EXISTENCE OF SOLUTIONS FOR A DISCRETE NON LINEAR EIGENVALUE PROBLEM *

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Abstract

In this article we expose some existence results on the solutions of the discrete non linear boundary value problem derived from Fisher's continuous partial differential equations in steady state.

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1. Introduction

The aim of this work is to establish the existence of solutions for the discrete non linear boundary value problem

$$(1.1) \qquad \triangle^2 u_{k-1} + \lambda u_k(1 - u_k) = 0 \ , \ k = 1, 2, \dots, n$$

$$(1.2) \qquad u_0 = u_{n+1} = 0$$

by means of the method of monotone solutions, where λ is a real parameter.

This problem derives from Fisher's partial differential equation

$$(1.3) \qquad \frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + Ku(1 - u) \quad D, K > 0$$

which is broadly well-known and extensively studied by many authors, since it is a basic reaction-diffusion equation that can be used as a model of populational dynamics with migration or as a model of distribution of temperature in chemical reactions.

The associated steady state equation

$$(1.4) \qquad D\frac{\partial^2 u}{\partial x^2} + Ku(1 - u) = 0$$

is also a study object since its solutions give information about the limit behavior of Fisher's equation.

The continuous form of these equations, as model for some of the mentioned phenomena, is not always possible or the most appropriate thing to be solved because the nature of many of these phenomena is, in fact, discrete. For example, the concentration of big populations in cities, or the births and seasonal deaths of a species are problems of discrete nature.

As a result, the discrete form of Fisher's equation, particularly that associated to the steady state equation(1.4), that is to say, the equations (1.1)-(1.2), is of special interest.

If we denote by $\triangle u_{k-1} = u_k - u_{k-1}$ and by $\triangle^2 u_{k-1} = u_{k+1} - 2u_k + u_{k-1}$, the problem (1.1)-(1.2) is written in the equivalent form

$$(1.5) \qquad u_{k+1} - 2u_k + u_{k-1} + \lambda u_k(1 - u_k) = 0 \ , \ k = 1, 2, \dots, n$$

$$(1.6) \qquad u_0 = u_{n+1} = 0$$

2. A first result

Motivated by the results we will show that the discrete boundary value problem (1.1)-(1.2) has a solution, a finite real sequence $\{u_i\}_{i=0}^{n+1}$ satisfying those equations. We will start from $n = 1, 2, 3$.

When $n = 1$ the problem reduces to the polynomial equation

$$P(\lambda, u_1) = \lambda u_1^2 + (\lambda - 2)u_1 = 0$$

$$u_0 = u_2 = 0$$

whose solutions are

$$u_1^{(1)} = 0, \ u_1^{(2)} = \frac{\lambda - 2}{\lambda}$$

so the non trivial solutions is $u = \{0, (\lambda - 2)/\lambda, 0\}, \ \lambda \in \mathbf{R} - \{0, 2\}$.

It is easy to observe that for $\lambda > 0$ it is concluded that $u_1 \leq 1$, and for $\lambda < 0$ it is $u_1 > 0$.
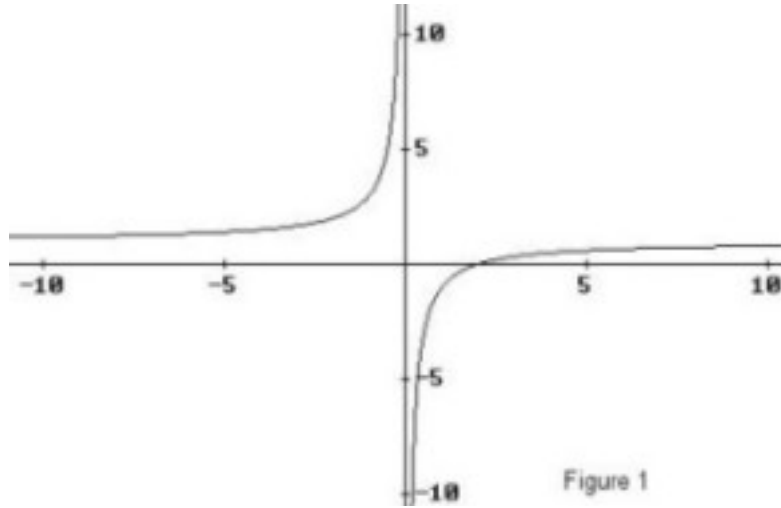

