Proyecciones
Vol. 23, N^o 3, pp. 205-234, December 2004.
Universidad Católica del Norte
Antofagasta - Chile
DOI: 10.4067/S0716-09172004000300002

UNIFORM STABILIZATION OF A PLATE EQUATION WITH NONLINEAR LOCALIZED DISSIPATION

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Received : April 2004. Accepted : October 2004

Abstract

We study the existence and uniqueness of a plate equation in a bounded domain of R^n, with a dissipative nonlinear term, localized in a neighborhood of part of the boundary of the domain. We use techniques from control theory, the unique continuation property and Nakao method to prove the uniform stabilization of the energy of the system with algebraic decay rates depending on the order of the nonlinearity of the dissipative term.

1. Introduction

Our goal in this paper is to investigate the qualitative properties of the following initial boundary value problem for a plate equation in a domain Ω of R^n, $1 \leq n \leq 3$:

$$(1.1) \qquad \begin{cases} u_{tt} + \triangle^2 u + \rho(x, u_t) = 0 & x \in \Omega,\ t \geq 0 \\ u(x,0) = u_0(x) & x \in \Omega \\ u_t(x,0) = u_1(x) & x \in \Omega \\ u(x,t) = 0 & x \in \partial\Omega,\ t \geq 0 \\ \frac{\partial u}{\partial \eta}(x,t) = 0 & x \in \partial\Omega,\ t \geq 0 \end{cases}$$

The domain Ω is a bounded open set of $R^n,\ 1 \leq n \leq 3,$ with regular boundary ($C^3 class$), $u = u(x,t),\ u_1 \in H_0^1(\Omega),\ \ u_0 \in H_0^2(\Omega) \cap H^3(\Omega)$ and $\rho : \bar{\Omega} \times R \to R$ a function specified as follows.

Let $a : \bar{\Omega} \to R^+$, $a \in L^\infty(\Omega)$ be a function with $a(x) \geq a_0 > 0$ in a neighborhood ω of part of the boundary of Ω, $\omega \subset \bar{\Omega}$. The hypotheses on the dissipative term $\rho(x, u_t)$, $\rho : \overline{\Omega} \times \mathbf{R} \to \mathbf{R}$, are:

i) $\rho(x,s)s \geq 0,\ s \in R,\ x \in \Omega$;

ii) ρ and $\frac{\partial \rho}{\partial s}$ continuous in $\overline{\Omega} \times R$;

iii) There exist constants $K_1 > 0,\ K_2 > 0$ and $p \in R,\ -1 < p \leq 2$, such that:

$$K_1\, a(x)\, |s|^{p+1} \;\leq\; |\rho(x,s)| \;\leq\; K_2\, a(x)\, [\, |s|^{p+1} \,+\, |s|\,],\ \forall s \in R,\ \forall x \in \Omega;$$

iv) $\dfrac{\partial \rho}{\partial s}(x,s) \geq 0,\ \ \forall s \in R,\ \ \forall x \in \Omega.$

In this paper, we show the uniform stabilization of the total energy for the system (1.1) with algebraic rates. To prove this result we use some energy identities associated with localized multipliers in order to construct special difference inequalities for the associated energy. These ideas come from Control Theory (see J.-L. Lions [11], V. Komornik [8], A. Haraux [6] and M. Nakao [13]). The main estimates in this work are obtained using the unique continuation principle (see Kim [9] and Tucsnak [14]) for the plate equation and Nakao's Lemma. This work generalizes a previous investigation of Tucsnak [14]) who studied the case with linear dissipation. The proof for this linear case is considerably simpler than the problem considered in our case.

In this work, we have considered for simplicity that the spatial dimension is $N = 1,\ 2$ or 3, but with slight modifications the results hold for

$N > 3$. In this case, the number p in hypothesis (iii) which give the growth of the function ρ, is to be such that $-1 < p \leq \frac{2}{N-2}$ for the case $N > 2$. Furthermore, we can only impose the condition (iii) on $\rho(x, s)$ for $|s| \geq 1$ and the additional condition

$$K_3 \, a\,(x) \ |s|^{r+1} \leq |\rho\,(x,s)| \leq K_4 \, a\,(x) \ \left(|s|^{r+1} + |s|\right)$$

for $|s| \leq 1$ with r some real constant such that $-1 < r < \infty$. Of course, in this case, the decay rates will depend in an explicit way on numbers p and r (see [13], [2]).

One of the first studies of stabilization of evolution models with locally distributed damping was performed by Zuazua [16], who studied the semi-linear wave equation with a linear locally distributed damping. Nakao [13] studied the wave equation with highly nonlinear locally distributed damping, where the function which localizes the dissipation has growth towards infinity similar to the case considered in the present paper. Similar problems were studied by several authors. We mention Martinez [12] and Tébou [15] for the wave equation and Alabau-Komornik [1], Horn [7] and Guesmia [5], Bisognin, Bisognin and Charão [2] for systems of elasticity.

2. Existence and Uniqueness

2.1. Local Solutions

Let $(w_k)_{k \in \mathbf{N}}$ be a basis of $V = H_0^2(\Omega) \cap H^3(\Omega)$ and $V_m = span(\{w_1, \cdots, w_m\})$.

The approximate problem is: find $u_m(t) = \sum_{j=1}^{m} g_{jm}(t)\, w_j$, defined in some interval $[0, t_m)$, which is the solution of the following system, associated to the problem (1.1):

$$\begin{cases} (u_m''(t), w_k)_{L^2(\Omega)} + b(u_m(t), w_k) + (\rho(x, u_m'(t)), w_k)_{L^2(\Omega)} = 0 \\ u_m(0) = u_{0m} \\ u_m'(0) = u_{1m} \end{cases}$$

with $k = 1, \cdots, m$, where $b : V \times V \to R$ is a bilinear function given by $b(u,v) = (\triangle u, \triangle v)_{L^2(\Omega)}$ and u_{0m} and u_{1m} are sequences in V_m such that $u_{0m} \to u_0$ strongly in V and $u_{1m} \to u_1$ strong in $H_0^1(\Omega)$.

The system above is equivalent to an initial value problem for a system of nonlinear ODE's of second order for the functions $g_{jm}(t)$. From Caratheodory's Theorem (see [4]), it follows that the approximate problem has a solution defined in the interval $[0, t_m)$.

2.2 Global Solutions

Using the hypotheses on the function $\rho(x,s)$ it is easy to show that there exists a constant $C>0$, independent of $t\in[0,t_m]$ and of $m\in\mathbf{N}$ such that

$$\|u_m^{'}(t)\|_{L^2(\Omega)}\ \le\ C\ \text{ and }\ \|\triangle u_m(t)\|_{L^2(\Omega)}\ \le\ C.$$

Using these estimates and Poincaré inequality, we obtain that $u_m(t)$ can be extended to an arbitrary interval $[0,\,T]$ and

$$u_m^{'}\ \text{ is bounded in }\ L^\infty(\,0,\,T;\,L^2(\Omega))$$
$$u_m\ \text{ is bounded in }\ L^\infty(\,0\,,T;\,H_0^2(\Omega)).$$

Furthermore, standard estimates show that

$$\|u_m^{''}(0)\|^2_{L^2(\Omega)}\le C$$

with C a positive constant independent of m. Therefore, we obtain the existence of a function $u=u(x,t)$ such that

$$\begin{aligned} u_m\ &\to u\ \text{ weak * in }\ L^\infty(0,T;H_0^2(\Omega))\subset L^2(0,T;H_0^2(\Omega))\\ u_m^{'}\ &\to u^{'}\ \text{ weak * in }\ L^\infty(0,T;L^2(\Omega))\subset L^2(0,T;L^2(\Omega))\\ u_m^{''}\ &\to u^{''}\ \text{ weak * in }\ L^\infty(0,T;L^2(\Omega))\subset L^2(0,T;L^2(\Omega))\end{aligned}$$

Using the convergences obtained above, Lions' Lemma and the Compactness Theorem of Aubin-Lions (Lions [11]) we can pass to the limit, with $m\to\infty$, in the approximate problem in order to obtain that the limit $u=u(x,t)$ is a solution of

$$(u^{''},\,v)_{L^2(\Omega)}\ +\ b(u,\,v)\ +\ (\rho(x,u^{'}),\,v)_{L^2(\Omega)}=0$$

for all $v\in V$ in the sense of $\mathcal{D}^{'}(0,T)$..

