohomology group $H^1(\mathcal{A}, \mathcal{M})$ is defined to be the quotient $Z^1(A, \mathcal{M})/B^1(A, \mathcal{M})$. If $H^1(\mathcal{A}, \mathcal{A}^{(n)})$=0, then $\mathcal{A}$ is called n-weakly amenable. If for all $n, \mathcal{A}$ is n-weakly amenable, $\mathcal{A}$ is said to be permanently weakly amenable. For instance, every C^*-algebra is permanently weakly amenable [2, Theorem 3.1].

Forrest and Marcoux investigated a relation between n-weak amenability of triangular Banach algebra $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ 0 & \mathcal{B} \end{bmatrix}$ and those of algebras $\mathcal{A}$ and $\mathcal{B}$. In particular, they proved permanently weak amenability of the triangular Banach algebra $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{A} \\ 0 & \mathcal{A} \end{bmatrix}$ where $\mathcal{A}$ is a C^*-algebra.

In this paper we give a simple proof of the $(2n)$-weak amenability of the Banach algebra $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{A} \\ 0 & \mathcal{A} \end{bmatrix}$ in which $\mathcal{A}$ is a C^*-algebra, cf. [3, Proposition 4.5].

2. Preliminaries

Let $\mathcal{P}, \mathcal{Q}$ and $\mathcal{R}$ be Banach spaces and $\phi : \mathcal{P}\times\mathcal{Q} \to \mathcal{R}$ be a bounded bilinear map. Define a bilinear map $\phi^* : \mathcal{R}^* \times \mathcal{P} \to \mathcal{Q}^*$ by $< q, \phi^*(r^*, p) >=< \phi(p, q), r^* >; p \in \mathcal{P}, q \in \mathcal{Q}, r^* \in \mathcal{R}^*$ where $< ., . >$ denotes the natural pairing. Then ϕ^{***} is called the dual of ϕ and has the following properties; cf. [1]:

(i)$\phi^{***}(p, q) = \phi(p, q);\ p \in \mathcal{P}, q \in \mathcal{Q}$,

(ii)for fixed $q^{**} \in \mathcal{Q}^{**}, p^{**} \mapsto \phi^{***}(p^{**}, q^{**})$ is weak*-continuous,

(iii)for fixed $p \in \mathcal{P}, q^{**} \mapsto \phi^{***}(p, q^{**})$ is weak*-continuous.

In general for fixed $p^{**} \in \mathcal{P}^{**}, q^{**} \mapsto \phi^{***}(p^{**}, q^{**})$ is not weak*-continuous.

We shall call the bounded bilinear map ϕ regular, when for fixed $p^{**} \in \mathcal{P}^{**}, q^{**} \mapsto \phi^{***}(p^{**}, q^{**})$ is weak*-continuous.

There are two important general examples as follows:

First, if $\mathcal{X}$ is a Banach left $\mathcal{A}$-module with the outer multiplication $\phi : \mathcal{A} \times \mathcal{X} \to \mathcal{X}$, then $\phi^{***} : \mathcal{A}^{**} \times \mathcal{X}^{**} \to \mathcal{X}^{**}$ defines a Banach left $\mathcal{A}^{**}$-module structure on $\mathcal{X}^{**}$ extending ϕ.

Second, if $\mathcal{A}$ is a Banach algebra, $\mathcal{P} = \mathcal{Q} = \mathcal{R} = \mathcal{A}$ and $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is defined by $\phi(a, b) = ab$, then ϕ^{***} is an associative multiplication on $\mathcal{A}^{**}$ making that into a Banach algebra. If ϕ is regular then $\mathcal{A}$ is called Arens regular.

Now, for each $a, b \in \mathcal{A}, f \in \mathcal{A}^*, F, G \in \mathcal{A}^{**}$, put

$(fa)(b) = f(ab),\ (Ff)(a) = F(fa),\ (FG)(f) = F(Gf)$;

$(a.f)(b) = f(ba),\ (f.F)(a) = F(a.f),\ (F.G)(f) = G(f.F)$.

We shall call FG and $F.G$ the first (left) and the second (right) Arens products, respectively. FG is nothing except ϕ^{***}. Indeed, $\phi^*(f, a)(b) = f(ab), \phi^{**}(F, f)(a) = F(f(a))$ and $\phi^{***}(F, G)(f) = F(G(f))$. Similarly $\psi^{***}(F, G) = G.F$, where $\psi : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is defined by $\psi(a, b) = ba$.