Figure 1

When $n = 2$ we obtain

$$u_2 + (\lambda - 2)u_1 - \lambda u_1^2 = 0$$

$$(\lambda - 2)u_2 + u_1 - \lambda u_2^2 = 0$$

$$u_0 = u_3 = 0$$

From the first equation we obtain $u_2 = (2 - \lambda)u_1 + \lambda u_1^2 = 0$ and, if we make their substitution in the second one, we have the following polynomial

in u_1 with coefficients depending on λ.

$$P(\lambda, u_1) = -\lambda^3 u_1^4 + (2\lambda^3 - 4\lambda^2)u_1^3 - (\lambda^3 - 5\lambda^2 + 6\lambda)u_1^2 - (\lambda^2 - 4\lambda + 3)u_1 = 0$$

From the factoring

$$-u_1\left[\lambda(u_1 - 1) + 1\right]\left[\lambda^2 u_1(u_1 - 1) + \lambda(3u_1 - 1) + 3\right] = 0$$

we find that the solutions for u_1 are

$$u_1^{(1)} = 0 \ , \ u_1^{(2)} = 1 - \tfrac{1}{\lambda}, \ u_1^{(3)} = \tfrac{1}{2} - \tfrac{3}{2\lambda} + \tfrac{\sqrt{(\lambda-3)(\lambda+1)}}{2\lambda} \ ,$$

$$u_1^{(4)} = \tfrac{1}{2} - \tfrac{3}{2\lambda} - \tfrac{\sqrt{(\lambda-3)(\lambda+1)}}{2\lambda}$$

therefore, the non trivial solutions for the problem are the following

$$\mathrm{u}^{(1)} = \left\{0, 1 - \tfrac{1}{\lambda}, 1 - \tfrac{1}{\lambda}, 0\right\}$$

$$u^{(2)} = \left\{0, \tfrac{1}{2} - \tfrac{3}{2\lambda} - \tfrac{\sqrt{(\lambda-3)(\lambda+1)}}{2\lambda}, \tfrac{1}{2} - \tfrac{3}{2\lambda} - \tfrac{\sqrt{(\lambda-3)(\lambda+1)}}{2\lambda}, 0\right\}$$

$$u^{(3)} = \left\{0, \tfrac{1}{2} - \tfrac{3}{2\lambda} + \tfrac{\sqrt{(\lambda-3)(\lambda+1)}}{2\lambda}, \tfrac{1}{2} - \tfrac{3}{2\lambda} + \tfrac{\sqrt{(\lambda-3)(\lambda+1)}}{2\lambda}, 0\right\}$$

$\lambda \in \mathbf{R} - \{0, 1\}$.

If we do the same for $n = 3$, we obtain a polynomial of degree seven in u_1 with coefficients depending on λ

$$P(\lambda, u_1) = -\lambda^7 u_1^8 + 4\lambda^6(\lambda - 2)u_1^7 + 2\lambda^5(2 - \lambda)(3\lambda - 7)u_1^6 +$$

$$2\lambda^4(2\lambda^3 - 15\lambda^2 + 36\lambda - 27)u_1^5 - \lambda^3(\lambda^4 - 14\lambda^3 + 61\lambda^2 - 105\lambda +$$

$$62)u_1^4 + 2\lambda^2(2 - \lambda)(\lambda^3 - 7\lambda^2 + 16\lambda + -11)u_1^3 - \lambda^2(\lambda^4 - 9\lambda^3 +$$

$$29\lambda^2 - 40\lambda + 20)u_1^2 + (\lambda^3 - 6\lambda^2 + 10\lambda - 4)u_1,$$

These examples allow us to conjecture that, except for a finite number of λ, it is always possible to find a solution for (1.1)-(1.2). Indeed, inductively we can obtain for each $n \in \mathbf{N}$ a polynomial of degree 2^n in u_1

$$P(\lambda, u_1) = -\lambda^{2^n - 1} u_1^{2^n} + q_{2^n - 1}(\lambda)u_1^{2^n - 1} + \cdots + q_1(\lambda)u_1$$

where the coefficients $q_k(\lambda)$ are at most polynomials of degree $2^n - 1$ in λ.