We also have that

$$u^{''}+\triangle^2u+\rho(x,u^{'})=0$$

in the sense of $\mathcal{D}^{'}(\Omega)$ for each $t\in[0,T]$.

Using the regularity of the initial data and the Elliptic Regularity Theorem we obtain that u is a solution of the equation in (1.1) such that

$$u\in L^\infty(0,\infty;H_0^2(\Omega)\cap H^3(\Omega))\quad\text{and}\quad u_t\in L^\infty(0,\infty;H_0^1(\Omega)).$$

The initial conditions are verified in a standard way. The uniqueness of solutions is shown using the mean value theorem and the hypothesis that $\frac{\partial\rho}{\partial s}(x,s)\ge 0$, for all $s\in R$.

3. Stabilization

We consider

$$\Gamma(x_0) = \{x \in \Gamma;\ (x - x_0).\eta(x) \geq 0\}$$

where $\eta = \eta(x)$ is the exterior unit normal vector at $x \in \Gamma = \partial\Omega$ and $x_0 \in R^n$ is a fixed vector.

The energy of the system (1.1) is given by

$$E(t) = \frac{1}{2}\int_\Omega \Big(|u_t|^2 + |\triangle u|^2\Big)\, dx.$$

We observe that $E(t)$ satisfies the following identity:

$$E(t) - E(t+T) = \int_t^{t+T}\int_\Omega \rho(x,u_t)u_t\, dxdt,\ \ t \geq 0,\ \ T > 0. \tag{3.1}$$

Thus, due to the hypothesis that $\rho(x,u_t)u_t \geq 0$ for all $t \geq 0$, it follows that the energy is a function decreasing with time.

Theorem 3.1 (Stabilization). *We suppose that the functions $a(x)$ and $\rho(x,s)$ satisfy the hypotheses in the introduction. Then, the energy associated with the solution $u = u(x,t)$ of the problem (1.1) has the following asymptotic behavior in time:*

$$E(t) = E\Big(u(x,t)\Big) \leq CE(0)(1+t)^{-\gamma_i},\ \ i = 1,2, \tag{3.2}$$

where C is a positive constant. The rates of decay γ_i are given according to the following cases:

case 1: $\gamma_1 = \frac{2}{p}$ if $0 < p \leq 2$ and $N \geq 3$ $(0 < p < \infty$ if $N = 1$ or $2)$
case 2: $\gamma_2 = \frac{2(p+1)}{-p}$ if $-1 < p < 0$.

If $p = 0$ the energy $E(t)$ decays exponentially.

To prove the stabilization of the energy $E(t)$, we show that $E(t)$ satisfies an inequality of the following form:

$$E(t)^{\epsilon_i} \leq C[E(t) - E(t+T)],\ \ t \geq 0 \tag{3.3}$$

where C is a positive constant, $T > 0$ is fixed and $\epsilon_i > 0$ is related with γ_i, which are given in Theorem 3.1.

After showing an estimate of this form for the energy, the desired result (3.2) of the Theorem of Stabilization follows from the next lemma:

Lemma 3.1 (Nakao [13]). *Let $\varphi(t)$ be a non negative function in R^+ which satisfies:*

$$\sup_{t\le s\le t+T} \varphi(s)^{1+\delta} \le g(t)[\varphi(t)-\varphi(t+T)]$$

for some $T>0$, $\delta>0$ fixed and for all $t\ge 0$, where $g(t)$ is a continuous non decreasing function. Then, $\varphi(t)$ satisfies

$$\varphi(t)\le \left\{\varphi(0)^{-\delta}+\int_T^t g(s)^{-1}\,ds\right\}^{\frac{-1}{\delta}},\quad t\ge T.$$

If $\delta=0$ then $\varphi(t)$ decays exponentially, that is

$$\varphi(t)\le C\,\varphi(0)\,\mathbf{e}^{-\lambda\,t},\qquad\qquad t\ge 0$$

for some $\lambda>0$.

We also include the following lemma, which will be used to estimate an integral involving the dissipative term $\rho(x,u_t)$.

Lemma 3.2 (Gagliardo-Niremberg). *Let $1\le r<p<\infty$, $1\le q\le p$ and $0\le m$. Then, $\|v\|_{W^{k,q}}\le C\,\|v\|^{\theta}_{W^{m,p}}\,\|v\|^{1-\theta}_{L^r}$ for $v\in W^{m,p}(\Omega)\cap L^r(\Omega)$, $\Omega\subset R^N$, where C is a positive constant and $\theta=\left(\frac{k}{N}+\frac{1}{r}-\frac{1}{q}\right)\left(\frac{m}{N}+\frac{1}{r}-\frac{1}{p}\right)^{-1}$ provided that $0<\theta\le 1$.*

In order to prove (3.3), we use the energy identities given in the following lemma:

Lemma 3.3. *Let $h:\mathbf{R}^n\to\mathbf{R}^n$ of class C^2, $m\in W^{2,\infty}(\Omega)$, u the solution of (1.1) and $T>0$ fixed. Then, the following identities are valid for all $t\ge 0$:*

$$\begin{aligned}\left[\int_\Omega u_t u\,dx\right]_t^{t+T}-\int_t^{t+T}\int_\Omega u_t^2\,dxds+\int_t^{t+T}\int_\Omega|\triangle u|^2\,dxds&\\ +\int_t^{t+T}\int_\Omega \rho(x,u_t)u\,dxds=0.&\end{aligned}\tag{3.4}$$

$$\begin{aligned}&\int_t^{t+T}\int_\Omega\Big[m(x)|\triangle u|^2-m(x)|u_t|^2\Big]\,dxds=\\ &-\left[\int_\Omega m(x)uu_t\,dx\right]_t^{t+T}-\int_t^{t+T}\int_\Omega m(x)u\rho(x,u_t)\,dxds\\ &-\int_t^{t+T}\int_\Omega\Big[u\triangle u\triangle m+2\triangle u\nabla u.\nabla m\Big]dxds.\end{aligned}\tag{3.5}$$

$$\left[\int_\Omega u_t(h.\nabla u)\,dx\right]_t^{t+T} + \frac{1}{2}\int_t^{t+T}\int_\Omega (\mathrm{div}h)|u_t|^2\,dxds$$

$$+\int_t^{t+T}\int_\Omega \rho(x,u_t)(h.\nabla u)\,dxds - \frac{1}{2}\int_t^{t+T}\int_\Omega (\mathrm{div}h)|\triangle u|^2\,dxds$$

$$+2\int_t^{t+T}\int_\Omega \sum_{j,k=1}^n D_j h^k(D_jD_ku)\triangle u\,dxds + \int_t^{t+T}\int_\Omega (\triangle h.\nabla u)\triangle u\,dxds$$

$$= \frac{1}{2}\int_t^{t+T}\int_\Gamma (h.\eta)|\triangle u|^2\,d\Gamma ds \tag{3.6}$$

$$\left[\int_\Omega u_t(x-x_0).\nabla u\,dx\right]_t^{t+T} + \frac{n}{2}\int_t^{t+T}\int_\Omega |u_t|^2\,dxds$$

$$+\int_t^{t+T}\int_\Omega \rho(x,u_t)((x-x_0).\nabla u)\,dxds + \left(2-\frac{n}{2}\right)\int_t^{t+T}\int_\Omega |\triangle u|^2\,dxds$$

$$= \frac{1}{2}\int_t^{t+T}\int_\Gamma (x-x_0).\eta|\triangle u|^2\,d. \tag{3.7}$$

where h^k indicates the k-th component of the field h, $D_j = \dfrac{\partial}{\partial x_j}$, $\triangle h = (\triangle h^1, \cdots, \triangle h^n)$, $\eta = \eta(x)$ is the normal at the point $x \in \Gamma = \partial\Omega$ and x_0 is a point in R^n, arbitrarily fixed.

