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COMPENSATORS FOR SINGULAR CONTROL SYSTEMS WITH DELAYS IN OUTPUTS

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Abstract

In this paper we study the design of dynamic compensators for linear singular control systems described by the equation $Ex'(t) = Ax(t) + Bu(t)$ *with time delayed observed output* $y(t) = Cx(t-r)$*. The proposed compensators are applied to solve the regulator problem for the mentioned systems with controlled output* $z(t) = Dx(t)$*.*

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1. Introduction

This paper is devoted to the dynamic compensation problem for finite-dimension singular systems.

Consider the linear invariant singular system

$$Ex'(t) = Ax(t) + Bu(t) \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$ denotes the state at time t of the system; $u(t) \in \mathbb{R}^m$, represent the input at time t and both E, $A \in \mathbb{R}^{n\times n}$ as $B \in \mathbb{R}^{n\times m}$ are constant matrices. When $E = I$ or, in general, E is an invertible matrix, the system (1.1) is called normal and singular (also called descriptor, semistate or generalized) in the case where $\det(E) = 0$.

Henceforth we will be concerned with a system (1.1) of singular type. In order to guarantee existence of solutions of equation (1.1), we will assume that the pencil (E, A) is regular in the following sense: there is $\alpha \in \mathbb{R}$ such that $\det(\alpha E - A) \neq 0$. These systems arise in the study of several control problems in science and technology. By this reason, in recent years it has been an increasing interest to study them. Readers are referred to Dai [7], Campbell [4, 5] and Favini and Yagi [15] as well as the references contained therein for the details.

On the other hand, the problem of stabilizing a linear invariant control system by a dynamic output feedback has a very extensive literature. At present the theory for normal control systems of finite dimension is well established and we refer to O'Reilly [34] and Wonham [49] for the most important part of the theory. The extension of these results to singular systems has attracted the attention of many authors last years. In particular, the pole assignment problem and the design of asymptotic observers and compensators for the system (1.1) has been considered in several works ([7, 13, 17, 29, 32, 36, 37, 38, 42, 50]). This work has been concentrated on systems with observed output given by

$$y(t) = Cx(t).$$

In these systems the observation is instantaneous. However in most of concrete systems their operation presents some time lag. The purpose of this note is to study the dynamic compensation for singular systems described by the equation (1.1) and having a time delay in the observed output. We will restrict us to consider a point delay in the observer variables. Thus, more specifically, our first objective is to determine a dynamic compensator for the system (1.1) with the output $y(t) \in I\!\!R^p$ at time t given by

$$y(t) = Cx(t-r), \tag{1.2}$$

where $C \in I\!\!R^{p\times n}$ is a time-invariant matrix and the constant $r > 0$ represents the time delay of the observation. After, we will apply this compensator to design a tracking controller for the system (1.1)-(1.2) with an appropriate controlled output. In a previous work ([22]) we have studied this problem in the case $r > 0$ small enough. Now we remove this condition by proposing a functional controller.

The theory of existence and uniqueness of solutions for the system (1.1) has been discussed by several authors (see Campbell [4] and Dai [7] and the references given therein). In particular, (1.1)-(1.2) is equivalent to the system defined by

$$x_1'(t) = A_1x_1(t) + B_1u(t), \tag{1.3}$$

$$Nx_2'(t) = x_2(t) + B_2u(t) \tag{1.4}$$

$$y(t) = C_1x_1(t-r) + C_2x_2(t-r) \tag{1.5}$$

where $x_1 \in I\!\!R^{n_1}$, $x_2 \in I\!\!R^{n_2}$, $n = n_1+n_2$ and the matrices A_1, B_1, B_2, C_1, C_2 and N have appropriate dimensions (see [7] for the terminology). Furthermore, the matrix N is nilpotent with index h. The system (1.3)-(1.4)-(1.5) is called standard form of (1.1)-(1.2) and is obtained applying a transformation of coordinates defined by invertible matrices P and Q so that

$$x = P\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \quad QEP = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}; \quad QAP = \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix}$$

$$QB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}; \qquad CP = [C_1, C_2].$$

The subsystems (1.3) and (1.4) are called slow and fast subsystem, respectively. Since (1.3) is a normal system, the state $x_1(t)$ can be obtained from the variation of constants formula. Thus,

$$x_1(t) = e^{A_1t}x_1(0) + \int_0^t e^{A_1(t-s)}B_1u(s)\,ds. \tag{1.6}$$

Furthermore, if $u(\cdot)$ is sufficiently smooth, the solution of (1.4) is given by

$$x_2(t) = -\sum_{i=0}^{h-1} N^iB_2u^{(i)}(t). \tag{1.7}$$

This paper is organized as follows. Section 2 is dedicated to the definition of the dynamic compensator. In section 3 we apply this compensator to the problem of tracking a signal with regulation of constant disturbances and in section 4 we extend the results of section 3 to include some non constant disturbances. Finally, we have included two appendixes with some technical results that will be needed in these sections.

2. Design of a dynamic compensator

In this section we shall be concerned with the design of an asymptotic compensator for singular systems with delayed observed output.

The problem of feedback stabilization of normal control systems with delays has been discussed in many works, employing different approaches. In particular, our purposes in this paper are related to the results obtained via the finite spectrum assignment method. Some authors have studied different aspects of the problem of stabilization for a fixed time delay (see [2, 6, 16, 28, 31, 33, 35, 40, 43, 44, 45, 46, 47, 41, 21]) while some others have considered the problem of stabilization independent of delays ([23, 30, 3, 11, 24, 25, 10, 12, 8, 48, 27, 14, 6]). On the other hand, some of these works are concentrated on the design of asymptotic observers with point delays (commensurate or noncommensurate) while some others consider distributed delays.

In this work we extend the approach of [28, 44, 21] to construct a dynamic compensator for a singular system with delayed observed output. We consider only a point delay and the proposed compensator will be a normal system with distributed delay.

To motivate our construction we begin by defining a retarded singular observer for the system (1.1)-(1.2).

We will say that the retarded singular system

$$Ez'(t) = Az(t) + GC\,z(t-r) - Gy(t) + Bu(t), \tag{2.1}$$

is a state observer of (1.1)-(1.2) if

$$\lim_{t\to\infty}(z(t) - x(t)) = 0.$$

Proposition 2.1. *Suppose that there exist a matrix G_1 such that the system*

$$w'(t) = A_1 w(t) + G_1 C_1\, w(t-r) \tag{2.2}$$

is asymptotically stable. Then there exists a matrix G such that the system (2.1) is a state observer of (1.1)-(1.2).

Proof. Let $e(t) := z(t) - x(t)$ be the estimation of the error. Then

$$Ee'(t) = Ae(t) + GC\,e(t-r). \tag{2.3}$$

Using the coordinate transformation to reduce the pencil (E, A) to the standard form, we obtain that (2.3) is equivalent to the equations

$$e_1'(t) = A_1 e_1(t) + Q_1 GCPe(t-r), \tag{2.4}$$

$$Ne_2'(t) = e_2(t) + Q_2 GCPe(t-r). \tag{2.5}$$

We can choose G so that QG has the block form $QG = \begin{bmatrix} G_1 \\ 0 \end{bmatrix}$.

Since $CP = [C_1, C_2]$, substituting these expressions into (2.4) and (2.5) we obtain

$$\begin{aligned} e_1'(t) &= A_1 e_1(t) + G_1 C_1 e_1(t-r) + G_1 C_2 e_2(t-r), \\ Ne_2'(t) &= e_2(t). \end{aligned}$$

From the last equation we infer that $e_2(t) = 0$, for every $t > 0$, which in turn implies that

$$e_1'(t) = A_1 e_1(t) + G_1 C_1 e_1(t-r), \quad t > r,$$

which completes the proof. ■

Related with this property we point out that applying a perturbation result of Halanay ([18], section 4.5) one can see that if (A_1, C_1) is detectable (see Wonham [49]) and $r > 0$ is enough small then there exists a matrix G such that the system (2.1) is a state observer of (1.1)-(1.2).

