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EXISTENCE OF SOLUTIONS FOR UNILATERAL PROBLEMS WITH L^1 DATA IN ORLICZ SPACES

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Abstract

This article is concerned with the existence result of the unilateral problem associated to equations of the type

$$Au + g(x, u, \nabla u) = f,$$

in Orlicz spaces, where $f \in L^1(\Omega)$, the term g is a nonlinearity having natural growth and satisfying the sign condition. Some stability and positivity properties of solutions are proved.

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1. Introduction

Let Ω be a bounded domain in $I\!R^N$ and let $Au = -\text{div}a(x,u,\nabla u)$ be a Leray-Lions operator defined on its domain $\mathcal{D}(A) \subset W_0^1L_M(\Omega)$, where M is an N-function which satisfies the Δ_2-condition and $f \in L^1(\Omega)$.

The author's in [6] proved the existence of at least one solution for the following nonlinear Dirichlet problem

$$(1.1)\quad \begin{cases} T_k(u) \in W_0^1L_M(\Omega), \quad g(x,u,\nabla u) \in L^1(\Omega) \\ \displaystyle\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx + \int_\Omega g(x,u,\nabla u)T_k(u-v)\,dx \\ \displaystyle\leq \int_\Omega fT_k(u-v)\,dx, \\ \forall\ \ v \in W_0^1L_M(\Omega) \cap L^\infty(\Omega),\ \forall k > 0. \end{cases}$$

Where g is a nonlinearity having natural growth with respect to $|\nabla u|$, and which satisfies the classical sign condition with respect to u.

It is our purpose, in this paper, to prove an existence theorem for the corresponding obstacle problem. More precisely, we prove the existence of at least one solution for the following unilateral problem

$$(1.2)\quad \begin{cases} u \geq \psi\ a.e.\ \text{ in }\ \Omega. \\ T_k(u) \in W_0^1L_M(\Omega), \quad g(x,u,\nabla u) \in L^1(\Omega) \\ \displaystyle\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx + \int_\Omega g(x,u,\nabla u)T_k(u-v)\,dx \\ \displaystyle\leq \int_\Omega fT_k(u-v)\,dx, \\ \forall\ \ v \in K_\psi \cap L^\infty(\Omega),\ \forall k > 0. \end{cases}$$

Where $K_\psi = \{u \in W_0^1L_M(\Omega)/u \geq \psi \ \ a.e.\ \text{ in }\ \Omega.)\}$, with ψ is a measurable function on Ω such that $\psi^+ \in W_0^1L_M(\Omega) \cap L^\infty(\Omega)$, and where T_k is the truncation operator at height $k > 0$, defined on $\mathbf{R}$ by $T_k(s) = \max(-k, \min(k,s))$.

Let us point out that another work in the L^p case can be found in [17] in the case of equation, and in [9] in the case of obstacle problems.

Note that this type of equations can be applied in sciences physics. Non-standard examples of $M(t)$ which occur in the mechanics of solids and fluids are $M(t) = t\log(1+t), M(t) = \displaystyle\int_0^t s^{1-\alpha}(\text{arcsinhs})^\alpha\,ds \ \ (0 \leq \alpha \leq 1)$ and $M(t) = t\log(1+\log(1+t))$ (see [11, 12, 13, 10]) for more details).

This paper is organized as follows, sections 2 contain some preliminaries and some technical lemmas. Section 3 is concerned with basic assumptions and the main result which is proved in section 4, finally, we study the stability and the positivity of solution.

2. Preliminaries

2-1 Let $M : \mathbf{R}^+ \to \mathbf{R}^+$ be an N-function, i.e. M is continuous, convex, with $M(t) > 0$ for $t > 0$, $\dfrac{M(t)}{t} \to 0$ as $t \to 0$ and $\dfrac{M(t)}{t} \to \infty$ as $t \to \infty$.

Equivalently, M admits the representation: $M(t) = \int_0^t a(s)\, ds$ where $a : \mathbf{R}^+ \to \mathbf{R}^+$ is nondecreasing, right continuous, with $a(0) = 0$, $a(t) > 0$ for $t > 0$ and $a(t)$ tends to ∞ as $t \to \infty$.

The N-function $\overline{M}$ conjugate to M is defined by $\overline{M} = \int_0^t \bar{a}(s)\, ds$, where $\bar{a}$:$\mathbf{R}^+ \to \mathbf{R}^+$ is given by $\bar{a}(t) = \sup\{s : a(s) \le t\}$.

The N-function M is said to satisfy the Δ_2-condition if, for some k

$$M(2t) \le kM(t) \quad \forall t \ge 0 \tag{2.1}$$

it is readily seen that this will be the case if and only if for every $r > 1$ there exists a positive constant $k = k(r)$ such that for all $t > 0$

$$M(rt) \le kM(t) \tag{2.2}$$

When (2.1) and (2.2) holds only for $t \ge t_0$ for some $t_0 > 0$ then M is said to satisfy the Δ_2-condition near infinity.

We will extend these N-functions as even functions on all $I\!R$. Moreover, we have the following Young's inequality

$$\forall s, t \ge 0, st \le M(t) + \overline{M}(s).$$

Let P and Q be two N-functions. $P << Q$ means that P grows essentially less rapidly than Q, i.e., for each $\epsilon > 0$, $\dfrac{P(t)}{Q(\epsilon t)} \to 0$ as $t \to \infty$. This is the case if and only if $\lim_{t\to\infty} \dfrac{Q^{-1}(t)}{P^{-1}(t)} = 0$.

2-2 Let Ω be an open subset of $I\!R^N$. The Orlicz class $K_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes modulo equality a. e.) real valued measurable functions u on Ω such that

$$\int_\Omega M(u(x))\, dx < +\infty(\text{ resp. } \int_\Omega M\left(\frac{u(x)}{\lambda}\right)\, dx < +\infty \text{ for some } \lambda > 0).$$

$L_M(\Omega)$ is a Banach space under the norm

$$\|u\|_{M,\Omega} = \inf\{\lambda > 0 : \int_\Omega M\left(\frac{u(x)}{\lambda}\right)\, dx \le 1\},$$

and $K_M(\Omega)$ is a convex subset of $L_M(\Omega)$.

The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$.

The dual of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_\Omega uv\,dx$, and the dual norm of $L_{\overline{M}}(\Omega)$ is equivalent to $\|.\|_{\overline{M},\Omega}$.

2-3 We now turn to the Orlicz-Sobolev space $W^1L_M(\Omega)$[resp. $W^1E_M(\Omega)$] which is the space of all functions u such that u and its distributional derivatives of order 1 lie in $L_M(\Omega)$ [resp. $E_M(\Omega)$]. It is a Banach space under the norm

$$\|u\|_{1,M} = \sum_{|\alpha|\leq 1} \|D^\alpha u\|_M.$$

Thus, $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of the product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by $\prod L_M$, we will use the weak topologies $\sigma(\prod L_M, \prod E_{\overline{M}})$ and $\sigma(\prod L_M, \prod L_{\overline{M}})$.

The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $D(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\prod L_M, \prod E_{\overline{M}})$ closure of $D(\Omega)$ in $W^1L_M(\Omega)$.

2-4 Let $W^{-1}L_{\overline{M}}(\Omega)$ [resp. $W^{-1}E_{\overline{M}}(\Omega)$] denote the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{M}}(\Omega)$ [resp. $E_{\overline{M}}(\Omega)$]. It is a Banach space under the usual quotient norm (for more details see [1]).

We now introduce the functional spaces we will need later.