These identities are proved using the multipliers $M(u) = u$, $M(u) = m(x)u$, $M(u) = h\cdot\nabla u$ and $M(u) = (x-x_0)\cdot\nabla u$, respectively. Here, $x\cdot y$ means the usual inner product in R^n.

4. Energy Estimates

In order to state the next results, we introduce a vector field $h = (h^1, h^2, ..., h^n)\colon \overline{\Omega} \to \mathbf{R}^n$ of $\mathcal{C}^2$ class satisfying

$$\begin{gathered} h(x) = \eta(x)\mathrm{in}\Gamma(x_0) \\ h(x).\eta(x) \geq 0\mathrm{in}\Gamma \\ h(x) = 0\mathrm{in}\Omega\backslash\hat{\omega} \end{gathered} \tag{4.1}$$

where $\hat{\omega}$ is an open set of R^n such that $\Gamma(x_0) \subset \hat{\omega} \cap \bar{\Omega} \subset \omega$ (regarding the existence of such a field h, see Haraux [6] and Lions [10]).

We observe that in all the estimates that follow, the letter C may indicate different positive constants.

The first estimate is given by the following lemma:

Lemma 4.1. *Let T be a fixed positive number. Then, there exist $\gamma > 0$ and $\beta > 0$ such that the solution $u(x,t)$ of (1.1) satisfies the following inequality:*

$$\gamma \int_t^{t+T} E(s)\,ds \le C\Big[E(t+T) + E(t)\Big] + \\ + \int_t^{t+T} \int_\Omega |\rho(x,u_t)| \Big[|u| + \beta M |\nabla u|\Big]\,dxds \\ + \frac{\beta}{2} \int_t^{t+T} \int_{\Gamma(x_0)} (x-x_0).\eta |\triangle u|^2\, d\Gamma ds$$

where $M = \sup_{\bar{\Omega}} |x - x_0|$ and $E = E(t)$ is the energy of the solution $u(x,t)$.

Proof. Let β be a fixed positive number satisfying $\dfrac{n\beta}{2} - 1 > 0$. Multiplying (3.7) by β and then adding (3.7) and (3.4) hand by hand we obtain

$$\int_t^{t+T} \int_\Omega \left[\left(\frac{n\beta}{2} - 1\right)|u_t|^2 + (1+2n)|\Delta u|^2\right] dxds \\ = - \int_\Omega [(x-x_0)\cdot \nabla u + u] u_t dx \Big|_t^{t+T} + \frac{\beta}{2} \int_t^{t+T} \int_\Gamma (x-x_0)\cdot \eta |\Delta u|^2 d\Gamma ds \\ - \frac{\beta}{2} \int_t^{t+T} \int_\Omega [\beta((x-x_0)\cdot\nabla u) + u] \rho(x,u_t) dxds.$$

Therefore, choosing $\gamma = min\Big\{2\Big(1 + \beta\dfrac{4-n}{2}\Big),\ 2\Big(\dfrac{n\beta}{2} - 1\Big)\Big\}$, it follows that

$$\gamma \int_t^{t+T} E(s)\,ds \le - \Big[\int_\Omega u_t [u + \beta(x-x_0).\nabla u]\,dx\Big]_t^{t+T} \\ - \int_t^{t+T} \int_\Omega \rho(x,u_t)[u + \beta(x-x_0).\nabla u]\,dxds \\ + \frac{\beta}{2} \int_t^{t+T} \int_\Gamma (x-x_0).\eta |\triangle u|^2\, d\Gamma ds \\ \le \Big[\int_\Omega |u_t| [|u| + \beta|x-x_0||\nabla u|]\,dx\Big]_t^{t+T} + \int_t^{t+T} \\ \int_\Omega |\rho(x,u_t)| [|u| + \beta|x-x_0||\nabla u|]\,dxds \\ + \frac{\beta}{2} \int_t^{t+T} \int_{\Gamma(x_0)} (x-x_0).\eta |\triangle u|^2\, d\Gamma ds$$

since $(x-x_0).\eta \leq 0$ in $\Gamma\backslash\Gamma(x_0)$.

From the estimate above, the fact that $u \in H_0^2(\Omega)$ and Poincaré inequality

$$\|u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)} \leq C\| \triangle u\|_{L^2(\Omega)}$$

valid for all $t \geq 0$, it follows that

$$\begin{aligned} \gamma \textstyle\int_t^{t+T} E(s)\,ds &\leq C\Big[\|u_t\|_{L^2(\Omega)}\| \triangle u\|_{L^2(\Omega)}\Big]_t^{t+T} \\ &+ \textstyle\int_t^{t+T}\int_\Omega |\rho(x,u_t)|[|u| + \beta M|\nabla u|]\,dxds \\ &+\frac{\beta}{2}\textstyle\int_t^{t+T}\int_{\Gamma(x_0)}(x-x_0).\eta| \triangle u|^2\,d\Gamma ds. \end{aligned} \tag{4.2}$$

Here, we observe that

$$\begin{aligned} &\Big[\|u_t\|_{L^2(\Omega)}\| \triangle u\|_{L^2(\Omega)}\Big]_t^{t+T} \\ \leq \frac{1}{2}\int_\Omega\Big(|u_t(t+T)|^2 + | \triangle u(t+T)|^2\Big)\,dx &+ \frac{1}{2}\int_\Omega\Big(|u_t(t)|^2 + | \triangle u(t)|^2\Big)\,dx \\ &= E(t+T) + E(t) \end{aligned}$$

follows from (4.2) the proof of the Lemma 4.1.

Lemma 4.2. *Let T be a fixed positive number and u the solution of (1.1). Then,*

$$\begin{aligned} \frac{1}{2}\int_t^{t+T}\int_{\Gamma(x_0)}|\triangle u|^2 d\Gamma ds &\leq \Big[\int_\Omega u_t(h.\nabla u)\,dx\Big]_t^{t+T} + \frac{1}{2}\int_t^{t+T}\int_\Omega (divh)|u_t|^2\,dxds \\ &+\int_t^{t+T}\int_\Omega \rho(x,u_t)(h.\nabla u)\,dxds - \frac{1}{2}\int_t^{t+T}\int_\Omega (divh)| \triangle u|^2\,dxds \\ &+2\int_t^{t+T}\int_\Omega \sum_{j,k=1}^{n}(D_jh^k)(D_jD_ku) \triangle u\,dxds \\ &+\int_t^{t+T}\int_\Omega (\triangle h.\nabla u) \triangle u\,dxds, \end{aligned}$$

where h is the field given in (4.1), h^k is the k-th component of h and $D_j = \dfrac{\partial}{\partial x_j}$.

Proof : The properties (4.1) of the vector field h and the identity (3.6) imply that

$$\frac{1}{2}\int_t^{t+T}\int_{\Gamma(x_0)} |\triangle u|^2 d\Gamma ds = \frac{1}{2}\int_t^{t+T}\int_{\Gamma(x_0)} (h.\eta)|\triangle u|^2 d\Gamma ds$$
$$\leq \frac{1}{2}\int_t^{t+T}\int_{\Gamma}(h.\eta)|\triangle u|^2 d\Gamma ds = \Big[\int_\Omega u_t(h.\nabla u)\,dx\Big]_t^{t+T}$$
$$+\frac{1}{2}\int_t^{t+T}\int_\Omega (divh)|u_t|^2\,dxds + \int_t^{t+T}\int_\Omega \rho(x,u_t)(h.\nabla u)\,dxds$$
$$-\frac{1}{2}\int_t^{t+T}\int_\Omega (divh)|\triangle u|^2\,dxds + 2\int_t^{t+T}\int_\Omega \sum_{j,k=1}^{n}(D_jh^k)(D_jD_ku)\triangle u\,dxds$$
$$+\int_t^{t+T}\int_\Omega (\triangle h.\nabla u)\triangle u\,dxds.$$

Thus, the Lemma 4.2 is proved.

We need estimate each term from the inequality which appears in Lemma 4.2.