The observer proposed in (2.1) is a system of singular type with delays in the state. These systems have been studied by Campbell [4]. To solve (2.1), one must specify the initial function $z(\cdot)$ on $[-r, 0]$ and this function must satisfy some strong consistency conditions. Thus, this approach is simple but has the disadvantage that, in general, the solutions are not continuous functions. To avoid this bad behavior, next we will show that, under appropriated hypotheses, we can define observers of normal type. In fact, following Uetake [42] we can transform the system (1.1) in the form

$$\widetilde{E}\,[\frac{dx}{dt} - \mu x(t)] = x(t) + \widetilde{B}u(t) \tag{2.6}$$

where $\widetilde{B}$ and $\widetilde{E}$ are defined by

$$\widetilde{E} := (A - \mu E)^{-1} E, \quad \widetilde{B} := (A - \mu E)^{-1} B \tag{2.7}$$

and μ is a constant such that $\mu E - A$ is an invertible matrix. If the system (E, A, C) is observable then there exists a matrix G such that $L := \widetilde{E} - GC$ is invertible and the polynomial $\det((s-\mu)L - I)$ is stable (Uetake [42], Theorem 2). Now, we introduce the system

$$\psi'(t) = L^{-1}(I+\mu L)\psi(t) - L^{-2}Gy(t) + L^{-1}\widetilde{B}u(t-r), \tag{2.8}$$

$$\phi(t) = \psi(t) - L^{-1}Gy(t), \tag{2.9}$$

where $L^{-2} = (L^{-1})^2$. Then (2.8)-(2.9) is an asymptotic observer of the system (1.1)-(1.2). In fact, if we define the error variable $e(t) := \phi(t) - x(t-r)$, from (2.8)-(2.9) we obtain that

$$L\phi'(t) = (I+\mu L)\phi(t) + \mu Gy(t) + \widetilde{B}u(t-r) - Gy'(t) \tag{2.10}$$

and collecting this expression with (2.6) follows that

$$Le'(t) = (I+\mu L)e(t).$$

Thus

$$e'(t) = L^{-1}(I+\mu L)e(t) \tag{2.11}$$

which is a stable system ([42]).

As a first application of the asymptotic observer (2.8)-(2.9) we consider the design of a dynamic compensator for the system (1.1)-(1.2).

We begin by rewriting the equation for ϕ. Using the relation

$$\mu Gy(t) - Gy'(t) = \mu GC(\phi(t) - e(t)) - GC(\phi'(t) - e'(t))$$

from (2.10) we obtain that

$$\phi'(t) = L^{-1}(I+\mu L+\mu GC)\phi(t) - L^{-1}GC\phi'(t) + L^{-1}GC(e'(t) - \mu e(t)) + L^{-1}\widetilde{B}u(t-r)$$

so that

$$(L+GC)\phi'(t) = (I+\mu L+\mu GC)\phi(t) + GC(e'(t) - \mu e(t)) + \widetilde{B}u(t-r)$$

which in turn implies that

$$E\phi'(t) = A\phi(t) + (A-\mu E)GC(e'(t) - \mu e(t)) + Bu(t-r) \tag{2.12}$$

and, applying the transformation of coordinates defined by P and Q, the above expression yields

$$\phi_1'(t) = A_1\phi_1(t) + [A_1 - \mu I,\, 0]\, P^{-1}GC\,(e'(t) - \mu e(t)) +$$

$$B_1\, u(t-r). \tag{2.13}$$

and

$$N\phi_2'(t) = \phi_2(t) \ + \ [0,\, I-\mu N]\, P^{-1}GC\, (e'(t)-\mu e(t)) +$$
$$B_2\, u(t-r). \tag{2.14}$$

In the sequel we denote by $C([-r,0];\mathbb{R}^n)$ the space of continuous functions defined on $[-r,0]$ and values in $\mathbb{R}^n$. Moreover, for a continuous function $x:[-r,\infty)\to\mathbb{R}^n$ and $t\geq 0$ we indicate by $x_t:[-r,0]\to\mathbb{R}^n$, called the history of x at t, the function given by $x_t(\theta):=x(t+\theta),\ -r\leq\theta\leq 0$.

Following Olbrot [33] and Pandolfi [35] we consider a control law $u(\cdot)$ defined by the equation

$$u'(t) = K_1(\phi_{1,\,t}) \ + \ K_2(u_t), \tag{2.15}$$

where K_1 and K_2 are bounded linear operators from $C([-r,\,0],\,\mathbb{R}^{n_1})$ and $C([-r,\,0],\,\mathbb{R}^{m})$, respectively, into $\mathbb{R}^m$.

We observe that (2.13)-(2.14)-(2.15) is a singular system with delays in the control variables but not a retarded singular system as (2.1) because equations (2.13)-(2.14)-(2.15) do not depend on $\phi_2(t-r)$ or, in general, on the history $\phi_{2,t}$.

Now we are able to establish the following result.

Theorem 2.1. *Assume that the system $(E,\,A,\,C)$ is observable and that the pair (A_1,B_1) is controllable. Then there exists a matrix G and there exist operators K_1 and K_2 such that the dynamical system (2.8)-(2.9)-(2.15) is an asymptotic compensator of (1.1)-(1.2).*

Proof. We define the delay control system

$$z'(t) = A_1 z(t) \ + \ B_1 u(t-r). \tag{2.16}$$

The controllability of (A_1,B_1) and the results of Olbrot [33] and Pandolfi [35] imply the existence of a control law $u(\cdot)$ defined by

$$u'(t) = K_1(z_t) \ + \ K_2(u_t) \tag{2.17}$$

such that $u(t)$ and $z(t)$ are exponentially convergent to zero as $t\to\infty$. Next we will prove that this selection of operators K_1 and K_2 turns the dynamical system (2.8)-(2.9)-(2.15) into an asymptotic compensator of (1.1)-(1.2). It is clear that the system (2.16)-(2.17) can be represented as

$$w'(t) = \Lambda(w_t) \tag{2.18}$$

where $w(t) := \begin{bmatrix} z(t) \\ u(t) \end{bmatrix}$ and $\Lambda : C([-r,0]; \mathbb{R}^{n_1} \times \mathbb{R}^m) \to \mathbb{R}^{n_1} \times \mathbb{R}^m$ is the operator defined by

$$\Lambda \begin{pmatrix} \varphi \\ \psi \end{pmatrix} := \begin{bmatrix} A_1\varphi(0) + B_1\psi(-r) \\ K_1(\varphi) + K_2(\psi) \end{bmatrix}.$$

Using this notation and the fact that $e'(t) - \mu e(t) = L^{-1}e(t)$, the system (2.13)-(2.15) can be rewritten as

$$w'(t) = \Lambda(w_t) \; + \; f(t) \tag{2.19}$$

where $w(t) := \begin{bmatrix} \phi_1(t) \\ u(t) \end{bmatrix}$ and $f(t) := \begin{bmatrix} [A_1 - \mu I,\, 0]\, P^{-1}GC\, L^{-1}e(t) \\ 0 \end{bmatrix}$.