Clearly this polynomial can be factored in u_1: $P(\lambda, u_1) = u_1 Q(\lambda, u_1)$. The solution $u_1 = 0$ for $P(\lambda, u_1) = 0$ gives the trivial solution $u = 0$ for (1.1)-(1.2). Being $Q(\lambda, u_1)$ of odd degree in u_1 for all but a finite number of $\lambda \in \mathbf{R}$, we see that there exists a non trivial solution u_1 of $P(\lambda, u_1) = 0$ and then, a solution of (1.1)-(1.2). From here we deduce the following result

Proposition 1 : For all but a finite number of non null real number λ there exists at least one non trivial solution and at most $2^n - 1$ non trivial solutions of (1.1)-(1.2).

Although this method has allowed us to establish a result of existence of solutions for our problem, it doesn't seem easy to obtain more information about solutions.

Below is the detailed application of the monotone solutions method

3. Some results of existence of solutions

Non-linear discrete boundary value problems such as

(3.1) $$\triangle^2 u_{k-1} + f(k, u_k) = 0 \ , \ k = 1, 2, \ldots, n$$

(3.2) $$u_0 = u_{n+1} = 0$$

have been studied by several authors[1-4], by means of techniques such as the method of *a priori estimates*, by theorems of contractive applications, Brouwer's fixed point theorem, the perturbation method and the method of monotone solutions.

In this work we will use the method of monotone solutions developed by [4], from where we will point out some concepts and results.

Definition 2 : If $v = \{v_1, v_2, \ldots, v_m\}$, $w = \{w_1, w_2, \ldots, w_m\}$ are finite sequences, it is said that v is less or equal than w (and we write $v \leq w$) if $v_i \leq w_i, \ \forall i \in \{1, 2, \ldots, m\}$.

Thus, a solution of the boundary value problem (3.1)-(3.2) is a real sequence $\{u_0, u_1, \ldots, u_{n+1}\}$ satisfying the equations (3.1) and the conditions (3.2).

Definition 3 : The real sequence $w = \{w_0, w_1, \ldots, w_{n+1}\}$ is said to be an upper solution of (3.1)-(3.2) if

1. $\triangle^2 w_{k-1} + f(k, w_k) \leq 0 \ , \ k = 1, 2, \ldots, n$
2. $w_0 \geq 0, \quad w_{n+1} \geq 0$

Similarly, the real sequence $v = \{v_0, v_1, \ldots, v_{n+1}\}$ is said to be a *lower solution* of (3.1)-(3.2) if:

1. $\triangle^2 v_{k-1} + f(k, v_k) \geq 0 \ , \ k = 1, 2, \ldots, n$
2. $v_0 \leq 0 \ , \ v_{n+1} \leq 0$

The following results from [4] assure the existence of solution for the boundary value problem (3.1)-(3.2).

Theorem 4 : Let $f(k, \cdot)$ be a continuous function on $\mathbf{R}$ for each $k = 0, 1, \ldots, n + 1$; suppose that $v = (v_0, v_1, \ldots, v_{n+1})$ is a lower solution and $w = (w_0, w_1, \ldots, w_{n+1})$ is an upper solution of the problem (3.1)-(3.2) satisfying $v \leq w$. Then there exist a solution $u = (u_0, u_1, \ldots, u_{n+1})$ of (3.1)-(3.2) such that $v \leq u \leq w$.

The existence results mentioned below are based on the methods of monotone solutions (S.S.Cheng, [4]) for (3.1)-(3.2).

Theorem 5 : For $n \geq 1$ the discrete boundary problem

$$\triangle^2 u_{k-1} + \lambda u_k(1 - u_k) = 0, \quad k = 1, 2, \ldots, n$$

$$u_0 = u_{n+1} = 0$$

has a solution for all but a finite number of $\lambda \in]1, +\infty[$

Proof : Let us notice that $f(k, u_k) = \lambda u_k(1 - u_k)$ is a continuous function for all $\lambda, u_k \in \mathbf{R}$. On the other hand, the sequence w defined by