Lemma 4.3. *Let T be a fixed positive number, $h : R^n \to R^n$ a vector field of class C^2 with the properties (4.1) and u the solution of (1.1). Then,*

$$\Big|\int_\Omega u_t(h.\nabla u)\,dx\Big|_t^{t+T} \leq C\Big(E(t+T)+E(t)\Big) \tag{4.3}$$

$$\Big|\frac{1}{2}\int_t^{t+T}\int_\Omega (divh)|u_t|^2\,dxds\Big| \leq C\int_t^{t+T}\int_{\bar\Omega\cap\hat\omega}|u_t|^2\,dxds \tag{4.4}$$

$$\Big|\int_t^{t+T}\int_\Omega \rho(x,u_t)(h.\nabla u)\,dxds\Big| \leq C\int_t^{t+T}\int_\Omega |\rho(x,u_t)||\nabla u|\,dxds \tag{4.5}$$

$$\Big|\int_t^{t+T}\int_\Omega (\triangle h.\nabla u)\triangle u\,dxds\Big| \leq C\int_t^{t+T}\int_{\bar\Omega\cap\hat\omega}|\triangle u|^2\,dxds \tag{4.6}$$

$$\Big|2\int_t^{t+T}\int_\Omega \sum_{j,k=1}^{n}(D_jh^k)(D_jD_ku)\triangle u\,dxds\Big|$$
$$\leq C\int_t^{t+T}\int_{\bar\Omega\cap\hat\omega}|\triangle u|^2\,dxds \tag{4.7}$$

$$\Big|-\frac{1}{2}\int_t^{t+T}\int_\Omega (divh)|\triangle u|^2\,dxds\Big| \leq C\int_t^{t+T}\int_{\bar\Omega\cap\hat\omega}|\triangle u|^2\,dxds \tag{4.8}$$

where $\hat\omega$ is mentioned in the properties (4.1) on the field h.

Proof :

Using the fact that the vector field h is C^2 and the Poincaré inequality we have:

$$\left|\int_\Omega u_t(h.\nabla u)\,dx\right|_t^{t+T} \le C\left[\int_\Omega |u_t||\nabla u|\right]_t^{t+T} \le C\left[\|u_t\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)}\right]_t^{t+T}$$

$$\le \left[\|u_t\|_{L^2(\Omega)}\|\triangle u\|_{L^2(\Omega)}\right]_t^{t+T} \le C\Big(E(t+T)+E(t)\Big),$$

Then, (4.3) is proved.

To prove the other estimates we use the fact that $h=0$ in $\bar{\Omega}\backslash\hat{\omega}$ and h is C^2 in $\bar{\Omega}$. So,

$$\left|\frac{1}{2}\int_t^{t+T}\int_\Omega (divh)|u_t|^2\,dxds\right|$$
$$\le \frac{1}{2}\int_t^{t+T}\int_{\bar{\Omega}\cap\hat{\omega}} |divh||u_t|^2\,dxds \le C\int_t^{t+T}\int_{\bar{\Omega}\cap\hat{\omega}} |u_t|^2 0\,dxds.$$

Thus (4.4) is proved, too.

Now, we note that

$$\left|\int_t^{t+T}\int_\Omega \rho(x,u_t)(h.\nabla u)\,dxds\right|$$

$$\le \int_t^{t+T}\int_\Omega |\rho(x,u_t)||h||\nabla u|\,dxds \le C\int_t^{t+T}\int_\Omega |\rho(x,u_t)||\nabla u|\,dxds.$$

Therefore, (4.5) holds.

To prove (4.6) we use Poincaré inequality. In fact

$$\left|\int_t^{t+T}\int_\Omega (\triangle h.\nabla u)\triangle u\,dxds\right| \le \int_t^{t+T}\int_{\bar{\Omega}\cap\hat{\omega}} |\triangle h||\nabla u||\triangle u|\,dxds$$

$$\le C\int_t^{t+T}\int_{\bar{\Omega}\cap\hat{\omega}} |\nabla u||\triangle u|\,dxds \le C\int_t^{t+T}\left(\int_{\bar{\Omega}\cap\hat{\omega}} |\nabla u|^2\,dx\right)^{\frac{1}{2}}$$

$$\left(\int_{\bar{\Omega}\cap\hat{\omega}} |\triangle u|^2\,dx\right)^{\frac{1}{2}} ds$$

$$\le C\int_t^{t+T}\left(\int_{\bar{\Omega}\cap\hat{\omega}} |\triangle u|^2\,dx\right)^{\frac{1}{2}}\left(\int_{\bar{\Omega}\cap\hat{\omega}} |\triangle u|^2\,dx\right)^{\frac{1}{2}} ds$$

$$= C\int_t^{t+T}\int_{\bar\Omega\cap\hat\omega} |\triangle u|^2\, dxds,$$

where the last inequality is due to Poincaré inequality applied in ∇u, since ∇u vanishes in a part of the boundary of $\bar\Omega\cap\hat\omega$ because $\Gamma(x_0)\subset\bar\Omega\cap\hat\omega$.

Then, the estimate (4.6) is valid.

Here, we estimate

$$\left|2\int_t^{t+T}\int_\Omega \sum_{j,k=1}^n (D_j h^k)(D_j D_k u)\triangle u\, dxds\right|$$

$$\leq 2\int_t^{t+T}\int_{\bar\Omega\cap\hat\omega}\sum_{j,k=1}^n |D_j h^k||D_j D_k u||\triangle u|\, dxds$$

$$\leq C\int_t^{t+T}\int_{\bar\Omega\cap\hat\omega}\sum_{j,k=1}^n |D_j D_k u||\triangle u|\, dxds$$

$$\leq C\int_t^{t+T}\left[\left(\int_{\bar\Omega\cap\hat\omega}\sum_{j,k=1}^n |D_j D_k u|^2\, dx\right)^{\frac12}\left(\int_{\bar\Omega\cap\hat\omega}|\triangle u|^2\, dx\right)^{\frac12}\right] ds$$

$$\leq C\int_t^{t+T}\left(\int_{\bar\Omega\cap\hat\omega}|\triangle u|^2\, dx\right)^{\frac12}\left(\int_{\bar\Omega\cap\hat\omega}|\triangle u|^2\, dx\right)^{\frac12} ds$$

$$= C\int_t^{t+T}\int_{\bar\Omega\cap\hat\omega}|\triangle u|^2\, dxds$$

due to Poincar'e's inequality for $\bar\Omega\cap\hat\omega$ because $\nabla u = 0$ in a part of the boundary of $\bar\Omega\cap\hat\omega$, that is, on $\Gamma\cap(\bar\Omega\cap\hat\omega)$. Thus, (4.7) is valid, too.

The proof of (4.8) follows from the fact that $h = 0$ outside $\bar\Omega\cap\hat\omega$ and $divh$ is bounded in $\bar\Omega$.

Lemma 4.4. *Let T be a fixed positive number and u the solution of (1.1). Then,*

$$\frac12\int_t^{t+T}\int_{\Gamma(x_0)}|\triangle u|^2\, d\Gamma ds \leq C\Big[E(t)+E(t+T)$$
$$+\int_t^{t+T}\int_{\bar\Omega\cap\hat\omega}|u_t|^2\, dxds+\int_t^{t+T}\int_{\bar\Omega\cap\hat\omega}|\triangle u|^2\, dxds$$
$$+\int_t^{t+T}\int_\Omega|\rho(x,u_t)||\nabla u|\, dxds\Big]$$

Proof : It follows substituting the estimates (4.3) - (4.8) from Lemma 4.3 in the estimate given in the Lemma 4.2.

Lemma 4.5. *Let T be a fixed positive number and u the solution of (1.1) it is valid that*

$$\int_t^{t+T}\int_{\bar{\Omega}\cap\hat{\omega}} |\triangle u|^2\,dxds \le C\Big[E(t)+E(t+T)$$

$$+\int_t^{t+T}\int_\Omega |\rho(x,u_t)||u|\,dxds+\int_t^{t+T}\int_\omega |\nabla u|^2\,dxds+\int_t^{t+T}\int_\omega |u|^2\,dxds\Big],$$

where ω is mentioned in the introduction and related is with the function $a(x)$ which localizes the dissipation.