For an N-function M, $\tau_0^{1,M}(\Omega)$ is defined as the set of measurable functions $u : \Omega \longrightarrow \mathbf{R}$ such that for all $k > 0$ the truncated functions $T_k(u) \in W_0^1L_M(\Omega)$.

We gives the following lemma this is a generalization of Lemma 2.1 [2] in Orlicz spaces.

Lemma 2.1. *For every $u \in \tau_0^{1,M}(\Omega)$, there exists a unique measurable function $v : \Omega \longrightarrow \mathbf{R}^N$ such that*

$$\nabla T_k(u) = v\chi_{\{|v|<k\}} \quad \textit{for all} \;\; k > 0.$$

Lemma 2.2. *Let $\lambda \in \mathbf{R}$ and let u and v be two measurable functions defined on Ω which are finite almost everywhere, and which are such that $T_k(u)$, $T_k(v)$ and $T_k(u+\lambda v)$ belong to $W_0^1L_M(\Omega)$ for every $k>0$ then*

$$\nabla(u+\lambda v)=\nabla(u)+\lambda\nabla(v) \text{ a.e. in } \Omega.$$

where $\nabla(u)$, $\nabla(v)$ and $\nabla(u+\lambda v)$ are the gradients of u, v and $u+\lambda v$ introduced in Lemma 2.1.

The proof of this lemma is similar to the proof of Lemma 2.12 [8] for the Sobolev spaces.

We recall some lemmas introduced in [4] which will be used later.

Lemma 2.3. *Let $F:\mathbf{R}\to\mathbf{R}$ be uniformly Lipschitzian, with $F(0)=0$. Let M be an N-function and let $u\in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). Then $F(u)\in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). Moreover, if the set D of discontinuity points of F' is finite, then*

$$\frac{\partial}{\partial x_i}F(u)=\begin{cases} F'(u)\frac{\partial}{\partial x_i}u \text{ a.e. in } \{x\in\Omega : u(x)\notin D\}, \\ 0 \qquad \text{a.e. in } \{x\in\Omega : u(x)\in D\}. \end{cases}$$

Lemma 2.4. *Let $F:\mathbf{R}\to\mathbf{R}$ be uniformly Lipschitzian, with $F(0)=0$. We suppose that the set of discontinuity points of F' is finite. Let M be an N-function, then the mapping $N_F: W^1L_M(\Omega)\to W^1L_M(\Omega)$ is sequentially continuous with respect to the weak* topology $\sigma(\prod L_M,\prod E_{\overline{M}})$.*

We give now the following lemma which concerns operators of Nemytskii type in Orlicz spaces (see [4]).

Lemma 2.5. *Let Ω be an open subset of $\mathbf{R}^N$ with finite measure.*

Let M, P and Q be N-functions such that $Q << P$, and let $F:\Omega\times\mathbf{R}\to\mathbf{R}$ be a Carathéodory function such that, for a.e. $x\in\Omega$ and all $s\in\mathbf{R}$:

$$|F(x,s)|\le c(x)+k_1P^{-1}M(k_2|s|),$$

where k_1, k_2 are real constants and $c(x)\in E_Q(\Omega)$.

Then the Nemytskii operator N_F defined by $N_F(u)(x)=F(x,u(x))$ is strongly continuous from $\mathcal{P}(E_M(\Omega),\frac{1}{k_2})=\{u\in L_M(\Omega) : d(u,E_M(\Omega))<\frac{1}{k_2}\}$ into $E_Q(\Omega)$.

3. Main results

Let Ω be an open bounded subset of $\mathbf{R}^N$, $N \geq 2$, with the segment property. Let M be an N-function satisfying the Δ_2-condition near infinity, and let P be an N-function such that $P << M$. We consider the Leray-Lions operator,

$$Au = -div(a(x, u, \nabla u)),$$

defined on $D(A) \subset W_0^1 L_M(\Omega)$ into $W^{-1} L_{\overline{M}}(\Omega)$ where $a : \Omega \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}^N$ is a Carathéodory function such that for a.e. $x \in \Omega$ and for all $\zeta, \xi \in \mathbf{R}^N (\zeta \neq \xi)$ and all $s \in \mathbf{R}$,

$$|a(x, s, \zeta)| \leq k(x) + k_1 \overline{P}^{-1} M(k_2|s|) + k_3 \overline{M}^{-1} M(k_4|\zeta|), \tag{3.1}$$

$$(a(x, s, \zeta) - a(x, s, \xi))(\zeta - \xi) > 0, \tag{3.2}$$

$$a(x, s, \zeta)\zeta \geq \alpha M\left(\frac{|\zeta|}{\delta}\right), \tag{3.3}$$

with $\alpha, \delta > 0 \quad k_1, k_2, k_3, k_4 \geq 0, k(x) \in E_{\overline{M}}(\Omega)$.

Furthermore let $g : \Omega \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}$ be a Carathéodory function such that for a.e. $x \in \Omega$ and for all $s \in \mathbf{R}$ and all $\zeta \in \mathbf{R}^N$,

$$g(x, s, \zeta)s \geq 0, \tag{3.4}$$

$$|g(x, s, \zeta)| \leq b(|s|)(c(x) + M(|\zeta|/\lambda)), \tag{3.5}$$

where $b : \mathbf{R}_+ \to \mathbf{R}_+$ is a continuous nondecreasing function, c is a given positive function in $L^1(\Omega)$, and $\lambda > 0$. Let the subset convex

$$K_\psi = \{u \in W_0^1 L_M(\Omega)/u \geq \psi \ \ a.e. \ \text{ in } \ \ \Omega.)\},$$

where $\psi : \Omega \to \overline{\mathbf{R}}$ is a measurable function on Ω such that

$$\psi^+ \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega). \tag{3.6}$$

Finally, we assume that

$$f \in L^1(\Omega), \tag{3.7}$$

In the next section, we will prove the following theorem.

Theorem 3.1. *Assume that the hypotheses (3.1)-(3.7) holds. Then, there exists at least one solution of the following unilateral problem*

$$(P)\begin{cases} u \geq \psi \text{ a.e. in } \Omega. \\ T_k(u) \in W_0^1 L_M(\Omega), \quad g(x,u,\nabla u) \in L^1(\Omega) \\ \displaystyle\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx + \int_\Omega g(x,u,\nabla u)T_k(u-v)\,dx \\ \qquad\qquad\qquad\qquad \leq \displaystyle\int_\Omega f T_k(u-v)\,dx, \\ \forall \;\; v \in K_\psi \cap L^\infty(\Omega), \;\; \forall k > 0. \end{cases}$$

Remark 3.1. *We obtain the same results of our theorem if we suppose that the sign condition (3.4) is fulfilled only near infinity.*

4. Proof of main result

To prove the existence theorem we proceed by steps.

STEP 1: A priori estimates.

Let us define

$$g_n(x,s,\xi) = \frac{g(x,s,\xi)}{1+\frac{1}{n}|g(x,s,\xi)|}$$

and let us consider the sequence of approximate variational inequalities

$$(P_n)\begin{cases} u_n \in K_\psi, \\ \langle Au_n, u_n - v\rangle + \displaystyle\int_\Omega g_n(x,u_n,\nabla u_n)(u_n - v)\,dx \\ \qquad\qquad\qquad\qquad \leq \displaystyle\int_\Omega f_n(u_n - v)\,dx, \\ \forall v \in K_\psi, \end{cases}$$

where f_n is a regular function such that f_n strongly converges to f in $L^1(\Omega)$. Since $g_n(x,s,\xi)$ is bounded and $g_n(x,s,\xi).s \geq 0$, then by using the Proposition 5 and Remark 6 of [15] (with $m = 1$ and $a_0(x,s,\xi) = g_n(x,s,\xi)$) it is easy to verifie that (P_n) has at least one solution.