Choosing G so that $L^{-1}(I + \mu L)$ is a stable matrix, from (2.11) and Appendix A we obtain that $e^{(i)}(t)$, $i = 0, 1, \cdots, h-1$, converge exponentially to zero as $t \to \infty$ so that f has the same property. Now we apply Proposition A.1 to compare the stability properties of systems (2.18) and (2.19). We infer that both $\phi_1(t)$ as $u(t)$, as well as their derivatives $\phi_1^{(i)}(t)$ and $u^{(i)}(t)$, $i = 1, 2, \cdots, h-1$, converge exponentially to zero as $t \to \infty$. Finally, from (2.14) and (1.7) we conclude that $\phi_2(t)$ also converges exponentially to zero as $t \to \infty$, which completes the proof. ■

3. Tracking and regulation of constant disturbances

Now we will employ our previous results to design a controller which regulates (i.e. remove the dependence on disturbances) and tracks (i. e. gets the control variables to follow a reference signal) the system. Usually, we will abbreviate our terminology saying that a controller with these properties is a regulator or that regulates the given system. In this section we only consider constant disturbances. Specifically we consider a control system

$$Ex'(t) = Ax(t) + Bu(t) + w, \tag{3.1}$$

$$y(t) = Cx(t-r), \tag{3.2}$$

$$z(t) = Dx(t) \tag{3.3}$$

where $z(t) \in \mathbb{R}^q$ denotes the controlled output, w designates an unknown constant which represents an external disturbance and $D \in \mathbb{R}^{q\times n}$ is a matrix. Our objective is to design a dynamic compensator for which the resulting closed-loop system with the perturbation $w = 0$ will be exponentially stable and the controlled output $z(t)$ will be regulated to a reference signal $\overline{z}$. Moreover, this property would occur for all w in a given class of perturbations. In this section we discuss this problem for constant perturbations.

Proceeding as in section 2, equation (3.1) can be expressed as

$$\widetilde{E}\,[\frac{dx}{dt} - \mu x(t)] = x(t) \ + \ \widetilde{B}u(t) \ + \ \widetilde{w} \tag{3.4}$$

where

$$\widetilde{w} := (A - \mu E)^{-1} w. \tag{3.5}$$

Next we shall show that, under certain conditions, we can use the previous construction, to define a PI feedback control to regulate the system. To this end, we introduce a new variable $\xi(t)$ defined by

$$\xi\,'(t) = D\phi(t) \ - M\ \psi\,'(t) \ - \ \overline{z}, \tag{3.6}$$

where M is an appropriate matrix to be determined.

Let ϕ_1 and ϕ_2 be the components of ϕ corresponding to the transformation $\phi := P \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$. Next we will show that the pair (ξ, ϕ_1) can be considered as the state variable of a normal control system with delays in the control variables so that the corresponding non delayed system is controllable.

Substituting $\psi\,'$ given by (2.8) into (3.6) and using both (2.9) and the definition of $e(t)$ we obtain that

$$\begin{aligned} \xi\,'(t) \ &= \ \Big(D - M\,L^{-1}(I + \mu L) \ - \ \mu M L^{-1} G C\Big)\,\phi(t) - \ M L^{-1}\widetilde{B}u(t - r) \\ &\quad + \mu M L^{-1} G C e(t) \ - \ \overline{z}. \end{aligned} \tag{3.7}$$

In order to obtain a normal system we need to avoid the dependence of the right hand side of (3.7) on ϕ_2. To achieve this objective we choose the matrix M as follows. If we put

$$P^{-1} L P := \begin{bmatrix} L_{1,1} & L_{1,2} \\ L_{2,1} & L_{2,2} \end{bmatrix},$$

then we can select the matrix G so that $L_{1,1}$ will be an invertible matrix (see Lemma B.1 in the appendix B). Hence, if T stands for the inverse of $P^{-1}LP$, then we can write

$$T := P^{-1}L^{-1}P := \begin{bmatrix} T_{1,1} & T_{1,2} \\ T_{2,1} & T_{2,2} \end{bmatrix}$$

where the block $T_{2,2}$ is also invertible. Since

$$\begin{aligned} I \,+\, \mu L \,+\, \mu GC &= I \,+\, \mu\widetilde{E} \\ &= (A - \mu E)^{-1}A, \end{aligned}$$

then

$$\begin{aligned} DP \,-\, ML^{-1}\,(I + \mu L \,+\, \mu GC)\,P &= DP \,-\, ML^{-1}(A - \mu E)^{-1}AP \\ &= DP \,-\, MPT\,\left(Q(A - \mu E)P\right)^{-1}\,QAP \\ &= [D_1,\, D_2] \\ &\quad -[M_1,\, M_2] \\ &\quad\; T \begin{bmatrix} (A_1 - \mu I)^{-1}A_1 & 0 \\ 0 & (I - \mu N)^{-1} \end{bmatrix} \\ &:= [\Omega_1,\, \Omega_2], \end{aligned}$$

where we have introduced the notations $DP := [D_1,\, D_2]$, $MP := [M_1,\, M_2]$ and

$$\begin{aligned} \Omega_1 &:= D_1 \,-\, M_1\,T_{1,1}\,(A_1 - \mu I)^{-1}A_1 \,-\, M_2\,T_{2,1}\,(A_1 - \mu I)^{-1}A_1 \\ \Omega_2 &:= D_2 \,-\, M_1\,T_{1,2}\,(I - \mu N)^{-1} \,-\, M_2\,T_{2,2}\,(I - \mu N)^{-1}. \end{aligned}$$

As $T_{2,2}$ is invertible we can choose M_2 so that $\Omega_2 = 0$. Consequently, henceforth we will assume that

$$D_2 \,-\, M_1\,T_{1,2}\,(I - \mu N)^{-1} \,-\, M_2\,T_{2,2}\,(I - \mu N)^{-1} = 0. \tag{3.8}$$

Substituting these expressions into (3.7) we obtain that

$$\xi\,'(t) = \Omega_1\,\phi_1(t) - \; ML^{-1}\widetilde{B}u(t - r) \;+\; \mu ML^{-1}GCe(t) \;-\; \overline{z}. \tag{3.9}$$

On the other hand, it is easy to see that the estimate of the error $e(t)$ satisfies

$$e\,'(t) = L^{-1}(I + \mu L)e(t) \;-\; L^{-1}\widetilde{\omega} \tag{3.10}$$

so that, in this case,

$$e'(t) - \mu e(t) = L^{-1}(e(t) - \widetilde{\omega}). \tag{3.11}$$

It is clear from this relation that equations (3.9) and (2.13) can be written as

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} \xi(t) \\ \phi_1(t) \end{bmatrix} &= \begin{bmatrix} 0 & \Omega_1 \\ 0 & A_1 \end{bmatrix}\begin{bmatrix} \xi(t) \\ \phi_1(t) \end{bmatrix} \\ &+ \begin{bmatrix} -ML^{-1}\widetilde{B} \\ B_1 \end{bmatrix} u(t-r) \ + \ f(t) \end{aligned} \tag{3.12}$$

where

$$f(t) := \begin{bmatrix} \mu\, ML^{-1}GCe(t) \ - \ \overline{z} \\ [A_1 - \mu I,\, 0]P^{-1}GCL^{-1}(e(t) - \widetilde{\omega}) \end{bmatrix}.$$

In order to study the stability of system (3.12) we introduce the following control system, with delayed control action,

$$\frac{d}{dt}\begin{bmatrix} \xi(t) \\ \phi_1(t) \end{bmatrix} = \begin{bmatrix} 0 & \Omega_1 \\ 0 & A_1 \end{bmatrix}\begin{bmatrix} \xi(t) \\ \phi_1(t) \end{bmatrix} + \begin{bmatrix} -ML^{-1}\widetilde{B} \\ B_1 \end{bmatrix} v(t-r), \tag{3.13}$$

with state space $\mathbb{R}^q \times \mathbb{R}^{n_1}$ and control space $\mathbb{R}^m$. Next we denote by S the $(q+n_1)\times(m+n_1)$ matrix

$$S := \begin{bmatrix} D_1 & D_2B_2 \\ -A_1 & B_1 \end{bmatrix}. \tag{3.14}$$

The following statement formalizes our assertion about the variables $\xi(t)$ and $\phi_1(t)$.