Proof : We bound each term that appears in the identity (3.5) of Lemma 3.3 with $m = m(x) \in W^{2,\infty}(\Omega)$ a function such that $\dfrac{|\nabla m|^2}{m}$ and $\dfrac{|\triangle m|^2}{m}$ are bounded functions and

$$\begin{aligned} 0 \le m \le 1 \text{in} \Omega \\ m = 1 \text{in} \tilde{\omega} \\ m = 0 \text{in} \bar{\Omega}\backslash\omega \end{aligned} \tag{4.9}$$

where $\tilde{\omega} \subset \bar{\Omega}$ is an open set in $\bar{\Omega}$ with $\Gamma(x_0) \subset \tilde{\omega} \subset \omega \subset \bar{\Omega}$. For the existence of a such function $m(x)$ see Lions [10], Haraux [6] and Tucsnak [14].

Using the fact that $m(x)$ is bounded, we obtain that

$$\left|\left[\int_\Omega m(x)uu_t\,dx\right]_t^{t+T}\right| \le C\Big[E(t)+E(t+T)\Big] \tag{4.10}$$

due to Poincaré inequality, since $u \in H_0^2(\Omega)$.

Furthermore,

$$\left|\int_t^{t+T}\int_\Omega m(x)u\rho(x,u_t)\,dxds\right| \le \int_t^{t+T}\int_\Omega |u||\rho(x,u_t)|\,dxds \tag{4.11}$$

since $0 \le m(x) \le 1$ on Ω.

Finally, using that $m = 0$ outside of ω and the fact that $\dfrac{|\nabla m|^2}{m}$ and $\dfrac{|\triangle m|^2}{m}$ are bounded, we obtain that there exists a positive constant C

such that

$$\begin{aligned}
&\left|\int_t^{t+T}\int_\Omega \Big[u \triangle u \triangle m + 2 \triangle u(\nabla u.\nabla m)\Big]\, dxds\right| \\
&\leq \int_t^{t+T}\left[C\int_\omega |u|^2\, dx + \frac{1}{4}\int_\omega m(x)|\triangle u|^2\, dx\right] ds \qquad (4.12)\\
&+C\int_t^{t+T}\left(\int_\omega |\nabla u|^2\, dx\right)^{\frac{1}{2}}\left(\int_\omega m(x)|\triangle u|^2\, dx\right)^{\frac{1}{2}} ds.
\end{aligned}$$

Substituting the estimates (4.10)-(4.12) in (3.5), we obtain:

$$\begin{aligned}
&\int_t^{t+T}\int_\Omega m(x)|\triangle u|^2\, dxds \\
&\leq C\left\{E(t)+E(t+T)+\int_t^{t+T}\int_\Omega |u||\rho(x,u_t)|\, dxds + \int_t^{t+T}\int_\omega |u|^2\, dxds\right\} \\
&+\frac{1}{4}\int_t^{t+T}\int_\Omega m(x)|\triangle u|^2\, dxds \\
&+C\int_t^{t+T}\left(\int_\omega |\nabla u|^2\, dx\right)^{\frac{1}{2}}\left(\int_\omega m(x)|\triangle u|^2\, dx\right)^{\frac{1}{2}} ds.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_t^{t+T}\int_\Omega m(x)|\triangle u|^2\, dxds \\
&\leq C\left\{E(t)+E(t+T)+\int_t^{t+T}\int_\Omega |u||\rho(x,u_t)|\, dxds\right. \\
&\left.+\int_t^{t+T}\int_\omega |u|^2\, dxds\right\}+C\int_t^{t+T}\int_\omega |\nabla u|^2\, dxds+\frac{1}{2}\int_t^{t+T}\int_\omega m(x)|\triangle u|^2\, dxds
\end{aligned}$$

that is,

$$\begin{aligned}
&\int_t^{t+T}\int_\Omega m(x)|\triangle u|^2\, dxds \leq C\left\{E(t)+E(t+T)\right. \qquad (4.13)\\
&\left.+\int_t^{t+T}\int_\Omega |u||\rho(x,u_t)|\, dxds + \int_t^{t+T}\int_\omega\left(|u|^2+|\nabla u|^2\right) dxds\right\}.
\end{aligned}$$

Using in (4.14) the fact that $0 \leq m(x) \leq 1$ on Ω and that $m(x) = 1$ in $\hat{\omega} \subset \bar{\Omega}$ (see (4.9)) the conclusion of the Lemma 4.5 follows.

Lemma 4.6. *Let $T > 0$ and u the solution of (1.1). Then,*

$$\frac{1}{2}\int_t^{t+T}\int_{\Gamma(x_0)} |\triangle u|^2 \, d\Gamma ds \leq C\Bigg\{ E(t) + E(t+T) +$$

$$+\int_t^{t+T}\int_\omega \Big[|u_t|^2 + |\nabla u|^2 + |u|^2\Big]\, dxds + \int_t^{t+T}\int_\Omega |\rho(x,u_t)|\Big[|u| + |\nabla u|\Big]\, dxds\Bigg\}$$

with C some positive constant.

Proof :

Combining the estimates from Lemmas 4.4 and 4.5 it follows that

$$\frac{1}{2}\int_t^{t+T}\int_{\Gamma(x_0)} |\triangle u|^2 \, d\Gamma ds \leq C\Bigg\{ E(t) + E(t+T) + \int_t^{t+T}\int_{\bar{\Omega}\cap\hat{\omega}} |u_t|^2 \, dxds$$

$$+\int_t^{t+T}\int_\Omega |\rho(x,u_t)||\nabla u| \, dxds\Bigg\}$$

$$+C\Bigg\{ E(t) + E(t+T) + \int_t^{t+T}\int_\Omega |\rho(x,u_t)||u| \, dxds$$

$$+\int_t^{t+T}\int_\omega |\nabla u|^2 \, dxds + \int_t^{t+T}\int_\omega |u|^2 \, dxds\Bigg\}.$$

Thus,

$$\frac{1}{2}\int_t^{t+T}\int_{\Gamma(x_0)} |\triangle u|^2 \, d\Gamma ds \leq C\Bigg\{ E(t) + E(t+T) + \int_t^{t+T}\int_\omega |u_t|^2 \, dxds$$

$$+\int_t^{t+T}\int_\Omega |\rho(x,u_t)|\Big(|\nabla u| + |u|\Big)\, dxds + \int_t^{t+T}\int_\omega |\nabla u|^2 \, dxds$$

$$+\int_t^{t+T}\int_\omega |u|^2 \, dxds\Bigg\}$$

$$= C\Bigg\{ E(t) + E(t+T) + \int_t^{t+T}\int_\omega \Big(|u_t|^2 + |\nabla u|^2 + |u|^2\Big)\, dxds$$

$$+\int_t^{t+T}\int_\Omega |\rho(x,u_t)|\Big(|\nabla u|+|u|\Big)\,dxds\Bigg\}$$

due to $\bar{\Omega}\cap\hat{\omega}\subset\omega$ (see (4.1)).

Therefore, the Lemma 4.6 is proved.

Now, we need the following lemma.

Lemma 4.7. *Let u be the solution of (1.1). Then, there exists $T>0$ such that*

$$E(t)\le C\Bigg\{E(t)-E(t+T)+\int_t^{t+T}\int_\Omega |\rho(x,u_t)|\Big(|\nabla u|+|u|\Big)\,dxds$$
$$+\int_t^{t+T}\int_\omega\Big(|u_t|^2+|\nabla u|^2+|u|^2\Big)\,dxds\Bigg\}$$

for some positive constant C and for all $t\ge 0$.

Notice that in this lemma, an energy difference appears.

Proof : From the estimate in Lemma 4.1, there exists $\gamma>0$ such that

$$\gamma\int_t^{t+T}E(s)\,ds\le C\Bigg\{E(t)+E(t+T)+\int_t^{t+T}\int_\Omega |\rho(x,u_t)|\Big(|\nabla u|+|u|\Big)\,dxds$$
$$+\int_t^{t+T}\int_{\Gamma(x_0)}|\triangle u|^2\,d\Gamma ds\Bigg\}$$

where $C>0$ is constant.