Let $w = u_n - T_k(u_n - \psi^+) \in K_\psi$. The choice of w as a test function in (P_n), we obtain

$$\begin{cases} \langle Au_n, T_k(u_n - \psi^+)\rangle + \displaystyle\int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n - \psi^+)\,dx \\ \qquad\qquad\qquad\qquad \leq \displaystyle\int_\Omega f_n T_k(u_n - \psi^+)\,dx. \end{cases}$$

which gives

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-\psi^+)\,dx+\int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n-\psi^+)\,dx \\ \leq \int_\Omega f_nT_k(u_n-\psi^+)\,dx.$$

Since $g_n(x,u_n,\nabla u_n)T_k(u_n-\psi^+)\geq 0$, we obtain

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-\psi^+)\,dx\leq \int_\Omega f_nT_k(u_n-\psi^+)\,dx,$$

Consequently, we deduce that

$$\int_{\{|u_n-\psi^+|\leq k\}} a(x,u_n,\nabla u_n)\nabla u_n\,dx\leq C_0k+\int_{\{|u_n-\psi^+|\leq k\}} a(x,u_n,\nabla u_n)\nabla\psi^+\,dx.$$

By using the Young's inequality and the Δ_2-condition we have

$$\int_{\{|u_n-\psi^+|\leq k\}} M(|\nabla u_n|/\delta)\,dx\leq C_1'+C_2'k,$$

which implies that

$$\int_\Omega M(|\nabla T_k(u_n)|/\delta)\,dx\leq \int_{\{|u_n-\psi^+|\leq k+\|\psi^+\|_\infty\}} M(|\nabla u_n|/\delta)\,dx\leq C_1'+C_2'k. \tag{4.1}$$

Now, we prove that u_n converges to some function u in measure (and therefore, we can always assume that the convergence is a.e. after passing to a suitable subsequence). We shall show that u_n is a Cauchy sequence in measure.

Thanks to Lemma 5.7 of [14], there exists two positive constants C_3' and C_4' such that

$$\int_\Omega M(u)\,dx\leq C_4'\int_{\Omega displaystyle \sum_{i=1}^N M(C_3'\frac{\partial u}{\partial x_i})\,dx} \quad\text{for all}\quad u\in W_0^1L_M(\Omega).$$

Taking $u=\frac{T_k(u_n)}{C_3'\delta}$, we have

$$\begin{aligned}\int_\Omega M(\frac{1}{C_3'\delta}T_k(u_n))\,dx &\leq C_4'\int_\Omega\sum_{i=1}^N M(\frac{1}{\delta}\frac{\partial T_k(u_n)}{\partial x_i})\,dx\\ &\leq NC_4'\int_\Omega M(\frac{\nabla T_k(u_n)}{\delta}\,dx\end{aligned}$$

then

$$\int_\Omega M(C_3 T_k(u_n))\,dx \le C_4 \int_\Omega M(|\nabla T_k(u_n)|/\delta)\,dx \le C_1 + C_2 k. \tag{4.2}$$

with $C_3 = \frac{1}{C_3'\delta}$ and $C_4 = NC_4'$. Then, we deduce by using (4.1) and (4.2) that
$M(C_3k)\text{meas}(\{|u_n| > k\}) = \int_{\{|u_n|>k\}} M(C_3 T_k(u_n))\,dx \le \int_\Omega M(C_3 T_k(u_n))\,dx \le$ $C_1 + C_2 k$, hence

$$\text{meas}(\{|u_n| > k\}) \le \frac{C_1 + C_2 k}{M(C_3 k)} \quad \forall n \quad \text{and} \quad \forall k > 0. \tag{4.3}$$

For every $\delta > 0$, we have

$$\begin{aligned} meas(\{|u_n - u_m| > \delta\}) \quad &\le \quad \text{meas}(\{|u_n| > k\}) + \text{meas}(\{|u_m| > k\}) \\ &\quad + \text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\}. \end{aligned} \tag{4.4}$$

Since $T_k(u_n)$ is bounded in $W_0^1 L_M(\Omega)$, there exists some $v_k \in W_0^1 L_M(\Omega)$, such that

$$\begin{aligned} &T_k(u_n) \rightharpoonup v_k \quad &&\text{weakly in } W_0^1 L_M(\Omega) \\ &T_k(u_n) \to v_k \quad &&\text{strongly in } E_M(\Omega) \text{ and a.e. in } \Omega. \end{aligned}$$

Consequently, we can assume that $T_k(u_n)$ is a Cauchy sequence in measure in Ω.

Let $\varepsilon > 0$. then, by (4.3) and (4.4), there exists some $k(\varepsilon) > 0$ such that $\text{meas}(\{|u_n - u_m| > \delta\}) < \varepsilon$ for all $n, m \ge n_0(k(\varepsilon), \delta,)$. This proves that (u_n) is a Cauchy sequence in measure in Ω, thus converges almost everywhere to some measurable function u. Then

$$\begin{aligned} &T_k(u_n) \rightharpoonup T_k(u) \quad &&\text{weakly in } W_0^1 L_M(\Omega) \text{ for } \sigma(\Pi L_M, \Pi E_{\overline{M}}), \\ &T_k(u_n) \to T_k(u) \quad &&\text{strongly in } E_M(\Omega). \end{aligned}$$

We shall prove that the sequence $(a(x, T_k(u_n), \nabla T_k(u_n)))_n$ is bounded in $(L_{\overline{M}}(\Omega))^N$ for all $k > 0$.

Let $w \in (E_M(\Omega))^N$ be arbitrary. By condition (3.2), we have

$$(a(x, u_n, \nabla u_n) - a(x, u_n, w))\,(\nabla u_n - w) \ge 0.$$

Consequently

$$\begin{aligned} &\int_{\{|u_n - \psi^+| \le k\}} a(x, u_n, \nabla u_n)(w - \nabla\psi^+)\,dx \\ &\le \int_{\{|u_n - \psi^+| \le k\}} a(x, u_n, \nabla u_n)(\nabla u_n - \nabla\psi^+)\,dx \\ &\quad + \int_{\{|u_n - \psi^+| \le k\}} a(x, u_n, w)(w - \nabla u_n)\,dx. \end{aligned} \tag{4.5}$$

Using the argument above, we may assume that the first term on the right remains bounded. Moreover, by (3.1), we have

$$|a(x,T_k(u_n),w)| \le c(x) + k_1\overline{M}^{-1}M(k_2|T_k(u_n)|) + k_3\overline{M}^{-1}M(k_4 w) + k_5.$$

Therefore,

$$\begin{aligned}\int_\Omega \overline{M}(\frac{|a(x,T_k(u_n),w)|}{\mu}\ dx \quad &\le \quad \tfrac{1}{\mu}\int_\Omega \overline{M}(c(x))\ dx + \frac{k_3}{\mu}\int_\Omega M(k_4 w) \\ &\qquad + \tfrac{k_6+M(k_2k)}{\mu}\mathrm{meas}(\Omega) \le 1,\end{aligned}$$

when $\mu > 0$ is large enough. Hence $a(x,T_k(u_n),w)$ is bounded in $L_{\overline{M}}(\Omega)$, which implies that the second term on the right in (4.5) is also bounded. By the theorem of Banach-Steinhaus, the sequence $(a(x,u_n,\nabla u_n)\chi_{\{|u_n-\psi^+|\le k\}})$ remains bounded in $L_{\overline{M}}(\Omega)$. Since k arbitrary, we deduce that $(a(x,T_k(u_n),\nabla T_k(u_n))$ also bounded in $L_{\overline{M}}(\Omega)$. Which implies that, for all $k>0$ there exists a function $h_k \in (L_{\overline{M}}(\Omega))^N$, such that

$$a(x,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup h_k \text{ weakly in } (L_{\overline{M}}(\Omega)^N \text{ for } \sigma(\Pi L_{\overline{M}}(\Omega),\Pi E_M(\Omega)) \tag{4.6}$$

STEP 2: Strong convergence of truncation.