Lemma 3.1. *If the pair* (A_1, B_1) *is controllable and the rank of* S *is* $q+n_1$, *then the system*

$$\frac{d}{dt}\begin{bmatrix} \xi(t) \\ \phi_1(t) \end{bmatrix} = \begin{bmatrix} 0 & \Omega_1 \\ 0 & A_1 \end{bmatrix}\begin{bmatrix} \xi(t) \\ \phi_1(t) \end{bmatrix} + \begin{bmatrix} -ML^{-1}\widetilde{B} \\ B_1 \end{bmatrix} v(t) \tag{3.15}$$

also is controllable.

Proof. This property is consequence of the Hautus test ([20]). In fact, if we use ρ to denote the rank of a matrix, since $\rho[\lambda I - A_1,\, B_1] = n_1$, for every $\lambda \in \mathbb{C}$, then

$$\begin{aligned} \rho\left[\lambda I - \begin{bmatrix} 0 & \Omega_1 \\ 0 & A_1 \end{bmatrix}, \begin{bmatrix} -ML^{-1}\widetilde{B} \\ B_1 \end{bmatrix}\right] &= \rho\begin{bmatrix} \lambda I & -\Omega_1 & -ML^{-1}\widetilde{B} \\ 0 & \lambda I - A_1 & B_1 \end{bmatrix} \\ &= q + n_1, \end{aligned}$$

for all $\lambda \neq 0$. On the other hand, in order to determine the rank in the case $\lambda = 0$ we begin by observing that

$$
\begin{aligned}
ML^{-1}\widetilde{B} &= [\,M_1,\, M_2\,]\, TP^{-1}\widetilde{B} \\
&= [\,M_1,\, M_2\,]\, TP^{-1}(A-\mu E)^{-1}B \\
&= [\,M_1,\, M_2\,]\, T\,(Q(A-\mu E)P)^{-1}\,QB \\
&= M_1\,\Big(T_{1,1}(A_1-\mu I)^{-1}B_1 + T_{1,2}(I-\mu N)^{-1}B_2\Big) \\
&\quad + M_2\,\Big(T_{2,1}(A_1-\mu I)^{-1}B_1 + T_{2,2}(I-\mu N)^{-1}B_2\Big)\,.
\end{aligned}
$$

This expression and the definition of Ω_1 allow us to write

$$
\begin{bmatrix} \Omega_1 & ML^{-1}\widetilde{B} \\ -A_1 & B_1 \end{bmatrix}
= \begin{bmatrix} I & (M_1T_{1,1}+M_2T_{2,1})(A_1-\mu I)^{-1} \\ 0 & I \end{bmatrix}
\begin{bmatrix} D_1 & (M_1T_{1,2}+M_2T_{2,2})(I-\mu N)^{-1}B_2 \\ -A_1 & B_1 \end{bmatrix}
$$

and combining this equality with (3.8) and applying our hypothesis about the rank of S, it follows that

$$
\begin{aligned}
\rho\begin{bmatrix} \Omega_1 & ML^{-1}\widetilde{B} \\ -A_1 & B_1 \end{bmatrix}
&= \rho\begin{bmatrix} D_1 & (M_1T_{1,2}+M_2T_{2,2})(I-\mu N)^{-1}B_2 \\ -A_1 & B_1 \end{bmatrix} \\
&= \rho\begin{bmatrix} D_1 & D_2B_2 \\ -A_1 & B_1 \end{bmatrix} \\
&= q + n_1,
\end{aligned}
$$

which completes the proof. ■

We close the system (3.12) by introducing a PI control law defined by

$$u'(t) = K_0(\xi_t) \;+\; K_1(\phi_{1,\,t}) \;+\; K_2(u_t) \tag{3.16}$$

where K_0, K_1 and K_2 are bounded linear operators from $C([-r,\,0],\ \boldsymbol{R}^q)$, $C([-r,\,0],\ \boldsymbol{R}^{n_1})$ and $C([-r,\,0], \boldsymbol{R}^m)$, respectively, into $\boldsymbol{R}^m$.

We can establish the following result. We refer to appendix A for the stability concepts.

Proposition 3.1. *Assume that* $(E,\,A,\,C)$ *is observable,* $(A_1,\,B_1)$ *is controllable and the rank of* S *is* $q + n_1$. *Then there exists a matrix* G *and there exist* *operators K_0, K_1 and K_2 such that the dynamical system (2.8)-(2.9)-(3.6) with the control law (3.16) is asymptotically stable and $\lim_{t\to\infty} D\phi(t) = \bar{z}$. If we assume in addition that the disturbance $w = 0$, then the system (2.8)-(2.9)-(3.6) with the control law (3.16) regulates the control system (3.1)-(3.2)-(3.3).*

Proof. We construct G as before. Now, proceeding as in the proof of Theorem 2.1 we introduce the auxiliary system

$$\alpha'(t) = \begin{bmatrix} 0 & \Omega_1 \\ 0 & A_1 \end{bmatrix} \alpha(t) + \begin{bmatrix} -ML^{-1}\widetilde{B} \\ B_1 \end{bmatrix} u(t-r) \tag{3.17}$$

controlled by

$$u'(t) = K(\alpha_t) \ + \ K_2(u_t) \tag{3.18}$$

where K is a bounded linear operator from $C([-r, 0], \boldsymbol{R}^{q+n_1}) = C([-r, 0], \boldsymbol{R}^q) \times C([-r, 0], \boldsymbol{R}^{n_1})$ into $\boldsymbol{R}^m$. From Lemma 3.1 and applying the results of Olbrot and Pandolfi already mentioned follow the existence of operators K and K_2 such that the solution of the close system (3.17)-(3.18) is exponentially convergent to zero as t goes to infinity. We can represent K in the block form as $K := [K_0, K_1]$ where K_0 and K_1 are defined on $C([-r, 0], \boldsymbol{R}^q)$ and $C([-r, 0], \boldsymbol{R}^{n_1})$, respectively. Next we will prove that this selection of operators K_0, K_1 and K_2 satisfies our assertions.

It is clear that system (3.17)-(3.18) can be represented as

$$\beta'(t) = \Lambda(\beta_t),$$

where $\beta(t) := \begin{bmatrix} \alpha(t) \\ u(t) \end{bmatrix}$ and Λ is the operator defined by

$$\Lambda \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \psi \end{pmatrix} := \begin{bmatrix} \Omega_1\varphi_1(0) - ML^{-1}\tilde{B}\psi(-r) \\ A_1\varphi_1(0) + B_1\psi(-r) \\ K_0(\varphi_0) + K_1(\varphi_1) + K_2(\psi) \end{bmatrix}.$$

Using this notation, equations (3.9), (2.13) and (3.16) can be reformulated as

$$\beta'(t) = \Lambda(\beta_t) \ + \ f(t), \tag{3.19}$$

where we have denoted $\beta(t) := \begin{bmatrix} \xi(t) \\ \phi_1(t) \\ u(t) \end{bmatrix}$ and

$$f(t) := \begin{bmatrix} \mu ML^{-1}GCe(t) \ - \ \overline{z} \\ [A_1 - \mu I, \, 0]P^{-1}GCL^{-1}(e(t) - \widetilde{\omega}) \\ 0 \end{bmatrix}.$$

From (3.10) and the Appendix A we obtain that $e(t)$ converges as $t \to \infty$ and that $e^{(i)}(t)$, $i = 1, 2, \cdots, h-1$, are exponentially convergent to zero as $t \to \infty$. This implies that $f(t)$ also converges when $t \to \infty$. Turning to apply Appendix A to system (3.19) we conclude that the functions $\xi(t)$, $\phi_1(t)$ and $u(t)$ are convergent when t goes to infinity. Furthermore, the derivatives $u^i(t)$, $i = 1, 2, \cdots h-1$, converge exponentially to zero as $t \to \infty$. Since $\phi_2(t)$ satisfies the fast subsystem (2.14), we can apply (1.7) to conclude that $\phi_2(t)$ also converges exponentially to zero as $t \to \infty$.