Using the estimate given in Lemma 4.6 it follows that

$$\gamma\textstyle\int_t^{t+T}E(s)\,ds\le C\Bigg\{E(t)+E(t+T)+\int_t^{t+T}\int_\Omega |\rho(x,u_t)|\Big(|\nabla u|+|u|\Big)\,dxds$$
$$+\textstyle\int_t^{t+T}\int_\omega\Big(|u_t|^2+|\nabla u|^2+|u|^2\Big)\,dxds\Bigg\}$$

(4.14)

with $T>0$ arbitrarily fixed, C a positive constant and for $t\ge 0$.

Now, we fix $T > \dfrac{2C}{\gamma} + 1$. The fact that $TE(t+T) \leq \displaystyle\int_t^{t+T} E(s)\,ds$, because $E(t)$ is decrease, and the above estimate (**??**) implies that

$$E(t) \leq \left(1+\frac{C}{\gamma}\right)\Big[E(t)-E(t+T)\Big]+\frac{C}{\gamma}\left[\int_t^{t+T}\int_\Omega |\rho(x,u_t)|\Big(|\nabla u|+|u|\Big)\,dxds\right.$$

$$\left.+\int_t^{t+T}\int_\omega \Big(|u_t|^2+|\nabla u|^2+|u|^2\Big)\,dxds\right].$$

So, the lemma 4.7 is proved.

It is necessary to estimate the following integral:

$$I = \int_t^{t+T}\int_\Omega |\rho(x,u_t)|\Big(|\nabla u|+|u|\Big)\,dxds$$

with $T > 0$ fixed by Lemma 4.7.

Lemma 4.8. *Let $T > 0$ be given by Lemma 4.7 and $u = u(x,t)$ be the solution of (1.1). If $0 \leq p \leq 2$, then*

$$I \leq C\Big[E(t)-E(t+T)\Big]^{\frac{1}{p+2}}\sqrt{E(t)}+C\Big[E(t)-E(t+T)\Big]^{\frac{p+1}{p+2}}\sqrt{E(t)}.$$

If $-1 < p < 0$ então,

$$I \leq C\Big[E(t)-E(t+T)\Big]^{\frac{p+1}{p+2}}\sqrt{E(t)}+C\Big[E(t)-E(t+T)\Big]^{\frac{2}{4-p}}\sqrt{E(t)}.$$

Proof :

We set $I = I_1 + I_2$ with

$$I_1 = \int_t^{t+T}\int_{\Omega_1} |\rho(x,u_t)|\Big(|\nabla u|+|u|\Big)\,dxds$$

and

$$I_2 = \int_t^{t+T}\int_{\Omega_2} |\rho(x,u_t)|\Big(|\nabla u|+|u|\Big)\,dxds\ ,$$

where $\Omega_1 = \Omega_1(t) = \{x \in \Omega;\ |u_t(x,t)| \leq 1\}$ and $\Omega_2 = \Omega\backslash\Omega_1$.

We need estimate the integrals I_1 and I_2 in according the two following cases for de number p related with the growth of the dissipative function $\rho(x,s)$.

Case 1: $0 \leq p \leq 2$ if $N = 3$ or $0 \leq p < \infty$ if $N = 1$ or $N = 2$.

Then, because $p \geq 0$ we obtain (using the fact that $E(t)$ is a nonincreasing function of t and the hypothesis (iii) on function $\rho(x, s)$): $\mathrm{I}_1 \leq \int_t^{t+T} \int_{\Omega_1} K_2 a(x) \Big[|u_t|^{p+1} + |u_t|\Big] \Big[|\nabla u| + |u|\Big]\, dxds$
$\leq 2K_2 \int_t^{t+T} \int_{\Omega_1} a(x)|u_t| \Big(|\nabla u| + |u|\Big)\, dxds$
$\leq 2K_2 \|a\|_\infty^{\frac{1}{2}} \int_t^{t+T} \int_{\Omega_1} \sqrt{a(x)}|u_t| \Big(|\nabla u| + |u|\Big)\, dxds$
$\leq C\Big(\int_t^{t+T} \int_{\Omega_1} a(x)|u_t|^2\, dxds\Big)^{\frac{1}{2}} \Big(\dfrac{1}{2} \int_t^{t+T} \int_{\Omega_1} [\, |\nabla u|^2 + |u|^2] dxds\Big)^{\frac{1}{2}}$
$\leq C\left(\int_t^{t+T} \int_{\Omega_1} a(x)|u_t|^2\, dxds\right)^{\frac{1}{2}} \left(\int_t^{t+T} E(s)\, ds\right)^{\frac{1}{2}}$
$\leq C\left(\int_t^{t+T} \int_{\Omega_1} a(x)|u_t|^2\, dxds\right)^{\frac{1}{2}} \sqrt{T}\sqrt{E(t)}$
$= C\left(\int_t^{t+T} \int_{\Omega_1} a(x)|u_t|^2\, dxds\right)^{\frac{1}{2}} \sqrt{E(t)}$

In the above estimate we have used that $\|\nabla u\|_{L^2(\Omega)}$ and $\|u\|_{L^2(\Omega)}$ is dominated by $\| \triangle u\|_{L^2(\Omega)}$ due to Poincaré inequality and the fact that $u \in H_0^2(\Omega)$.

Since $\dfrac{1}{\frac{p+2}{2}} + \dfrac{1}{\frac{p+2}{p}} = 1$, Hölder's inequality implies that

$$I_1 \leq C\left(\int_t^{t+T} \int_{\Omega_1} \Big(a(x)|u_t|^2\Big)^{\frac{p+2}{2}}\, dxds\right)^{\frac{1}{p+2}} \left(\int_t^{t+T} \int_{\Omega} dxds\right)^{\frac{p}{2(p+2)}} \sqrt{E(t)}$$

$$\leq C\left(\int_t^{t+T} \int_{\Omega_1} a(x)|u_t|^{p+2}\, dxds\right)^{\frac{1}{p+2}} \sqrt{E(t)}$$

with C a positive constant which depends on $\|a\|_\infty$, T and $|\Omega|$, the measure of Ω.

In this point we use the hypotheses (i) and (iii) on the function $\rho(x, s)$ to obtain

$$I_1 \leq C\Big(\int_t^{t+T} \int_{\Omega_1} \rho(x, u_t)u_t\, dxds\Big)^{\frac{1}{p+2}} \sqrt{E(t)} = C\Big[E(t) - E(t+T)\Big]^{\frac{1}{p+2}} \sqrt{E(t)}$$

due to the energy identity (3.1).

Now we estimate I_2. Since $|u_t| \geq 1$ in Ω_2, from hypothesis (iii) on $\rho(x,s)$ we get

$$\begin{aligned}
I_2 &\leq \int_t^{t+T}\int_{\Omega_2} K_2 a(x)\Big[|u_t|^{p+1}+|u_t|\Big]\Big[|\nabla u|+|u|\Big]\,dxds \\
&\leq 2K_2\int_t^{t+T}\int_{\Omega_2} a(x)|u_t|^{p+1}\Big[|\nabla u|+|u|\Big]\,dxds \\
&\leq 2K_2\Big(\int_t^{t+T}\int_{\Omega_2} a(x)^{\frac{p+2}{p+1}}|u_t|^{p+2}\,dxds\Big)^{\frac{p+1}{p+2}} \\
&\Big(\int_t^{t+T}\int_{\Omega}\Big(|\nabla u|+|u|\Big)^{p+2}\,dxds\Big)^{\frac{1}{p+2}}
\end{aligned}$$

because $\frac{p+1}{p+2}+\frac{1}{p+2}=1$ e $\frac{p+2}{p+1}, > 1$ for $p \geq 0$.