We fix $k>0$. Let $\Omega_r = \{x \in \Omega : |\nabla T_k(u(x))| \le r\}$ and denote by χ_r the characteristic function of Ω_r. Clearly, $\Omega_r \subset \Omega_{r+1}$ and $\mathrm{meas}(\Omega\backslash\Omega_\mathrm{r}) \longrightarrow 0$ as $r \longrightarrow \infty$.

Fix r and let $s \ge r$, we have,

$$\begin{aligned} 0 \le \quad & \int_{\Omega_r}[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n) \\ &\qquad\qquad\qquad\qquad -\nabla T_k(u)]\ dx \\ \le \quad & \int_{\Omega_s}[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n) \\ &\qquad\qquad\qquad\qquad -\nabla T_k(u)]\ dx \\ = \quad & \int_{\Omega_s}[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)][\nabla T_k(u_n) \\ &\qquad\qquad\qquad\qquad -\nabla T_k(u)\chi_s]\ dx \\ \le \quad & \int_{\Omega}[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)][\nabla T_k(u_n) \\ &\qquad\qquad\qquad\qquad -\nabla T_k(u)\chi_s]\ dx. \end{aligned} \tag{4.7}$$

Now, consider the following function

$$\varphi(s) = se^{\gamma s^2}, \quad \text{where} \quad \gamma > (K\frac{b(k)}{\alpha})^2.$$

It is well known that

$$\varphi'(s) - K\frac{b(k)}{\alpha}|\varphi(s)| \geq \frac{1}{2}, \quad \forall s\, \mathbf{R}, \tag{4.8}$$

where K is a constant which will be used later.

Let $k \geq \|\psi^+\|_\infty$, we define $w_n = T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))$ where $h > 2k > 0$. For $\eta = \exp(-4\gamma \mathrm{k}^2)$, we define the following function as

$$v_{n,h} = u_n - \eta\varphi(w_n). \tag{4.9}$$

We take $v_{n,h}$ as test function in (P_n) (for more explication concerned this test function see the appendix II), we obtain,

$$\langle A(u_n), \eta\varphi(w_n)\rangle + \int_\Omega g_n(x, u_n, \nabla u_n)\eta\varphi(w_n)\, dx \leq \int_\Omega f_n\eta\varphi(w_n)\, dx.$$

Which, implies that

$$\langle A(u_n), \varphi(w_n)\rangle + \int_\Omega g_n(x, u_n, \nabla u_n)\varphi(w_n)\, dx \leq \int_\Omega f_n\varphi(w_n)\, dx. \tag{4.10}$$

It follows that

$$\begin{aligned}\int_\Omega a(x, u_n, \nabla u_n)\nabla w_n\varphi'(w_n)\, dx + \int_\Omega g_n(x, u_n, \nabla u_n)\varphi(w_n)\, dx& \\ \leq \int_\Omega f_n\varphi(w_n)\, dx.&\end{aligned} \tag{4.11}$$

Note that, $\nabla w_n = 0$ on the set where $|u_n| > h + 4k$, therefore, setting $m = 4k + h$, and denoting by $\varepsilon_h^1(n), \varepsilon_h^2(n), ...$ various sequences of real numbers which converge to zero as n tends to infinity for any fixed value of h, we get, by (4.11),

$$\begin{aligned}\int_\Omega a(x, T_m(u_n), \nabla T_m(u_n))\nabla w_n\varphi'(w_n)\, dx + \int_\Omega g_n(x, u_n, \nabla u_n)\varphi(w_n)\, dx& \\ \leq \textstyle\int_\Omega f_n\varphi(w_n)\, dx,&\end{aligned}$$

and since for x in the set $\{x \in \Omega : |u_n(x)| > k\}$, we have that $\varphi(w_n)g(x, u_n, \nabla u_n) \geq 0$, we deduce from (4.11) that

$$\begin{aligned}\int_\Omega a(x, T_m(u_n), \nabla T_m(u_n))\nabla w_n\varphi'(w_n)\, dx + \int_{\{|u_n|\leq k\}} g_n(x, u_n, \nabla u_n)\varphi(w_n)\, dx& \\ \leq \int_\Omega f_n\varphi(w_n)\, dx.&\end{aligned} \tag{4.12}$$

Splitting the first integral on the left hand side of (4.12) where $|u_n| \leq k$ and $|u_n| > k$, we can write, using (3.3) and the fact that $a(x, s, 0) = 0$ for all $s \in I\!R$.:

$$\begin{aligned} &\int_\Omega a(x, T_m(u_n), \nabla T_m(u_n)) \nabla w_n \varphi'(w_n)\, dx \\ (4.13) \quad &\geq \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))[\nabla T_k(u_n) - \nabla T_k(u)]\varphi'(w_n)\, dx \\ &-C_k \int_{\{|u_n|>k\}} |a(x, T_m(u_n), \nabla T_m(u_n))||\nabla T_k(u)|\, dx, \end{aligned}$$

where $C_k = \varphi'(2k)$. Since, as n tends to infinity, $|\nabla T_k(u)|\chi_{\{|u_n|>k\}}$ strong converges to zero in $E_M(\Omega)$ while $|a(x, T_m(u_n), \nabla T_m(u_n))|$ is bounded in $L_{\overline{M}}(\Omega)$, the last term in the previous inequality tends to zero for every h fixed.

Now, observe that

$$\begin{aligned} &\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))[\nabla T_k(u_n) - \nabla T_k(u)]\varphi'(w_n)\, dx \\ &= \int_\Omega [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi_s)] \\ (4.14) \quad &\qquad \times [\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\varphi'(w_n)\, dx \\ &+ \int_\Omega a(x, T_k(u_n), \nabla T_k(u)\chi_s)[\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\varphi'(w_n)\, dx \\ &- \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\nabla T_k(u)\chi_{\Omega\setminus\Omega_s}\varphi'(w_n)\, dx. \end{aligned}$$

The second terms of the right hand side of (4.14) tend to 0. Indeed

$$\begin{aligned} &\int_\Omega a(x, T_k(u_n), \nabla T_k(u)\chi_s)[\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\varphi'(w_n)\, dx = \\ &\int_\Omega a(x, T_k(u_n), \nabla T_k(u)\chi_s)\nabla T_k(u_n)\varphi'(T_k(_n) - T_k(u))\, dx \\ &- \int_\Omega a(x, T_k(u_n), \nabla T_k(u)\chi_s)\nabla T_k(u)\chi_s\varphi'(w_n)\, dx \end{aligned}$$

by using Lemma 2.5, we have

$$a(x, T_k(u_n), \nabla T_k(u)\chi_s)\varphi'(T_k(_n) - T_k(u)) \to a(x, T_k(u), \nabla T_k(u)\chi_s)\varphi'(0)$$

$$\text{strongly in } \ (E_{\overline{M}}(\Omega))^N.$$

and

$$a(x, T_k(u_n), \nabla T_k(u)\chi_s) \to a(x, T_k(u), \nabla T_k(u)\chi_s)$$

$$\text{strongly in } (E_{\overline{M}}(\Omega))^N.$$

Moreover,

$$\nabla T_k(u_n) \rightharpoonup \nabla T_k(u) \text{ weakly in } (L_M(\Omega))^N, \text{ for } \sigma(\Pi L_M(\Omega), \Pi E_{\overline{M}}(\Omega)),$$

and

$$\nabla T_k(u)\chi_s\varphi'(w_n) \to \nabla T_k(u)\chi_s\varphi'(w_n) \text{ strongly in } (L_M(\Omega))^N$$

by using the Lebesgue's theorem. Combining the previous statement, we deduce the result.