Following [3], we describe the action of $\mathcal{T}$ upon $\mathcal{T}^{(n)}$:

Let $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ 0 & \mathcal{B} \end{bmatrix}$ be a triangular Banach algebra and $\begin{bmatrix} F_1 & H_1 \\ 0 & G_1 \end{bmatrix}, \begin{bmatrix} F_2 & H_2 \\ 0 & G_2 \end{bmatrix} \in \mathcal{T}^{**}$. Suppose that $\begin{bmatrix} F_1 & H_1 \\ 0 & G_1 \end{bmatrix} = weak^* - \lim_i \begin{bmatrix} a_i & m_i \\ 0 & b_i \end{bmatrix}$ and $\begin{bmatrix} F_2 & H_2 \\ 0 & G_2 \end{bmatrix} = weak^* - \lim_j \begin{bmatrix} a_j & m_j \\ 0 & b_j \end{bmatrix}$. Then $F_1 = weak^* - \lim_i a_i, F_2 = weak^* - \lim_j a_j, G_1 = weak^* - \lim_i b_i, G_2 = weak^* - \lim_j b_j, H_1 = weak^* - \lim_i m_i$ and $H_2 = weak^* - \lim_i m_j$. Therefore

$$\begin{bmatrix} F_1 & H_1 \\ 0 & G_1 \end{bmatrix} \begin{bmatrix} F_2 & H_2 \\ 0 & G_2 \end{bmatrix} = weak^* - \lim_i \lim_j \begin{bmatrix} a_i & m_i \\ 0 & b_i \end{bmatrix} \begin{bmatrix} a_j & m_j \\ 0 & b_j \end{bmatrix} =$$

$$weak^* - \lim_i \lim_j \begin{bmatrix} a_i a_j & m_i m_j \\ 0 & b_i b_j \end{bmatrix}$$
$$= \begin{bmatrix} weak^* - \lim_i \lim_j a_i a_j & weak^* - \lim_i \lim_j m_i m_j \\ 0 & weak^* - \lim_i \lim_j b_i b_j \end{bmatrix}$$
$$= \begin{bmatrix} F_1F_2 & F_1H_2 + H_1G_2 \\ 0 & G_1G_2 \end{bmatrix}.$$

Hence the first Arens product on $\mathcal{T}^{**}$ is a matrix-like multiplication.

By repeatedly applying the first (or second) Arens product, one can similarly make $\mathcal{T}^{(2n)}$ for $n > 1$ into a Banach algebra. Thus an action of T upon $\mathcal{T}^{(2n)}$ is obtained when we restrict the Arens product to the image of $\mathcal{T}$ in $\mathcal{T}^{(2n)}$ under the canonical embedding. This action looks like standard matrix multiplication.

It is easy to see that the actions of $\mathcal{T}$ upon $\mathcal{T}^*$ is given by
$\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\begin{bmatrix} f & h \\ 0 & g \end{bmatrix} = \begin{bmatrix} af + mh & bh \\ 0 & bg \end{bmatrix}$ and
$\begin{bmatrix} f & h \\ 0 & g \end{bmatrix}\begin{bmatrix} a & m \\ 0 & b \end{bmatrix} = \begin{bmatrix} fa & ha \\ 0 & hm + gb \end{bmatrix}$, in which
$\begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \in \mathcal{T}, \begin{bmatrix} f & h \\ 0 & g \end{bmatrix} \in \mathcal{T}^*$.

The action of $\mathcal{T}$ upon $\mathcal{T}^{(3)}$ is the restriction to $\mathcal{T}$ of the dual action $\mathcal{T}^{(2)}$ upon $\mathcal{T}^{(3)}$. In fact, the action of $\mathcal{T}$ upon $\mathcal{T}^*$ can be generalized to the action of $\mathcal{T}$ upon $\mathcal{T}^{(2m-1)}, m \geq 2$.

3. Main Result

We start by reviewing some results on $(2n-1)$-weak amenability of triangular Banach algebras:

Theorem 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras and $\mathcal{M}$ be a unital Banach $\mathcal{A}-\mathcal{B}-$module. Let $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ 0 & \mathcal{B} \end{bmatrix}$ be the corresponding triangular Banach algebra. Let n be a positive integer. Then*

$$H^1(\mathcal{T}, \mathcal{T}^{(2n-1)}) \simeq H^1(\mathcal{A}, \mathcal{A}^{(2n-1)}) \oplus H^1(\mathcal{B}, \mathcal{B}^{(2n-1)}).$$

It follows that $\mathcal{T}$ is $(2n-1)$-weakly amenable if and only if both $\mathcal{A}$ and $\mathcal{B}$ are.

[3, Theorem 3.7]

Corollary 3.2. $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{A} \\ 0 & \mathcal{A} \end{bmatrix}$ *is* $(2n-1)$*-weakly amenable if* $\mathcal{A}$ *is a* C^**-algebra.*

Proof. Apply above theorem and permanently weak amenability of $\mathcal{A}$ [2, Theorem 3.1].