Thus,

$$\begin{aligned}
I_2 &\leq C\Big(\textstyle\int_t^{t+T}\int_{\Omega_2} a(x)|u_t|^{p+2}\,dxds\Big)^{\frac{p+1}{p+2}}\Big(\textstyle\int_t^{t+T}\int_{\Omega}\Big[|\nabla u|^{p+2}+|u|^{p+2}\Big]\,dxds\Big)^{\frac{1}{p+2}} \\
&\leq C\Big(\textstyle\int_t^{t+T}\int_{\Omega_2} a(x)|u_t|^{p+2}\,dxds\Big)^{\frac{p+1}{p+2}}\Big(\textstyle\int_t^{t+T}\int_{\Omega}|\nabla u|^{p+2}\,dxds\Big)^{\frac{1}{p+2}}
\end{aligned} \tag{4.15}$$

due to Poincaré inequality in $W_0^{1,p+2}(\Omega)$. The constant $C > 0$ depends on $\|a\|_\infty$ and the Poincaré constant for Ω.

Using Gagliardo-Niremberg Lemma and Poincaré inequality we obtain $\|\nabla u\|_{L^{p+2}(\Omega)} \leq C\,\|\nabla u\|^{\theta}_{H^1(\Omega)}\,\|\nabla u\|^{1-\theta}_{L^2(\Omega)} \leq C\,\|u\|^{\theta}_{H^2(\Omega)\cap H^1_o(\Omega)}\,\|\nabla u\|^{1-\theta}_{L^2(\Omega)}$ $\leq C\,\|\Delta u\|^{\theta}_{L^2(\Omega)}\,\|\nabla u\|^{1-\theta}_{L^2(\Omega)} \leq C||\Delta u||_{L^2} \leq CE(t)^{\frac{1}{2}}$ with $\theta = \frac{N\,p}{2(p+2)}$.

Of course, because the solution u, for each t, is in $H_0^2(\Omega)$ then $u \in W_0^{1,p+2}(\Omega), 0 \leq p \leq 2$ (if $N = 3$ or $0 \leq p < \infty$ for $N = 1,2$) .

From the last estimate, we have

$$\begin{aligned}
\Big(\textstyle\int_t^{t+T}\int_{\Omega}|\nabla u|^{p+2}\,dxds\Big)^{\frac{1}{p+2}} &\leq C\Big(\textstyle\int_t^{t+T} E(s)^{\frac{p+2}{2}}\,ds\Big)^{\frac{1}{p+2}} \\
&\leq CT^{\frac{1}{p+2}}E(t)^{\frac{1}{2}}
\end{aligned} \tag{4.16}$$

because $E(t)$ is decreasing.

Substituting (4.16) in (4.15) we obtain that

$$I_2 \leq C\Big(\int_t^{t+T}\int_{\Omega_2} a(x)|u_t|^{p+2}\,dxds\Big)^{\frac{p+1}{p+2}}E(t)^{\frac{1}{2}}.$$

Using the hypotheses (i) and (iii) on the function $\rho(x,s)$ it follows that

$$
\begin{aligned}
&I_2 \le C\Big(\int_t^{t+T}\int_{\Omega_2}\rho(x,u_t)u_t\,dxds\Big)^{\frac{p+1}{p+2}}E(t)^{\frac{1}{2}}\\
&= C\Big[E(t)-E(t+T)\Big]^{\frac{p+1}{p+2}}E(t)^{\frac{1}{2}}
\end{aligned}
$$

where the last equality is due to energy identity (3.1).

Combining the estimates for I_1 and I_2 the conclusion of Lemma 4.8 follows, for this first case.

Case 2:$-1 < p < 0$.

We write again $I = I_1 + I_2$. Then, using the hypotheses (i) and (iii) on $\rho(x,s)$, Hölder inequality and Poincaré inequality in $W_0^{1,p+2}(\Omega)$, we have from the fact that $0 < p+1 < 1$: $\mathrm{I}_1 = \int_t^{t+T}\int_{\Omega_1}|\rho(x,u_t)|\Big(|\nabla u|+|u|\Big)\,dxds$
$\le K_2\int_t^{t+T}\int_{\Omega_1}a(x)\Big(|u_t|^{p+1}+|u_t|\Big)\Big(|u|+|\nabla u|\Big)\,dxds$
$\le 2K_2\int_t^{t+T}\int_{\Omega_1}a(x)|u_t|^{p+1}\Big(|u|+|\nabla u|\Big)\,dxds$
$\le C\Big(\int_t^{t+T}\int_{\Omega_1}a(x)|u_t|^{p+2}\,dxds\Big)^{\frac{p+1}{p+2}}\Big(\int_t^{t+T}\int_{\Omega}|\nabla u|^{p+2}\,dxds\Big)^{\frac{1}{p+2}}$
$\le C\Big(\int_t^{t+T}\int_{\Omega_1}|\rho(x,u_t)|\,|u_t|\,dxds\Big)^{\frac{p+1}{p+2}}\Big(\int_t^{t+T}\int_{\Omega}|\nabla u|^2\,dxds\Big)^{\frac{1}{2}}$
$\Bigg(\Big(\int_t^{t+T}\int_{\Omega}dxds\Big)^{\frac{-p}{2}}\Bigg)^{\frac{1}{p+2}}$
$\le C\Big(\int_t^{t+T}\int_{\Omega}\rho(x,u_t)u_t\,dxds\Big)^{\frac{p+1}{p+2}}\Big(\int_t^{t+T}\int_{\Omega}|\nabla u|^2\,dxds\Big)^{\frac{1}{2}}\Big(T|\Omega|\Big)^{\frac{-p}{2(p+2)}}$.

Thus, using again Poincaré inequality, we conclude that
$\mathrm{I}_1 \le C\Big[\int_t^{t+T}\int_{\Omega}\rho(x,u_t)u_t\,dxds\Big]^{\frac{p+1}{p+2}}\sqrt{E(t)}$. The final constant C, in the above estimate, depends on Poincaré constant for Ω, measure of Ω, T, $\|a\|_\infty$ and p.

Now,we estimate I_2. To do this, we use the Hölder and Poincaré inequalities. Then, because $0 < p+1 < 1$ we have

$$\begin{aligned}
I_2 &= \int_t^{t+T}\int_{\Omega_2} |\rho(x,u_t)|\Big(|\nabla u| + |u|\Big)\,dxds\\
&\le K_2\int_t^{t+T}\int_{\Omega_2} a(x)\Big(|u_t|^{p+1} + |u_t|\Big)\Big(|u| + |\nabla u|\Big)\,dxds\\
&\le 2K_2\int_t^{t+T}\int_{\Omega_2} a(x)|u_t|\Big(|u| + |\nabla u|\Big)\,dxds\\
&\le C\Big(\int_t^{t+T}\int_{\Omega_2} a(x)|u_t|^2\,dxds\Big)^{\frac{1}{2}}\Big(\int_t^{t+T}\int_{\Omega} |\nabla u|^2\,dxds\Big)^{\frac{1}{2}}\\
&\le C\Big(\int_t^{t+T}\int_{\Omega_2} a(x)|u_t|^2\,dxds\Big)^{\frac{1}{2}}\sqrt{E(t)}\\
&= C\left(\int_t^{t+T}\int_{\Omega_2} a(x)|u_t|^{2-\alpha}|u_t|^{\alpha}\right)^{\frac{1}{2}}\sqrt{E(t)}
\end{aligned}$$

where α is a positive constant to be chosen.

Then, Hölder inequality implies that

$$I_2 \le C\left[\left(\int_t^{t+T}\int_{\Omega_2}\Big(a(x)|u_t|^{2-\alpha}\Big)^{\frac{4-p}{4}}\,dxds\right)^{\frac{4}{4-p}}\right]^{\frac{1}{2}}$$

$\left[\left(\int_t^{t+T}\int_{\Omega_2}\Big(|u_t|^{\alpha}\Big)^{\frac{4-p}{-p}}\,dxds\right)^{\frac{-p}{4-p}}\right]^{\frac{1}{2}}\sqrt{E(t)}\mathit{because}\dfrac{-p}{4-p}+\dfrac{4}{4-p} = 1$ and $\dfrac{-p}{4-p}, \dfrac{4}{4-p} > 1$ since $-1 < p < 0$.