The third term of the right hand side of (4.14) tends to the quantity $-\int_\Omega h_k \nabla T_k(u)\chi_{\Omega\setminus\Omega_s}\varphi'(T_{2k}(u - T_h(u)))\,dx$ as $n \to \infty$, since by (4.6), we have

$$a(x, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup h_k \quad \text{weakly in } (L_{\overline{M}}(\Omega))^N \text{ for } \sigma(\Pi L_{\overline{M}}(\Omega), \Pi E_M(\Omega))$$

while

$$\chi_{\Omega\setminus\Omega_s}\nabla T_k(u)\varphi'(w_n) \to \chi_{\Omega\setminus\Omega_s}\nabla T_k(u)\varphi'(T_{2k}(u - T_h(u))) \quad \text{strongly in } (E_M(\Omega))^N.$$

So that (4.13) yields

$$\begin{aligned}
\int_\Omega a(x, T_m(u_n), \nabla T_m(u_n))\nabla w_n \varphi'(w_n)\,dx \\
\geq \int_\Omega [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi_s)] \\
\times[\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\varphi'(w_n)\,dx \\
-\int_\Omega h_k \nabla T_k(u)\chi_{\Omega\setminus\Omega_s}\varphi'(T_{2k}(u - T_h(u)))\,dx + \varepsilon_h^1(n).
\end{aligned} \tag{4.15}$$

Since the N-function M satisfies the Δ_2-condition near infinity, then there exist two positive constants K and K' such that

$$M(t/\lambda) \leq KM(t/\delta) + K', \quad \forall t \geq 0. \tag{4.16}$$

Endeed : if $\delta \leq \lambda$, we get $M(\frac{t}{\lambda}) \leq M(\frac{t}{\delta}) \quad \forall\, t \geq 0$ and if $\frac{\delta}{\lambda} > 1$, under the condition (2.2) there exists $k(\frac{\delta}{\lambda}) > 0$ such that $M(\frac{\delta}{\lambda}t) \leq k(\frac{\delta}{\lambda})M(t) \quad \forall t \geq t_0$ for some $t_0 > 0$, hence $M(\frac{t}{\lambda}) \leq k(\frac{t}{\delta})M(t) \quad \forall t \geq \delta t_0$ and $M(\frac{t}{\lambda}) \leq K' \;\; \forall 0 \leq$

$t \leq \delta t_0$.

Finally, taking $K = \max(1, k(\frac{\delta}{\lambda}))$, we deduce (4.16).

For the second term of the left hand side of (4.12), we can estimate as follows

$$\begin{aligned}
&\int_{\{|u_n|\leq k\}} g_n(x, u_n, \nabla u_n)\varphi(w_n)\, dx \\
&\leq \int_{\{|u_n|\leq k\}} b(k)(c(x) + K' + KM(\nabla u_n/\delta))|\varphi(w_n)|\, dx \\
&\leq b(k)\int_\Omega (c(x) + K')|\varphi(w_n)|\, dx \\
&+K\tfrac{b(k)}{\alpha}\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\nabla T_k(u_n)|\varphi(w_n)|\, dx,
\end{aligned} \tag{4.17}$$

let us consider the last integral in the previous inequality

$$\begin{aligned}
&\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\nabla T_k(u_n)|\varphi(w_n)|\, dx \\
&= \int_\Omega [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi_s)] \\
&\times[\nabla T_k(u_n) - \nabla T_k(u)\chi_s]|\varphi(w_n)|\, dx \\
&+\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\nabla T_k(u)\chi_s|\varphi(w_n)|\, dx \\
&+\int_\Omega a(x, T_k(u_n), \nabla T_k(u)\chi_s)[\nabla T_k(u_n) - \nabla T_k(u)\chi_s]|\varphi(w_n)|\, dx.
\end{aligned} \tag{4.18}$$

It is easy to see that the second term of the right hand side tends to the quantity

$$\int_\Omega h_k\nabla T_k(u)\chi_s|\varphi(T_{2k}(u - T_h(u)))|\, dx,$$

since $a(x, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup h_k$ in $(L_{\overline{M}}(\Omega))^N$ for $\sigma(\Pi L_{\overline{M}}(\Omega), \Pi E_M(\Omega))$ and

$$\nabla T_k(u)\chi_s|\varphi(w_n)| \to \chi_s\nabla T_k(u)|\varphi(T_{2k}(u - T_h(u)))| \text{ strongly in } (E_M(\Omega))^N.$$

Reasoning as in (4.14), the third term of the right hand side of (4.18) tends to 0. From (4.17) and (4.18), we obtain

$$
\begin{aligned}
&\int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n)\varphi(w_n)\,dx\\
&\le K\tfrac{b(k)}{\alpha}\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)]\\
&\times[\nabla T_k(u_n) - \nabla T_k(u)\chi_s]|\varphi(w_n)|\,dx\\
&+K\tfrac{b(k)}{\alpha}\int_\Omega h_k\nabla T_k(u)\chi_s|\varphi(T_{2k}(u-T_h(u)))|\,dx\\
&+b(k)\int_\Omega (c(x)+K')|\varphi(T_{2k}(u-T_h(u)))|\,dx + \varepsilon_h^3(n).
\end{aligned}
\tag{4.19}
$$

Combining (4.12), (4.15) and (4.19), we obtain

$$
\begin{aligned}
&\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)]\\
&\qquad\times[\nabla T_k(u_n) - \nabla T_k(u)\chi_s](\varphi'(w_n) - K\tfrac{b(k)}{\alpha}|\varphi(w_n)|)\,dx\\
&\le \varepsilon_h^4(n) + \int_\Omega h_k\nabla T_k(u)\chi_{\Omega\setminus\Omega_s}\varphi'(T_{2k}(u-T_h(u)))\,dx\\
&\quad+K\tfrac{b(k)}{\alpha}\int_\Omega h_k\nabla T_k(u)\chi_s|\varphi(T_{2k}(u-T_h(u)))|\,dx\\
&\quad+b(k)\int_\Omega (c(x)+K')|\varphi(T_{2k}(u-T_h(u)))|\,dx\\
&\quad+\int_\Omega f\varphi(T_{2k}(u-T_h(u)))|\,dx
\end{aligned}
$$

which implies, by using (4.8),

$$
\begin{aligned}
&\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)]\\
&\qquad\times[\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\,dx\\
&\le \varepsilon_h^5(n) + 2\int_\Omega h_k\nabla T_k(u)\chi_{\Omega\setminus\Omega_s}\varphi'(T_{2k}(u-T_h(u)))\,dx\\
&\quad+2K\tfrac{b(k)}{\alpha}\int_\Omega h_k\nabla T_k(u)\chi_s|\varphi(T_{2k}(u-T_h(u)))|\,dx\\
&\quad+2b(k)\int_\Omega (c(x)+K')|\varphi(T_{2k}(u-T_h(u)))|\,dx\\
&\quad+2\int_\Omega f\varphi(T_{2k}(u-T_h(u)))|\,dx.
\end{aligned}
\tag{4.20}
$$