Now let $\mathcal{M}$ be a Banach $\mathcal{A}-\mathcal{B}-$module and $\rho_{x,y} : \mathcal{M} \longrightarrow \mathcal{M}^{(2n)}$ be defined by $\rho_{x,y}(m) = xm - my$, where $x \in \mathcal{A}^{(2n)},\ m \in \mathcal{M}$ and $y \in \mathcal{B}^{(2n)}$. Recall that if $\mathcal{A}$ is a Banach algebra, so is $\mathcal{A}^{(2n)}$ equipped with the (first) Arens product $\Gamma_1 \circ \Gamma_2 = w^* - \lim_i \lim_j a_i a_j$ where $\{a_i\}$ and $\{a_j\}$ are nets in $\mathcal{A}^{(2n-2)}$ converging in weak*-topology to $\Gamma_1, \Gamma_2 \in \mathcal{A}^{(2n-2)}$, respectively. We could therefore define the centeralizer of $\mathcal{A}$ in $\mathcal{A}^{(2n)}$ as $Z_{\mathcal{A}}(\mathcal{A}^{(2n)}) = \{x \in \mathcal{A}^{(2n)}; xa = ax \text{ for all } a \in \mathcal{A}\}$ and the central Rosenblum operator on $\mathcal{M}$ with coefficients in $\mathcal{M}^{(2n)}$ as $ZR_{\mathcal{A},\mathcal{B}}(\mathcal{M}, \mathcal{M}^{(2n)}) = \{\rho_{x,y}; x \in Z_{\mathcal{A}}(\mathcal{A}^{(2n)}), y \in Z_{\mathcal{B}}(\mathcal{A}^{(2n)})\}$. The later space is clearly a subspace of $Hom_{\mathcal{A},\mathcal{B}}(\mathcal{M}, \mathcal{M}^{(2n)}) = \{\phi : \mathcal{M} \longrightarrow \mathcal{M}^{(2n)}; \phi(amb) = a\phi(m)b, \text{ for all } a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B}\}$.
The following theorem play a key role in the subject:

Theorem 3.3. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be unital Banach algebras and* $\mathcal{M}$ *be a unital Banach* $\mathcal{A}-\mathcal{B}-$*module. Let* $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ 0 & \mathcal{B} \end{bmatrix}$ *be the corresponding triangular Banach algebra. If* n *is a positive integer and both* $\mathcal{A}$ *and* $\mathcal{B}$ *are* $(2n)$*-weakly amenable, then*

$$H^1(\mathcal{T}, \mathcal{T}^{(2n)}) = H^1(\mathcal{M}, \mathcal{M}^{(2n)})/ZR_{\mathcal{A},\mathcal{B}}(\mathcal{M}, \mathcal{M}^{(2n)}).$$

We are ready to give our proof of Proposition 4.5 of [3]:

Theorem 3.4. $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{A} \\ 0 & \mathcal{A} \end{bmatrix}$ *is* $(2n)$*-weakly amenable if* $\mathcal{A}$ *is a* C^**-algebra.*

Proof. For each n, the von Neumann algebra $\mathcal{A}^{(2n)}$ is unital with the unit denoted by $1_{\mathcal{A}^{(2n)}}$. Clearly $ZR_{\mathcal{A},\mathcal{A}}(\mathcal{A},\mathcal{A}^{(2n)}) \subseteq Hom_{\mathcal{A},\mathcal{A}}(\mathcal{A},\mathcal{A}^{(2n)})$. For the converse, assume that $\phi \in Hom(\mathcal{A},\mathcal{A}^{(2n)})$. Since $\phi(a) = \phi(a.1.1) = a\phi(a)$ and $\phi(a) = \phi(1.1.a) = \phi(1)a$, we have $\phi(1) \in Z_{\mathcal{A}}(\mathcal{A}^{(2n)})$ and $1_{\mathcal{A}^{(2n)}} - \phi(1) \in Z_{\mathcal{A}}(\mathcal{A}^{(2n)})$. Hence

$$\phi(a) = 1_{\mathcal{A}^{(2n)}}.a - 1_{\mathcal{A}^{(2n)}}.a + \phi(a) = \rho_{1_{\mathcal{A}^{(2n)}},1_{\mathcal{A}^{(2n)}}-\phi(a)}(a).$$

It follows that $\phi = \rho_{1_{\mathcal{A}^{(2n)}},1_{\mathcal{A}^{(2n)}}-\phi(a)} \in ZR_{\mathcal{A},\mathcal{A}}(\mathcal{A},\mathcal{A}^{(2n)})$.Therefore

$$ZR_{\mathcal{A},\mathcal{A}}(\mathcal{A},\mathcal{A}^{(2n)}) = Hom_{\mathcal{A},\mathcal{A}}(\mathcal{A},\mathcal{A}^{(2n)}).$$

Applying Theorem 2.5, we conclude that

$$H^1(\mathcal{T},\mathcal{T}^{(2n)}) = H^1(\mathcal{A},\mathcal{A}^{(2n)})/ZR_{\mathcal{A},\mathcal{A}}(\mathcal{A},\mathcal{A}^{(2n)}) = 0.$$

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Mohammad Sal Moslehian
Department of Mathematics
Ferdowsi University
P. O. Box 1159
Mashhad 91775
Irán
e-mail : msalm@math.um.ac.ir

Fatemeh Negahban
Department of Mathematics
Ferdowsi University
P. O. Box 1159
Mashhad 91775
Irán