Next we represent by $\overline{e}$, $\overline{\xi}$, $\overline{\phi}_1$, $\overline{\phi}_2$ and $\overline{u}$ the limit at infinity of $e(t)$, $\xi(t)$, $\phi_1(t)$, $\phi_2(t)$ and $u(t)$, respectively. Consequently, from (3.10) we obtain that

$$\overline{e} = (I + \mu L)^{-1}\widetilde{w}$$

and, substituting this value in (3.9), (2.13) and (2.14) we obtain the following set of equations:

$$\Omega_1 \overline{\phi}_1 + \mu M L^{-1} G C (I + \mu L)^{-1}\widetilde{w} - M L^{-1}\widetilde{B}\overline{u} - \overline{z} = 0, \tag{3.20}$$

$$A_1\overline{\phi}_1 - \mu[A_1 - \mu I,\, 0]P^{-1} G C (I + \mu L)^{-1}\widetilde{w} + B_1\overline{u} = 0, \tag{3.21}$$

$$\overline{\phi}_2 - \mu[0,\, I - \mu N]P^{-1} G C (I + \mu L)^{-1}\widetilde{w} + B_2\overline{u} = 0. \tag{3.22}$$

Since (3.21) and (3.22) are equivalent to

$$A\overline{\phi} - \mu(A - \mu E) G C (I + \mu L)^{-1}\widetilde{w} + B\overline{u} = 0,$$

multiplying this expression by $M L^{-1}(A - \mu E)^{-1}$ it yields that

$$M L^{-1}(A - \mu E)^{-1} A\overline{\phi} = \mu M L^{-1} G C (I + \mu L)^{-1}\widetilde{w} - M L^{-1}\widetilde{B}\overline{u}.$$

Substituting the right hand side of the above expression in (3.20) we see that

$$\Omega_1 \overline{\phi}_1 + M L^{-1}(A - \mu E)^{-1} A\overline{\phi} - \overline{z} = 0$$

which, by the definition of Ω_i, $i = 1, 2$, implies that $D\overline{\phi} = \overline{z}$. Finally, since $\overline{e} = \overline{\phi} - \overline{x}$ and $\overline{e} = (I + \mu L)^{-1}\widetilde{w}$ we conclude that $D\overline{x} = \overline{z} - (I + \mu L)^{-1}\widetilde{w}$, which completes the proof. ■

4. Tracking and regulation of some nonconstant disturbances

The conclusion of section 3 can be strengthen for some not zero, yet non constant, disturbances. To obtain this extension, next we consider the regulator problem for the singular system

$$Ex'(t) = Ax(t) + Bu(t) + Ww(t), \tag{4.1}$$

$$y(t) = Cx(t - r), \tag{4.2}$$

$$z(t) = Dx(t), \tag{4.3}$$

where the disturbance $w(t) \in \boldsymbol{R}^d$ satisfies the equation

$$w'(t) = Vw(t), \tag{4.4}$$

for some matrices W and V of appropriate dimensions.

Proceeding as in Dai ([7]) we can treat the regulator problem for this system as the regulator problem for a system without perturbations. In fact, using the definition $x_a := \begin{bmatrix} x \\ w \end{bmatrix}$ we represent (4.1-4.4) as the augmented system

$$E_a x_a'(t) = A_a x_a(t) + B_a u(t), \tag{4.5}$$

$$y(t) = C_a x_a(t-r), \tag{4.6}$$

$$z(t) = D_a x_a(t), \tag{4.7}$$

where we have introduced the notations

$$E_a := \begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix}; \quad A_a := \begin{bmatrix} A & W \\ 0 & V \end{bmatrix}; \quad B_a := \begin{bmatrix} B \\ 0 \end{bmatrix};$$

$$C_a := [C, 0]; \qquad D_a := [D, 0].$$

In what follows we show that, with appropriated hypotheses, we can apply the approach developed in section 3 to design an asymptotic regulator for the augmented system (4.5)-(4.6)-(4.7).

We begin by observing that (E_a, A_a) is a regular pair. In order to follow the scheme established in section 3, next we will assume that P, Q, A_1, N, B_1 and B_2 represent the matrices previously defined in connection with the standard form of (E, A). Proceeding in similar way as in section 2 we define $\widetilde{E_a} := (A_a - \mu E_a)^{-1} E_a$ and $\widetilde{B_a} := (A_a - \mu E_a)^{-1} B_a$. If we assume that the system (E_a, A_a, C_a) is observable then there exists a matrix G_a such that $L_a := \widetilde{E_a} - G_a C_a$ is invertible and the polynomial $\det((s-\mu)L_a - I)$ is stable. Furthermore, the system

$$\psi_a'(t) = L_a^{-1}(I + \mu L_a)\psi_a(t) - L_a^{-2} G_a y(t) + L_a^{-1}\widetilde{B_a} u(t-r), \tag{4.8}$$

$$\phi_a(t) = \psi_a(t) - L_a^{-1} G_a y(t) \tag{4.9}$$

is an asymptotic observer of (4.5)-(4.6). Specifically, if

$$e_a(t) := \phi_a(t) - x_a(t-r) \tag{4.10}$$

stands for the estimation of the error then

$$e_a'(t) = L_a^{-1}(I + \mu L_a) e_a(t), \tag{4.11}$$

which is a stable system.

We divide our development in several steps. In a first step, we reduce the system (4.5)-(4.6)-(4.7) to its standard form by repeated transformation of coordinates. Initially we apply the transformation which is performed by multiplying the equation (4.5) on the left by the matrix $Q^1_a := \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix}$ and by substituting x_a by $\begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ w \end{bmatrix}$. Next we apply a transformation of coordinates in such way that the second and third components of the vector $(x_1,\, x_2,\, w)^T$ are permuted. It is clear that relative to the new coordinates the equation (4.5) takes the form

$$x_1'(t) = A_1 x_1(t) + W_1 w(t) + B_1 u(t), \tag{4.12}$$
$$w'(t) = V w(t), \tag{4.13}$$
$$N x_2'(t) = x_2(t) + W_2 w(t) + B_2 u(t), \tag{4.14}$$

where $\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = QW$. However, this singular system is not yet in standard form. By this reason we proceed to perform a new transformation of coordinates which is constructed as in Lemma B.2 of the Appendix B. To apply this result we put $H := \begin{bmatrix} A_1 & W_1 \\ 0 & V \end{bmatrix}$ and $R := [0,\, W_2]$ and we define the matrix X by means of (B.3). The new transformation of coordinates is obtained by multiplying the system (4.12-4.14) on the left by $Q^2_a := \begin{bmatrix} I & 0 \\ -NX & I \end{bmatrix}$ and substituting the vector $(x_1,\, w, x_2)^T$ according to