Consequently, from (4.7), we have

$$\begin{aligned}\int_{\Omega_r} & [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))] \\ & \qquad\qquad \times[\nabla T_k(u_n) - \nabla T_k(u)]\,dx \\ & \qquad \le \varepsilon_h^6(n) + 2\int_\Omega h_k \nabla T_k(u)\chi_{\Omega\setminus\Omega_s}\varphi'(T_{2k}(u-T_h(u)))\,dx \\ & \qquad\quad +2K\tfrac{b(k)}{\alpha}\int_\Omega h_k\nabla T_k(u)\chi_s|\varphi(T_{2k}(u-T_h(u)))|\,dx \\ & \qquad\quad +b(k)\int_\Omega (c(x)+K')|\varphi(T_{2k}(u-T_h(u)))|\,dx \\ & \qquad\quad +2\int_\Omega f\varphi(T_{2k}(u-T_h(u)))|\,dx.\end{aligned}$$

By passing to the lim sup over n, and letting h, s tend to infinity, we obtain by the same method used in [4] that

$$\nabla u_n \to \nabla u \quad a.e. \text{ in } \Omega, \tag{4.21}$$

which implies that $h_k = a(x,T_k(u),\nabla T_k(u))\ \ \forall k>0$.

Again by (4.20), we get

$$\begin{aligned} & \limsup_{n\to\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\,dx \\ & \le \limsup_{n\to\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u)\chi_s\,dx \\ & +\limsup_{n\to\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi_s)(\nabla T_k(u_n)-\nabla T_k(u)\chi_s)\,dx \\ & +2\int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)\chi_{\Omega\setminus\Omega_s}\varphi'(T_{2k}(u-T_h(u)))\,dx \\ & +2K\tfrac{b(k)}{\alpha}\int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)\chi_s|\varphi(T_{2k}(u-T_h(u)))|\,dx \\ & +b(k)\int_\Omega (c(x)+K')|\varphi(T_{2k}(u-T_h(u)))|\,dx \\ & +2\int_\Omega f\varphi(T_{2k}(u-T_h(u)))|\,dx.\end{aligned} \tag{4.22}$$

The first term of the right hand side of the last inequality tend to $\int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)\chi_s\,dx$ and since $a(x,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup a(x,T_k(u),\nabla T_k(u))$ weakly in $(L_{\overline{M}}(\Omega))^N$ for $\sigma(\Pi L_{\overline{M}}(\Omega),\Pi E_M(\Omega))$ while $\nabla T_k(u)\chi_s \in E_M(\Omega)$.

The second term of the right hand side of (4.22) tends to 0, since $a(x,T_k(u_n),\nabla T_k(u)\chi_s) \to a(x,T_k(u),\nabla T_k(u)\chi_s)$ strongly in $(E_{\overline{M}}(\Omega))^N$ while

$\nabla T_k(u_n)$ tends weakly to $\nabla T_k(u)$. We deduce that

$$
\begin{aligned}
&\limsup_{n\to\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\,dx\\
&\le \int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)\chi_s\,dx\\
&+2\int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)\chi_{\Omega\setminus\Omega_s}\varphi'(T_{2k}(u-T_h(u)))\,dx\\
&+2K\tfrac{b(k)}{\alpha}\int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)\chi_s|\varphi(T_{2k}(u-T_h(u)))|\,dx\\
&+b(k)\int_\Omega (c(x)+K')|\varphi(T_{2k}(u-T_h(u)))|\,dx\\
&+2\int_\Omega f\varphi(T_{2k}(u-T_h(u)))|\,dx.
\end{aligned}
$$

Passing again to the limsup but now over h, and by using that the functions $a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)$,
$c(x)+K'$ and f belong to $L^1(\Omega)$ and that

$$|\varphi(T_{2k}(u-T_h(u)))|\to 0,$$

$\varphi'(T_{2k}(u-T_h(u)))\to\varphi'(0)$ as $h\to\infty$, one easily obtains by Lebesgue's theorem

$$
\begin{aligned}
\limsup_{h\to\infty}\ \limsup_{n\to\infty}&\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\,dx\\
&\le \int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)\chi_s\,dx\\
&+2\varphi'(0)\int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)\chi_{\Omega\setminus\Omega_s}\,dx.
\end{aligned}
$$

Using again the fact that $a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)\in L^1(\Omega)$ and letting $s\to\infty$ we get, since $\operatorname{meas}(\Omega\setminus\Omega_s)\to 0$,

$$
\begin{aligned}
\limsup_{h\to\infty}\ \limsup_{n\to\infty}&\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\,dx\\
&\le \int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)\,dx.
\end{aligned}
$$

On the other hand, by Fatou's lemma,

$$
\begin{aligned}
&\int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)\,dx\\
&\le \limsup_{h\to\infty}\ \limsup_{n\to\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\,dx,
\end{aligned}
$$

which implies

$$\lim_{n\to\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\,dx = \int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)\,dx. \tag{4.23}$$

Thanks to (4.16), we have

$$M(|\nabla T_k(u_n)|/\lambda) \le K' + KM(|\nabla T_k(u_n)|/\delta)$$

and by using (4.23), one obtains, by Vitali's theorem,

$$M(|\nabla T_k(u_n)|/\lambda) \to M(|\nabla T_k(u)|/\lambda) \ \text{ in } \ L^1(\Omega), \tag{4.24}$$

STEP 3: Passing to the limit. Let $v \in K_\psi \cap L^\infty(\Omega)$, we take $u_n - T_k(u_n - v)$ as test function in (P_n), we can write

$$\begin{aligned}\int_\Omega a(x,T_{k+\|v\|_\infty}(u_n),\nabla T_{k+\|v\|_\infty}(u_n))\nabla T_k(u_n-v)\,dx&\\ +\int_\Omega g(x,u_n,\nabla u_n)T_k(u_n-v)\,dx&\\ \le \int_\Omega f_n T_k(u_n-v)\,dx.&\end{aligned} \tag{4.25}$$

By Fatou's lemma and the fact that

$$a(x,T_{k+\|v\|_\infty}(u_n),\nabla T_{k+\|v\|_\infty}(u_n)) \rightharpoonup a(x,T_{k+\|v\|_\infty}(u),\nabla T_{k+\|v\|_\infty}(u))$$

weakly in $(L_{\overline{M}}(\Omega))^N$ for $\sigma(\Pi L_{\overline{M}},\Pi E_M)$ on easily see that

$$\begin{aligned}&\int_\Omega a(x,T_{k+\|v\|_\infty}(u),\nabla T_{k+\|v\|_\infty}(u))\nabla T_k(u-v)\,dx\\ &\le \liminf_{n\to\infty}\int_\Omega a(x,T_{k+\|v\|_\infty}(u_n),\nabla T_{k+\|v\|_\infty}(u_n))\nabla T_k(u_n-v)\,dx.\end{aligned} \tag{4.26}$$