$$w_k = 1 , \quad k = 1, 2, \ldots, n$$
$$w_0 = w_{n+1} = 0$$

is an upper solution of (1.1)-(1.2) in fact,

$$w_2 - 2w_1 + \lambda w_1(1 - w_1) = -1 \leq 0$$

$$w_{k+1} - 2w_k + w_{k-1} + \lambda w_k(1 - w_k) = 0 \leq 0, \quad \text{for} \quad k = 2, 3, \ldots, n - 1$$

$$-2w_n + w_{n-1} + \lambda w_n(1 - w_n) = -1 \leq 0$$

so that $\quad^2 w_{k-1} + \lambda w_k(1 - w_k) \leq 0 \quad, k = 1, 2, \ldots, n$ Similarly, the sequence
$$w_0 \geq 0 , w_{n+1} \geq 0$$
v defined by

$$v_k = \frac{\lambda - 1}{\lambda} , \quad k = 1, 2, \ldots, n$$
$$v_0 = v_{n+1} = 0$$

is a lower solution in fact,

$$v_2 - 2v_1 + \lambda v_1(1 - v_1) = 0 \geq 0$$

$$v_{k+1} - 2v_k + v_{k-1} + \lambda v_k(1 - v_k) = \frac{\lambda - 1}{\lambda} \geq 0, \text{ for } k = 2, 3, ..., n - 1$$

$$-2v_n + v_{n-1} + \lambda v_n(1 - v_n) = 0 \geq 0$$

Therefore, $\quad {}^2 v_{k-1} + \lambda v_k(1 - v_k) \geq 0 \quad , k = 1, 2, \ldots, n$

$$w_0 \leq 0 , w_{n+1} \leq 0$$

Finally, from

$$0 < \frac{\lambda - 1}{\lambda} \leq 1 \text{ for } \lambda > 1$$

we deduce that $v \leq w$ and from Theorem 3, it is verified that $v \leq u \leq w$, so we conclude that u is a solution.

Corollary 6 : If $\{u_k\}_{k=0}^{n+1}$ is a solution of (1.1)-(1.2) for some $\lambda \in]1, +\infty[$, then

$$n(1 - 1/\lambda) \leq \sum_{k=1}^{n} u_k \leq n$$

Proof : From Theorem 5 we have $(1 - 1/\lambda) \leq u_k \leq 1$ and therefore, the announced inequality.

Theorem 7 : The problem (1.1)-(1.2) has a solution for all but a finite number of $\lambda \in]0, 1]$.

Proof : Let us consider the upper solution
$$w_k = 1, \quad k = 1, 2, \ldots, n$$
$$w_0 = w_{n+1} = 0$$
for $\lambda > 0$.

We claim that the sequence

$$v_k = v_{n+1-k} = \frac{\lambda k}{n}, \quad 1 \leq k \leq \frac{n}{2}$$

if n is even or

$$1 \leq k \leq \left[\frac{n}{2}\right] + 1$$

if n is odd

$$v_0 = v_{n+1} = 0$$

is a lower solution in fact,

$$v_2 - 2v_1 + \lambda v_1(1 - v_1) = \frac{2\lambda}{n} - 2\frac{\lambda}{n} + \lambda\frac{\lambda}{n}\left(1 - \frac{\lambda}{n}\right) = \frac{\lambda^2}{n}\left(1 - \frac{\lambda}{n}\right) \geq 0$$

$$v_{k+1} - 2v_k + v_{k-1} + \lambda v_k(1 - v_k) = \frac{k\lambda^2}{n}\left(1 - \frac{k\lambda}{n}\right) \geq 0,$$

$$\text{if } 1 \leq k \leq \frac{n}{2} \ (n \text{ even}) \ \text{ or } 1 \leq k \leq \left[\frac{n}{2}\right] + 1 \ (n \text{ odd})$$

for λ being in the interval $]0, 1]$.

We obtain
$$\triangle^2 w_{k-1} + \lambda w_k(1 - w_k) \geq 0 \ , \ k = 1, 2, \ldots, n$$
$$w_0 \leq 0 \ , \ w_{n+1} \leq 0$$

Finally, from

$$0 < \frac{\lambda k}{n} \leq 1 \quad \text{for } 0 < \lambda \leq 1 \ , \qquad 1 \leq k \leq \frac{n}{2} \ (n \text{ even})$$
$$\text{or } 1 \leq k \leq \left[\frac{n}{2}\right] + 1 \ (n \text{ odd})$$

we deduce that $v \leq w$ and so, from Theorem 4 there exist a solution u of (1.1)-(1.2) satisfying $v \leq u \leq w$.