$$\begin{bmatrix} x_1 \\ w \\ x_2 \end{bmatrix} := \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} \begin{bmatrix} x_1 \\ w \\ \overline{x_2} \end{bmatrix},$$

where $\overline{x_2}$ is a new variable. The system obtained by applying to (4.12)-(4.13)-(4.14) this transformation is given by

$$\begin{aligned} x_1'(t) &= A_1 x_1(t) + W_1 w(t) + \overline{B_1} u(t), \\ w'(t) &= V w(t), \\ N \overline{x_2'}(t) &= \overline{x_2}(t) + \overline{B_2} u(t), \end{aligned}$$

where $\begin{bmatrix} \overline{B_1} \\ 0 \\ \overline{B_2} \end{bmatrix} = \begin{bmatrix} I & 0 \\ -NX & I \end{bmatrix} \begin{bmatrix} B_1 \\ 0 \\ B_2 \end{bmatrix}$. Furthermore, since

$$X = -\sum_{i=0}^{h-1} N^i\, [0,\, W_2] \begin{bmatrix} A_1 & W_1 \\ 0 & V \end{bmatrix}^i,$$

then

$$X \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = -\sum_{i=0}^{h-1} N^i\, [0,\, W_2] \begin{bmatrix} A_1 & W_1 \\ 0 & V \end{bmatrix}^i \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = 0,$$

which implies that $\overline{B_1} = B_1$ and $\overline{B_2} = B_2$.

We denote $Q_a := Q_a^2 Q_a^1$ and define P_a as the product of the transformations of this sequence that take x_a into $(x_1, w, \overline{x_2})^T$. It is clear that the transformation defined by Q_a and P_a transfer the system (4.5) into its standard form.

As second step we define the regulator. We begin by introducing a variable $\xi(t)$ given by

$$\xi\,'(t) = D_a\phi_a(t) \; - M_a \; \psi_a'(t) \; - \; \overline{z}. \tag{4.15}$$

Moreover, using the transformation P_a we set $\phi_a := P_a \begin{bmatrix} \phi_{11} \\ \phi_{12} \\ \phi_2 \end{bmatrix}$ and we close the system by defining the control law

$$u(t) \;=\; F_{12}\phi_{12}(t) \;+\; v(t), \tag{4.16}$$

$$v\,'(t) \;=\; K_0(\xi_t) \;+\; K_1(\phi_{11,\,t}) \;+\; K_2(v_t). \tag{4.17}$$

In these equations M_a is a $q \times (n+d)$ matrix, F_{12} is a $m \times d$ matrix and K_0, K_1 and K_2 are bounded linear operators from $C([-r,\,0],\,\boldsymbol{R}^q)$, $C([-r,\,0],\,\boldsymbol{R}^{n_1})$ and $C([-r,\,0],\,\boldsymbol{R}^m)$, respectively, into $\boldsymbol{R}^m$. These matrices and operators must be determined appropriately in order to obtain an asymptotic regulator.

As third step we are going to show that it is possible to choose the matrices M_a and F_{12} so that the variables ξ and ϕ_{11} be decoupled from ϕ_{12} and ϕ_2. As consequence of this selection we also obtain that the pair $(\xi,\,\phi_{11})$ is the state variable of a system of type (3.13). We begin by studying the equation for ϕ_{11}. Proceeding in the same way as we obtained (2.12), it follows that

$$\begin{aligned} E_a\phi_a'(t) = A_a\phi_a(t) \;+\; (A_a - \mu E_a)G_aC_a(e_a'(t) \\ -\mu e_a(t)) \;+\; B_a u(t-r). \end{aligned} \tag{4.18}$$

Multiplying this equation on the left by the transformation Q_a, it is easy to see that the system (4.18) is changed into

$$\begin{aligned} \phi_{11}'(t) = A_1\phi_{11}(t) \;+\; W_1\phi_{12}(t) \;+\; B_1(F_{12}\phi_{12}(t-r) + v(t-r)) \\ +\, f_{11}(t), \end{aligned} \tag{4.19}$$

$$\phi_{12}'(t) = V\phi_{12}(t) \;+\; f_{12}(t), \tag{4.20}$$

$$\begin{aligned} N\phi_2'(t) = \phi_2(t) \;+\; B_2(F_{12}\phi_{12}(t-r) + v(t-r)) \;+ \\ f_2(t) \end{aligned} \tag{4.21}$$

where we have abbreviated our notations by introducing the functions $f_{11}(t)$, $f_{12}(t)$ and $f_2(t)$. It is worth to point out that these functions, as well as g_{11}, g_{12} and f, which will be defined later, are obtained by algebraic operations from $e_a(t)$ by which, assuming that $e_a(t)$ is exponentially convergent to zero as $t \to \infty$, we infer that all them are also exponentially convergent to zero as $t \to \infty$.

On the other hand, it follows from (4.20) that

$$\begin{aligned} \phi_{12}(t) &= e^{Vr}\phi_{12}(t-r) + \int_{t-r}^{t} e^{V(t-s)} f_{12}(s)\, ds \\ &= e^{Vr}\phi_{12}(t-r) + g_{12}(t) \end{aligned} \tag{4.22}$$

where $g_{12}(t) \to 0$ as $t \to \infty$. Substituting this expression in (4.19) we infer that

$$\phi_{11}'(t) = A_1\phi_{11}(t) + (W_1 e^{Vr} + B_1 F_{12})\, \phi_{12}(t-r) + B_1 v(t-r) + g_{11}(t),$$

where g_{11} is a function such that $g_{11}(t) \to 0$ as $t \to \infty$.

In order to avoid the dependence of ϕ_{11} on ϕ_{12}, the above considerations suggest to introduce the additional hypothesis

$$\rho[B,\, W] = \rho[B].$$

Under this condition we can choose F_{12} so that

$$W e^{Vr} + B F_{12} = 0 \tag{4.23}$$

and utilizing this property in the preceding expression for ϕ_{11} we obtain

$$\phi_{11}'(t) = A_1\phi_{11}(t) + B_1 v(t-r) + g_{11}(t). \tag{4.24}$$

Next we derive the equation for ξ. From (4.15) we can write

$$\begin{aligned} \xi'(t) &= [D_a - M_a L_a^{-1}(I + \mu L_a + \mu G_a C_a)]\phi_a(t) \\ &\quad + \mu M_a L_a^{-1} G_a C_a e_a(t) - M_a L_a^{-1} \widetilde{B_a} u(t-r) - \overline{z}. \end{aligned}$$

This expression can be modified by replacing

$$I + \mu L_a + \mu G_a C_a = (A_a - \mu E_a)^{-1} A_a$$

and by observing that

$$D_a\phi_a(t) = D_a P_a \begin{bmatrix} \phi_{11}(t) \\ \phi_{12}(t) \\ \phi_2(t) \end{bmatrix} = [D_1,\, D_{12},\, D_2] \begin{bmatrix} \phi_{11}(t) \\ \phi_{12}(t) \\ \phi_2(t) \end{bmatrix}$$

where $D_{12} = -\sum_{i=0}^{h-1} D_2 N^i W_2 V^i$. In fact, from the definition of P_a we know that

$$D_a P_a = [D_1,\, 0,\, D_2] \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} = [\,[D_1,\, 0] + D_2 X,\; D_2]$$

and since

$$\begin{aligned} D_2 X &= -\sum_{i=0}^{h-1} D_2 N^i\, [0,\, W_2] \begin{bmatrix} A_1 & W_1 \\ 0 & V \end{bmatrix}^i \\ &= [0,\, -\sum_{i=0}^{h-1} D_2 N^i W_2 V^i] \end{aligned}$$