Now, we need to prove that

$$g_n(x,u_n,\nabla u_n) \to g(x,u,\nabla u) \ \text{ strongly in } \ L^1(\Omega), \tag{4.27}$$

in particular it is enough to prove that the functions $g_n(x,u_n,\nabla u_n)$ are equiintegrable of $g_n(x,u_n,\nabla u_n)$. To this purpose. We take $u_n-(T_{l+1}(u_n)-T_l(u_n))$ (with $l \ge \|\psi^+\|_\infty$) as test function in (P_n) (see Appendix III), we obtain

$$\int_{\{|u_n|>l+1\}} |g_n(x,u_n,\nabla u_n)|\,dx \le \int_{\{|u_n|>l\}} |f_n|\,dx.$$

Let $\varepsilon > 0$, then there exists $l(\varepsilon) \geq 1$ such that

$$\int_{\{|u_n|>l(\varepsilon)\}} |g(x, u_n, \nabla u_n)|\, dx < \varepsilon/2. \tag{4.28}$$

For any measurable subset $E \subset \Omega$, we have

$$\begin{aligned}\int_E |g_n(x, u_n, \nabla u_n)|\, dx \quad &\leq \quad \int_E b(l(\varepsilon))(c(x) + M(\nabla T_{l(\varepsilon)}(u_n)/\lambda))\, dx \\ &\quad + \int_{\{|u_n|>l(\varepsilon)\}} |g(x, u_n, \nabla u_n)|\, dx.\end{aligned}$$

In view by (4.24) there exists $\eta(\varepsilon) > 0$ such that

$$\begin{array}{c}\int_E b(l(\varepsilon))(c(x) + M(|\nabla T_{l(\varepsilon)|}(u_n)/\lambda))\, dx < \varepsilon/2 \\ \text{for all} \\ E \\ \text{such that} \quad \text{meas}(E) < \eta(\varepsilon).\end{array} \tag{4.29}$$

Finally, by combining (4.28) and (4.29), one easily has

$$\int_E |g_n(x, u_n, \nabla u_n)|\, dx < \varepsilon \quad \text{for all} \quad E \quad \text{such that} \quad \text{meas}(E) < \eta(\varepsilon),$$

which allows us, by using (4.26) and (4.27), to pass to the limit in (4.25). This completes the proof of Theorem.

Remark 4.1. *:The results obtained in Theorem* 3.1, *remains true if we replace (3.1) by the general growth condition*

$$|a(x, s, \xi)| \leq \overline{b}(s)(\overline{h}(x) + \overline{M}^{-1} M(k|\xi|))$$

where $k \geq 0, \overline{h} \in E_{\overline{M}}(\Omega)$ *and* $\overline{b} : I\!R_+ \to I\!R$ *is a nondecreasing continuous function.*

Remark 4.2. *: Note that we obtain the existence result without assuming the coercivity condition. However one can overcome this difficulty by introduced the function* $w_n = T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))$ *in the test function (4.9).*

Corollary 4.1. *Assume that the hypothesis (3.1)-(3.7) holds. Let f_n any sequence of function in $L^1(\Omega)$ that converges to f weakly in $L^1(\Omega)$ and let u_n the solution of the following unilateral problem*

$$(P'_n)\begin{cases} u_n \geq \psi \text{ a.e. in } \Omega. \\ u_n \in \tau_0^{1,M}(\Omega), \quad g(x,u_n,\nabla u_n) \in L^1(\Omega) \\ \displaystyle\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-v)\,dx + \int_\Omega g(x,u_n,\nabla u_n)T_k(u_n-v)\,dx \\ \qquad\qquad\qquad\qquad \leq \displaystyle\int_\Omega f_n T_k(u_n-v)\,dx, \\ \forall\ \ v \in K_\psi \cap L^\infty(\Omega),\ \forall k > 0. \end{cases}$$

Then, there exists a subsequence of u_n still denoted u_n such that u_n converges to u almost everywhere and $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $W_0^1L_M(\Omega)$. Further u is a solution of the unilateral problem (P).

Proof. We give the proof briefly.

Step 1. A priori estimates

We proceed as previous, we take $v = \psi^+$ as test function in (P'_n), we get

$$\int_\Omega M(|\nabla T_k(u_n)|/\delta)\,dx \leq C_1. \tag{4.30}$$

Hence, by the same method used in the first step in the proof of Theorem 3.1 there exists a function u (with $T_k(u) \in W_0^1L_M(\Omega)\ \forall k > 0$) and a subsequence still denoted by u_n such that

$$T_k(u_n) \rightharpoonup T_k(u), n \to \infty \text{ weakly in } \ W_0^1L_M(\Omega),\ \forall k > 0.$$

Step 2. Strong convergence of truncation

The choice of $v = T_h(u_n - \eta\phi(w_n)),\ h > \|\psi^+\|_\infty$ as test function in (P'_n), we get, for all $l > 0$

$$\begin{aligned} \int_\Omega a(x,u_n,\nabla u_n)&\nabla T_l(u_n - T_h(u_n-\eta\phi(w_n))) \\ &+\int_\Omega g(x,u_n,\nabla u_n)T_l(u_n - T_h(u_n-\eta\phi(w_n)))\,dx \\ &\leq \int_\Omega f_n T_l(u_n - T_h(u_n-\eta\phi(w_n)))\,dx. \end{aligned}$$

Which implies that

$$\int_{\{|u_n-\eta\phi(w_n)|\leq h\}} a(x,u_n,\nabla u_n)\nabla T_l(\eta\phi(w_n))\,dx$$
$$+\int_\Omega g(x,u_n,\nabla u_n)T_l(u_n-T_h(u_n-\eta\phi(w_n)))\,dx$$
$$\leq \int_\Omega f_nT_l(u_n-T_h(u_n-\eta\phi(w_n)))\,dx.$$

Letting h tends to infinity and choosing l large enough, we deduce

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla\phi(w_n)+\int_\Omega g(x,u_n,\nabla u_n)\phi(w_n)\,dx\leq\int_\Omega f_n\phi(w_n)\,dx.$$

The rest of the proof of this step is the same as in step 2 of the proof of Theorem 3.1.

Step 3. Passing to the limit

This step is similarly to the step 3 in the proof of Theorem 3.1, by using the Egorov's theorem in the last term of (P'_n).

Remark 4.3. *If the assumptions of Theorem* 3.1 *hold and* $f\geq 0$*, then* $u\geq 0$.

The use $v=T_h(u^+)$ as function test in (P), we obtain

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-T_h(u^+))\,dx$$
$$+\int_\Omega g(x,u,\nabla u)T_k(u-T_h(u^+))\,dx$$
$$\leq\int_\Omega fT_k(u-T_h(u^+))\,dx.$$

Since $g(x,u,\nabla u)T_k(u-T_h(u^+))\geq 0$, we deduce

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-T_h(u^+))\,dx\leq\int_\Omega fT_k(u-T_h(u^+))\,dx,$$

we remark also, using $f\geq 0$

$$\int_\Omega fT_k(u-T_h(u^+))\,dx\leq\int_{\{u\geq h\}} fT_k(u-T_h(u))\,dx.$$

On the other hand, by using (3.3), we conclude

$$\alpha \int_{\Omega} M(|\nabla T_k(u^-)/\delta|)\, dx \leq \int_{\{u \geq h\}} f T_k(u - T_h(u))\, dx.$$

Letting h tend to infinity, we conclude

$$\int_{\Omega} M(|\nabla T_k(u^-)/\delta|)\, dx = 0,$$

so that $\nabla T_k(u^-) = 0$ a.e. in Ω, which gives $T_k(u^-) = c$ a.e.. in Ω, where c is a real constant which depends on k. Since $T_k(u^-) \in W_0^1 L_M(\Omega)$, hence $T_k(u^-) = 0$ a.e. in Ω which gives $u^- = 0$ a.e. in Ω, implying that $u \geq 0$ a.e. in Ω.