Corollary 8 : The discrete boundary value problem (1.1)-(1.2) has at least a positive solution for $\lambda > 0$.

Proof : This is a direct consequence from the previous results.

Theorem 9 : The discrete boundary value problem (1.1)-(1.2) has at least a solution for all but a finite number of $\lambda < 0$.

Proof : Let us $\lambda < 0$ and let us put $\lambda' = -\lambda > 0$. From previous results there exists a non trivial solution $u = (0, u_1, u_2, \cdots, u_n, 0)$. of (1.1)-(1.2) for λ'.

Let us consider the sequence $z = (0, z_1, z_2, \ldots, z_n, 0)$, defined by $z_k = 1 - u_k$, $k = 1, \ldots, n$. We claim that z is a (non trivial) solution of (1.1)-(1.2) for λ.

In fact,

$z_{k+1} - 2z_k + z_{k-1} + \lambda z_k (1 - z_k) = -[u_{k+1} - 2u_k + u_{k-1} + (-\lambda) u_k (1 - u_k)] = 0$, for $k = 1, \ldots, n$

Then, z is a solution of the problem for negative λ.

Corollary 10 : If $\{u_k\}_{k=0}^{n+1}$ is a solution of (1.1)-(1.2) for $\lambda < 0$, then $0 \leq u_k \leq -1/k \leq 1 - 1/k$, $k = 0, 1, \ldots, n + 1$

Proof : If u is a solution of (1.1)-(1.2) for $\lambda < 0$, then $z = (0, z_1, \ldots, z_n, 0)$, $z_k = 1 - u_k$, $1 \leq k \leq n$, is also a solution but for $-\lambda$.

As in Corollary 6 we have

$$(1 + 1/\lambda) \leq z_k \leq 1, \quad 1 \leq k \leq n$$

Therefore,

$$0 \leq u_k \leq -1/\lambda \leq 1 - 1/\lambda, \quad k = 1, \ldots, n$$

Proposition 11 : Let us suppose that $\{u_k\}_{k=0}^{n+1}$ is a nontrivial solution of (1.1)-(1.2) for $\lambda > 0$ and assume that $u_{\overline{k}} = 0$ for some $\overline{k}$, $2 \leq \overline{k} \leq n - 1$. Then $u_{\overline{k}-1}, u_{\overline{k}+1} \neq 0$ and $u_{\overline{k}-1} = -u_{\overline{k}+1}$.

Proof : If $u_{\overline{k}} = 0$ for some $\overline{k}$, $2 \leq \overline{k} \leq n - 1$, then, since u is a nontrivial solution of the boundary problem (1.1)-(1.2), we have $u_{\overline{k}-1} u_{\overline{k}+1} \neq 0$ (otherwise, if one of the factors was equal to zero then we would have $u = 0$). Furthermore, from (1.1) for $k = \overline{k}$ we have $u_{\overline{k}-1} + u_{\overline{k}+1} = 0$, therefore $u_{\overline{k}-1} = -u_{\overline{k}+1}$.

Remark 12 : Note that if $u_1 = 0$ or $u_n = 0$ in the previous proposition, then $u_k = 0$ for $k = 0, \ldots, n + 1$.

4. Conclusions

By means of computer experiments we have verified some facts about the existence results for the problem (1.1)-(1.2). First, there exists solutions for all but a finite number of $\lambda \in \mathbf{R}$, $\lambda \neq 0$. It is not easy to specify λ values where is not a solution since they depend on the value of n. In general, there exists symmetrical solutions: $u_k = u_{n+1-k}$ for $1 \leq k \leq n/2$, if n is even or $1 \leq k \leq [n/2] + 1$, if n is odd.

The following conjectures are of interest:

- For $\lambda < 0$ all the solutions are positives and for $\lambda > 0$ there exists at least a positive solution.

- For all but a finite number of $\lambda > 0$, all the solutions u of the problem (1.1)-(1.2) satisfy $|u_k| \leq 1$, $k = 0, 1, \ldots n + 1$.

- For all but a finite number of $\lambda \in [-1, 1]$, $\lambda \neq 0$, there exists one and only one non trivial solution of (1.1)-(1.2).

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