the assertion follows. Thus, we find that

$$\begin{aligned}
\xi'(t) &= D_a P_a \begin{bmatrix} \phi_{11}(t) \\ \phi_{12}(t) \\ \phi_2(t) \end{bmatrix} - M_a L_a^{-1}(A_a - \mu E_a)^{-1} A_a P_a \begin{bmatrix} \phi_{11}(t) \\ \phi_{12}(t) \\ \phi_2(t) \end{bmatrix} \\
&\quad - M_a L_a^{-1} \widetilde{B_a} u(t-r) + \mu M_a L_a^{-1} G_a C_a e_a(t) - \overline{z} \\
&= D_1 \phi_{11}(t) + D_{12}\phi_{12}(t) + D_2 \phi_2(t) - M_a L_a^{-1} \widetilde{B_a} v(t-r) \\
&\quad + \mu M_a L_a^{-1} G_a C_a e_a(t) - \overline{z} \\
&\quad - [M'_{11},\, M'_{12},\, M'_2] \begin{bmatrix} A_1\phi_{11}(t) + W_1\phi_{12}(t) + B_1 F_{12}\phi_{12}(t-r) \\ V\phi_{12}(t) \\ \phi_2(t) + B_2 F_{12} \phi_{12}(t-r) \end{bmatrix}
\end{aligned}$$

where we have used the notation

$$[M'_{11},\, M'_{12},\, M'_2] := M_a L_a^{-1}(A_a - \mu E_a)^{-1} Q_a^{-1}.$$

Replacing in the last expression for $\xi'(t)$ the value of $\phi_{12}(t)$ given by (4.22) and designating $D_1 - M'_{11} A_1$ as Ω_1 we can write

$$\begin{aligned}
\xi'(t) = \Omega_1 \phi_{11}(t) - M_a L_a^{-1} \widetilde{B_a} v(t-r) + (D_2 - M'_2)\phi_2(t) + f(t) - \overline{z} \\
+ [(D_{12} - M'_{11} W_1 - M'_{12} V) e^{Vr} - M'_{11} B_1 F_{12} - M'_2 B_2 F_{12}]\, \phi_{12}(t-r)
\end{aligned} \tag{4.25}$$

where $f(t)$ is a function that vanishes at ∞.

We select the matrix M_a so that $M'_{12} = -D_2 N \sum_{i=0}^{h-2} N^i W_2 V^i$ and $M'_2 = D_2$. It is clear that

$$\begin{aligned}
M'_{12} V &= -D_2 \sum_{i=0}^{h-2} N^{i+1} W_2 V^{i+1} \\
&= -D_2 \sum_{i=0}^{h-1} N^i W_2 V^i + D_2 W_2 \\
&= D_{12} + D_2 W_2.
\end{aligned}$$

We denote temporarily by H the matrix that multiplies $\phi_{12}(t-r)$ in (4.25). Using the condition (4.23) and the last property we obtain that

$$\begin{aligned}
H &= (-M'_{11} W_1 - D_2 W_2) e^{Vr} - M'_{11} B_1 F_{12} - M'_2 B_2 F_{12} \\
&= -[M'_{11}, M'_2] Q (W e^{Vr} + B F_{12}) \\
&= 0,
\end{aligned}$$

which implies that the expression (4.25) yields

$$\xi'(t) = \Omega_1 \phi_{11}(t) \; - \; M_a \, L_a^{-1} \widetilde{B_a} v(t-r) \; + \; f(t) \; - \; \overline{z} \tag{4.26}$$

Finally, it is clear that, with minor changes in the notations, equations (4.24) and (4.26) can be reformulated as the system (3.12). The matrix S that arises in this case is the same already defined in (3.14). Furthermore, proceeding as in the proof of Lemma 3.1 we obtain that the system

$$\frac{d}{dt} \begin{bmatrix} \xi(t) \\ \phi_{11}(t) \end{bmatrix} = \begin{bmatrix} 0 & \Omega_1 \\ 0 & A_1 \end{bmatrix} \begin{bmatrix} \xi(t) \\ \phi_{11}(t) \end{bmatrix} + \begin{bmatrix} -M_a L_a^{-1} \widetilde{B_a} \\ B_1 \end{bmatrix} v(t)$$

is controllable. Taking in consideration that with this approach the perturbed original system is transformed into an augmented system of type (3.1-3.3) which is free of disturbances, applying Proposition 3.1 we can state the main result of this work.

Theorem 4.1. *Assume that the following conditions hold:*

(i) *The system* $(E_a,\, A_a,\, C_a)$ *is observable;*

(ii) *The pair* $(A_1,\, B_1)$ *is controllable;*

(iii) $\rho[B,\, W] = \rho[B]$;

(iv) $\rho S = q + n_1$.

Then there exist matrices G_a *and* F_{12} *and there exist operators* $K_0,\ K_1$ *and* K_2 *such that the system (4.8)-(4.9)-(4.15) with the control law defined by (4.16)-(4.17) is an asymptotic regulator of system (4.1)-(4.2)-(4.3)-(4.4).*

A. Appendix

In this section we establish some properties of the asymptotic behavior of solutions of retarded differential equations that are essential for our development. We think that these properties are well known though we have not found them in the literature. We thus include them for reference.

We begin by observing that if A is a stable matrix and f is a continuous function such that $\lim_{t\to\infty} f(t) = f_0$, then the solution of equation

$$x'(t) = Ax(t) \; + \; f(t), \quad t \geq 0,$$

converges to $-A^{-1} f_0$ as $t \to \infty$. Moreover, if $\|f^{(i)}(t)\| \leq C_1 e^{-\alpha t},\ t \geq 0,\ i = 1,\, 2, \cdots, k,$ for some constants $C_1,\ \alpha > 0,$ then there exist $C_2 \geq 0$ and $\beta > 0$ such that $\|x^{(i)}(t)\| \leq C_2 e^{-\beta t},\ t \geq 0,$ for all $i = 1,\, 2, \cdots k$ (see [39], Theorem 4.4.4). Next we establish similar properties for retarded differential equations. In what follows we use the terminology of [19]. In particular, we denote by $X(\cdot)$ the fundamental solution of the homogeneous retarded differential equation

$$x'(t) = \Lambda(x_t) \tag{A.1}$$

where $x(t) \in \boldsymbol{R}^n$ and $\Lambda : C([-r,0];\boldsymbol{R}^n) \to \boldsymbol{R}^n$ is a bounded linear operator . Here, as is usual in the theory of retarded functional differential equations, we denote by $x_t \in C([-r,0];\boldsymbol{R}^n)$ the function defined by $x_t(\theta) := x(t+\theta)$. We define $\overline{\Lambda}$ on $\boldsymbol{R}^n$ by $\overline{\Lambda}(x) := \Lambda(\overline{x})$, where $\overline{x}$ denotes the constant function $\overline{x}(\theta) := x$, $-r \le \theta \le 0$. Moreover, we denote by $x(\cdot;\varphi,f)$ the solution of the nonhomogeneous initial value problem

$$x'(t) = \Lambda(x_t) + f(t) \tag{A.2}$$

$$x(\theta) = \varphi(\theta), \quad -r \le \theta \le 0, \tag{A.3}$$

where φ is continuous on $[-r,0]$ and f is an appropriate function.

We say that a system (A.2) is asymptotically stable if there exist constants $C \ge 0$ and $\alpha > 0$ such that

$$\|X(t)\| \le Ce^{-\alpha t}, \quad t > 0. \tag{A.4}$$

As a consequence of the variation of constants formula ([19]) and the asymptotic behavior of the Laplace transform we can establish.