Appendix I : Our hypotheses satisfies the conditions $(A_1) - (A_4)$ of Proposition 5 of [15], with $\varphi = b_\beta(x) = b(x) = 0$, and in (A_4) it suffices to impose with $\sum_{|\beta|=m}$ instead of $\sum_{|\beta|\leq m}$ in the right hand side (see Remark 6 of [15]).

Endeed :

In our paper we take, $(A_\beta(x,s,\xi))_{|\beta|=1} = a(x,s,\xi)$ and $A_0(x,s,\xi) = g_n(x,s,\xi)$.

In the one hand, since $g_n(x,s,\xi)$ is bounded and $c(x) \in E_{\overline{M}}$ it is clear our hypotheses verifies the conditions $(A_1) - (A_3)$ of proposition 5 [15]. On the other hand from (3.3), we have $a(x,s,\xi).\xi \geq \alpha M(\frac{|\xi|}{\delta})$ and since $g_n(x,s,\xi).s \geq 0$ (see (3.4)), then

$$\begin{aligned} a(x,s,\xi).\xi &\geq \alpha M(\tfrac{|\xi|}{\delta}) = \tfrac{\alpha}{N} N M(\tfrac{|\xi|}{\delta}) \\ &\geq \tfrac{\alpha}{N} \sum_{i=1}^{N} M(\frac{|\xi_i|}{\delta}). \end{aligned}$$

Which implies

$$(A_\beta(x,s,\xi))_{|\beta|=1}.\xi + A_0(x,s,\xi).s \geq \frac{\alpha}{N} \sum_{i=1}^{N} M(\frac{|\xi_i|}{\delta}).$$

Appendix II : We shall prove that $v_{n,h} \in K_\psi$.

It is clear that $v_{n,h} \in W_0^1 L_M(\Omega)$, then it remains to verified that $v_{n,h} \geq \psi$.

-If $w_n \leq 0$, we get $v_{n,h} \geq u_n \geq \psi$.

-If $w_n \geq 0$, we have $w_n^2 \leq 4k^2$ which implies that $w_n e^{\gamma w_n^2} \leq w_n e^{4\gamma k^2}$, hence $u_n - \eta w_n e^{\gamma w_n^2} \geq u_n - \eta w_n e^{4\gamma k^2} = u_n - w_n$.

Remark that $u_n - w_n \geq T_h(u_n) - T_k(u_n) + T_k(u_n)$, on the one hande $T_h(u_n) - T_k(u_n) + T_k(u_n) \geq \psi$. Endeed

$$
\begin{aligned}
T_h(u_n) - T_k(u_n) + T_k(u_n) &= \begin{cases} T_k(u) & \text{if} \quad |u_n| \leq k \\ u_n - k + T_k(u) & \text{if} \quad k \leq u_n \leq h \\ h - k + T_k(u) & \text{if} \quad u_n \geq h \\ u_n + k + T_k(u) & \text{if} \quad -h \leq u_n \leq -k \\ -h + k + T_k(u) & \text{if} \quad u_n \leq -h \end{cases} \\
&\geq \begin{cases} T_k(u) & \text{if} \quad |u_n| \leq k \\ T_k(u) & \text{if} \quad k \leq u_n \leq h \\ T_k(u) & \text{if} \quad u_n \geq h \\ u_n & \text{if} \quad -h \leq u_n \leq -k \\ -h \geq u_n & \text{if} \quad u_n \leq -h. \end{cases}
\end{aligned}
$$

Finally, since $k \geq \|\psi^+\|_\infty, u \geq \psi$ and $u_n \in K_\psi$, we deduce the result.

Appendix III : As in Appendix II, it is easy to see that the following function

$$
u_n - (T_{l+1}(u_n) - T_l(u_n)) = \begin{cases} u_n & \text{if} \quad |u_n| \leq l \\ l & \text{if} \quad l \leq u_n \leq l+1 \\ u_n - 1 \geq l & \text{if} \quad u_n \geq l+1 \\ u_n & \text{if} \quad -l-1 \leq u_n \leq -l \\ u_n + 1 \geq u_n & \text{if} \quad u_n \leq -l-1. \end{cases}
$$

belongs to K_ψ.

References

[1] R. Adams , *Sobolev espaces,*Academic Press, New York, (1975).

[2] P. Bénilan, L. Boccardo, T. Gallouet, R. Gariepy, M. Pierre and J. L. Vázquez, *An L^1-theory of existence and uniqueness of non-linear elliptic equations.,* Ann. Scuola Norm. Sup. Pisa **22**, pp. 240-273, (1995).

[3] A. Benkirane , *Approximation de type de Hedberg dans les espaces $W^m LlogL(\Omega)$ et application,* Ann. Fac. Sci. Toulouse. **11**, 4 , pp. 67-78, (1990).

[4] A. Benkirane and A. Elmahi, *An existence theorem for a strongly nonlinear elliptic problems in Orlicz spaces,* Nonlinear Anal. T. M. A., **36**, pp. 11-24, (1999).

[5] A. Benkirane and A. Elmahi, *A strongly nonlinear elliptic equation having natural growth terms and L^1 data,* Nonlinear Anal. T. M. A., **39**, pp. 403-411, (2000).

[6] A. Benkirane, A. Elmahi, and D. Meskine , *An existence theorem for a class of elliptic problems in L^1,* Applicationes Mathematicae., **29**, 4 , pp. 439-457, (2002).

[7] A. Benkirane and J. P. Gossez, *An approximation theorem for higher order Orlicz-Sobolev spaces,* Studia Math., **92**, pp. 231-255, (1989).

[8] G. Dalmaso, F. Murat, L. Orsina and A. Prignet, *Renormalized solutions of elliptic equations with general measure data,* Ann. Scuola Norm. Sup Pisa, Cl. Sci. 12, 4, pp. 741-808, (1997).

[9] A. Elmahi, D. Meskine, *Unilateral elliptic problems in L^1 with natural growth terms,* To appear Nonlinear and convex analysis.,

[10] M. Fuchs and L. Gongbao, *L^∞-bounds for elliptic equations on Orlicz-Sobolev spaces,* Archiv der Mathematik, **72**, pp. 293-297, (1999).

[11] M. Fuchs and G. Seregin, *Variational methods for fluids for Prandtl-Eyring type and plastic materials with logarithmic hardening,* Preprint No. 476. SFB256, Universität Bonn, Math. Methods Appl. Sci. in press.

[12] M. Fuchs and G. Seregin, *A reguralit y theory for variational integrals with LlnL-growth,* Calc. of Variations , **6**, pp. 171-187, (1998).

[13] M. Fuchs and G. Seregin, *Regurality for solutions of variational problems in the deformation theory of plasticity with logarithmic hardening,* Preprint No. 421, SFB256, Universität Bonn.

[14] J. P. Gossez, *Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients,* Trans. Amer. Math. Soc., **190**, pp. 163-205, (1974).

[15] J. P. GOSSEZ and V. MUSTONEN, *Variational inequalities in Orlicz-Sobolev spaces,* Nonlinear Anal. **11**, pp. 379-392, (1987).

[16] M. A. KRASNOSEL'SKII and Y. B. RUTIKII , *Convex functions and Orlicz spaces, Noordhoff Groningen,* (1969).

[17] A. PORRETTA , *Existence for elliptic equations in* L^1 *having lower order terms with natural growth,* Portugal. Math. **57**, pp. 179-190, (2000).

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