Proposition A.1. *Assume that (A.1) is asymptotically stable and $f : [0,\infty) \to \boldsymbol{R}^n$ is a continuous function.*

(a) If $f(t)$ converges to f_0 as $t \to \infty$, then $x(t;\varphi,f) \to -\bar{\Lambda}^{-1} f_0$.

(b) If $f(t)$ converges exponentially to zero as $t \to \infty$, then the same occurs with $x(t;\varphi,f)$.

(c) If f is a function of class $C^{(k)}$ such that $f(t)$ converges to f_0 as $t \to \infty$, and $f^{(i)}(t) \to 0$ as $t \to \infty$ for every $i = 1, 2, \cdots, k$, then $x^{(i)}(t) \to 0$ as $t \to \infty$ for every $i = 1, 2, \cdots, k$.

(d) If f is a function of class $C^{(k)}$ such that $f(t)$ converges to f_0 as $t \to \infty$, and $f^{(i)}(t)$ converges exponentially to zero as $t \to \infty$ for every $i = 1, 2, \cdots, k$, then $x^{(i)}(t)$ also converges exponentially to zero as $t \to \infty$ for every $i = 1, 2, \cdots, k$.

Proof. Since $X(t)$ satisfies the condition (A.4) then the Laplace transform $\hat{X}(\lambda)$ of $X(t)$ is defined for $\mathrm{Re}(\lambda) > -\alpha$. Furthermore, since $X(\cdot)$ is the solution of equation

$$X'(t) = \Lambda(X_t)$$

with initial condition $X_0(\theta) := \begin{cases} I, & \theta = 0, \\ 0, & \theta < 0 \end{cases}$ we obtain

$$\lambda\hat{X}(\lambda) - I = \Lambda(\hat{X}_t(\lambda)) = \Lambda(e^{-\lambda\theta}\hat{X}(\lambda))$$

which implies that $\lim_{\lambda\to 0}\hat{X}(\lambda) = -\Lambda(I)^{-1}$. From the variation of constants formula ([19], Theorem 1.6.1) it follows that

$$x(t;\varphi,f) = y(t) + \int_0^t X(t-s)f(s)\,ds \tag{A.5}$$

where y denotes the solution of (A.1) with initial condition φ. Since $y(t)\to 0$ as $t\to\infty$ only remains to prove that

$$u(t) := \int_0^t X(t-s)f(s)\,ds$$

converges to $-\overline{\Lambda}^{-1}f_0$ as t goes to infinity. Using the Cauchy criterion we easily derive that $u(t)$ is convergent as $t\to\infty$. On the other hand, applying theorem 34.2 and theorem 34.3 in [9] we can write

$$\begin{aligned}\lim_{t\to\infty} u(t) &= \lim_{\lambda\to 0}\lambda\,\hat{u}(\lambda)\\ &= \lim_{\lambda\to 0}\lambda\hat{X}(\lambda)\hat{f}(\lambda)\\ &= -\overline{\Lambda}^{-1}f_0,\end{aligned}$$

which completes the proof of (a). The assertions (b), (c) and (d) are quite easy to prove as consequence of the variation of constant formula (A.5) and the properties of the solution $y(t)$ of the homogeneous equation (A.1). In fact, if $T(t)$ denotes the solution semigroup of (A.1), then $T(t)$ is exponentially stable and differentiable for $t > r$ ([19]). Since $y(t) = [T(t)\varphi](0)$, from the properties of differentiable semigroups ([39]) we obtain that

$$\|y^{(i)}(t)\| \le C_1 e^{-\alpha t}, \quad i = 0, 1, \cdots, k, \;\; t \ge 0,$$

for some constant $C_1 \ge 0$. Furthermore, it follows from (A.5) that

$$x'(t;\varphi,f) = y'(t) + X(t)f(0) + \int_0^t X(t-s)f'(s)\,ds.$$

Turning to apply the previous arguments, but utilizing the last expression, we establish the assertions for $x'(t;\varphi,f)$. We complete the proof proceeding inductively. ■

B. Appendix

In this section we collect some simple results of purely algebraic nature, which have been used in the previous sections.

Lemma B.1. *Assume that (E, A, C) is observable. Let $L := \widetilde{E} - GC$ and set*

$$P^{-1}LP := \begin{bmatrix} L_{1,1} & L_{1,2} \\ L_{2,1} & L_{2,2} \end{bmatrix}.$$

Then we can select the matrix G so that both L and $L_{1,1}$ are invertible matrices and $\det((s-\mu)L - I) \neq 0$, for $Re(s) \geq 0$.

Proof. In fact, if we select G so that matrix $P^{-1}G := \begin{bmatrix} G_1 \\ 0 \end{bmatrix}$, from the definition of L and $\widetilde{E}$ and using the transformation of coordinates we can write

$$\begin{aligned}
P^{-1}LP &= P^{-1}(A-\mu E)^{-1}EP - P^{-1}GCP \\
&= P^{-1}(A-\mu E)^{-1}Q^{-1}QEP - P^{-1}GCP \\
&= (Q(A-\mu E)P)^{-1}QEP - P^{-1}GCP \\
&= \begin{bmatrix} A_1-\mu I & 0 \\ 0 & I-\mu N \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} - P^{-1}GCP \\
&= \begin{bmatrix} (A_1-\mu I)^{-1} & 0 \\ 0 & (I-\mu N)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} - \begin{bmatrix} G_1C_1 & G_1C_2 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} (A_1-\mu I)^{-1} - G_1C_1 & -G_1C_2 \\ 0 & (I-\mu N)^{-1}N \end{bmatrix}.
\end{aligned}$$

Thus $L_{1,1} = (A_1-\mu I)^{-1} - G_1C_1$; $L_{1,2} = -G_1C_2$; $L_{2,1} = 0$ and $L_{2,2} = (I-\mu N)^{-1}N$. In view of N is a nilpotent matrix, the above expression implies that

$$\begin{aligned}
\det((s-\mu)L - I) &= \det((s-\mu)L_{1,1} - I)\det((s-\mu)L_{2,2} - I) \\
&= \pm \det((s-\mu)L_{1,1} - I).
\end{aligned}$$

Since (A_1, C_1) is observable the pair $((A_1-\mu I)^{-1}, C_1)$ also is observable and we can choose G_1 so that $L_{1,1}$ is invertible and the solutions of the equation $\det((s-\mu)L_{1,1} - I) = 0$ are located in $\mathrm{Re}(s) < 0$. This completes the proof. ■

Next we establish the existence of a suitable coordinate transformation to reduce a singular system in block form to its standard form.

Lemma B.2. *Let N, H and R be $n\times n$, $m\times m$ and $n\times m$ matrices, respectively, such that N is nilpotent. Then there exists a $n\times m$ matrix X for which the following conditions hold:*

$$\begin{bmatrix} I_m & 0 \\ -NX & I_n \end{bmatrix} \begin{bmatrix} I_m & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} I_m & 0 \\ X & I_n \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & N \end{bmatrix} \tag{B.1}$$

and

$$\begin{bmatrix} I_m & 0 \\ -NX & I_n \end{bmatrix} \begin{bmatrix} H & 0 \\ R & I_n \end{bmatrix} \begin{bmatrix} I_m & 0 \\ X & I_n \end{bmatrix} = \begin{bmatrix} H & 0 \\ 0 & I_n \end{bmatrix}. \tag{B.2}$$

Proof. Let h be the index of N. We define X as the matrix

$$X := -\sum_{i=0}^{h-1} N^i R H^i. \tag{B.3}$$

It is clear from this definition that $N\,X\,H = R + X$ and that, in turn, relations (B.1) and (B.2) are easy consequence of this property. ■

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