We chose $\alpha = \dfrac{-6p}{4-p} > 0$. Then, the condition $-1 < p < 0$, implies that

$$\begin{aligned}
I_2 &\le C\Big(\int_t^{t+T}\int_{\Omega_2} a(x)|u_t|^{2+p}\,dxds\Big)^{\frac{2}{4-p}}\Big(\int_t^{t+T}\int_{\Omega_2} |u_t|^{6}\,dxds\Big)^{\frac{-p}{8-2p}}\sqrt{E(t)}\\
&\le C\Big(\int_t^{t+T}\int_{\Omega_2} a(x)|u_t|^{2+p}\,dxds\Big)^{\frac{2}{4-p}}\sqrt{E(t)}
\end{aligned}$$

due to $u_t \in L^\infty(0,\infty; H_0^1(\Omega))$ and the Sobolev imbedding which says that $u_t \in L^\infty((0,\infty, L^6(\Omega))$ for $1 \le n \le 3$ (Ω is bounded).

Thus, using the hypothesis (iii)on $\rho(x,s)$, we obtain

$\mathrm{I}_2 \le C\left(\int_t^{t+T}\int_{\Omega_2}\rho(x,u_t)u_t\,dxds\right)^{\frac{2}{4-p}}\sqrt{E(t)}.$

Finally, combining the estimates (4) and (4) for I_1 and I_2, respectively, and the energy identity (3.1) we conclude the proof of Lemma 4.8.

5. Main Estimates

By combining the results from Lemma 4.8 with the estimate from Lemma 4.7 and using the Young's inequality we obtain:

Proposition 5.1. *Let u be the solution of (1.1) and $T > 0$ fixed and given by Lemma 4.7. Then, the energy of the solution u satisfies for $t \geq 0$:*

$$E(t) \leq C\left\{D_i(t)^2 + \int_t^{t+T}\int_\omega \left[|u_t|^2 + |u|^2 + |\nabla u|^2\right] dxds\right\}$$

for $i = 1, 2$, where C is a positive constant which is independent of u and

$$D_1(t)^2 = E(t) - E(t+T) + \left[E(t) - E(t+T)\right]^{\frac{2}{p+2}} + \left[E(t) - E(t+T)\right]^{\frac{2(p+1)}{p+2}}$$

for the case $0 \leq p \leq 2$ (if N=3 and $0 \leq p < \infty$ if $N = 1, 2$)

$$D_2(t)^2 = E(t) - E(t+T) + \left[E(t) - E(t+T)\right]^{\frac{2(p+1)}{p+2}} + \left[E(t) - E(t+T)\right]^{\frac{4}{4-p}},$$

for the case $-1 < p < 0$.

At this point, using the estimate from proposition 5.1, we show the following result.

Proposition 5.2. *Let $R > 0$ fixed and u the solution of (1.1) with initial data u_0 and u_1 such that $E(0) \leq R$. Let $T > 0$ be given by Lemma 4.7. Then, there exists $C > 0$ such that*

$$\int_t^{t+T}\int_\Omega \left[|u|^2 + |\nabla u|^2\right] dxds \leq C\left\{\tilde{D}_i(t)^2 + \int_t^{t+T}\int_\omega |u_t|^2\, dxds\right\}$$

with $i = 1$ or 2 according to the cases $0 \leq p \leq 2$ and $-1 < p < 0$, respectively. The constant C depends on R.

Proof : We prove this proposition by contradiction. We follow Zuazua [16] and Nakao [13] to combine appropriate estimates and the unique continuation property (see Kim [9] and Tucsnak [14]) for the plate equation.

We observe that, in our case, the proof of proposition 5.2 is a little more delicate because the integral $\int_\omega \left(|u|^2 + |\nabla u|^2\right) dx$ is estimated instead of the integral $\int_\omega |u|^2\,dx$, which appear in the references mentioned above. This difficulty appears since here we deal with a equation with a term involving $\triangle^2 u$ instead of $\triangle u$ as in previous works already cited.

We suppose that the estimate in Proposition (5.2) is false. Then, there exist a sequence of solutions $\{u_n\}_{n\in\mathbf{N}}$ associated to initial data u_0^n and u_1^n and a sequence of points $\{t_n\}_{n\in\mathbf{N}}$ such that

$$\lim_{n\to\infty} \frac{\int_{t_n}^{t_n+T}\int_\Omega \Big[\,|u_n|^2+|\nabla u_n|^2\Big]\,dxds}{D_i(t_n)^2+\int_{t_n}^{t_n+T}\int_\omega |(u_n)_t|^2\,dxds}=\infty. \tag{5.1}$$

Now, we denote

$$\lambda_n^2=\int_{t_n}^{t_n+T}\int_\Omega \Big[\,|u_n|^2+|\nabla u_n|^2\Big]\,dxds \tag{5.2}$$

and

$$I_n(t_n)^2=\frac{1}{\lambda_n^2}\left[D_i(t_n)^2+\int_{t_n}^{t_n+T}\int_\omega |(u_n)_t|^2\,dxds\right]. \tag{5.3}$$

Thus, from (5.1) we have

$$I_n(t_n)^2\to 0,\ \text{ as }\ n\to\infty. \tag{5.4}$$

We set $v_n(x,t)=\dfrac{u_n(x,t+t_n)}{\lambda_n}$, $0\le t\le T$. Then, from (5.2) we obtain that $1=\dfrac{1}{\lambda_n^2}\displaystyle\int_{t_n}^{t_n+T}\int_\Omega\Big[\,|u_n(x,s)|^2+|\nabla u_n(x,s)|^2\Big]\,dxds$
$=\dfrac{1}{\lambda_n^2}\displaystyle\int_0^T\int_\Omega\Big[\,|u_n(x,t+t_n)|^2+|\nabla u_n(x,t+t_n)|^2\Big]\,dxdt$
$=\displaystyle\int_0^T\int_\Omega\Big[\,|v_n(x,t)|^2+|\nabla v_n(x,t)|^2\Big]\,dxdt.$

That is,

$$\int_0^T\int_\Omega\Big[\,|v_n(x,t)|^2+|\nabla v_n(x,t)|^2\Big]\,dxdt=1, \tag{5.5}$$

for all $n\in\mathbf{N}$.

From the estimate given by Proposition (5.1) and (5.5) it follows that $\mathrm{E}(\mathrm{v}_n(t))=E(\frac{u_n(t+t_n)}{\lambda_n})=\dfrac{1}{\lambda_n^2}E(u_n(t+t_n))\le\dfrac{1}{\lambda_n^2}E(u_n(t_n))$
$\le\dfrac{C}{\lambda_n^2}\left\{D_i(t_n)^2+\int_{t_n}^{t_n+T}\int_\omega|(u_n)_t|^2\,dxds+\int_{t_n}^{t_n+T}\int_\Omega\Big[\,|u_n|^2+|\nabla u_n|^2\Big]\,dxds\right\}$

$= CI_n(t_n)^2 + \dfrac{C}{\lambda_n^2} \int_{t_n}^{t_n+T} \int_\Omega \Big[\, |u_n(x,s)|^2 + |\nabla u_n(x,s)|^2 \Big] \, dxds$
$= CI_n(t_n)^2 + C \int_0^T \int_\Omega \Big[\, |v_n(x,t)|^2 + |\nabla v_n(x,t)|^2 \Big] \, dxdt = C \Big[\, I_n(t_n)^2 + 1 \Big].$

But, $I_n(t_n)^2$ is a bounded sequence due to (5.4). Thus, we have that

$$E(v_n(t)) \leq C$$

for all $0 \leq t \leq T$ and for all $n \in \mathbf{N}$, where $C > 0$ is independent of t and n.

Therefore,

$$\|(v_n)_t(t)\|_{L^2(\Omega)} \leq C \quad \text{e} \quad \| \triangle v_n(t)\|_{L^2(\Omega)} \leq C \tag{5.6}$$

for all $0 \leq t \leq T$ and for all $n \in \mathbf{N}$.

In this point, we use Poincaré inequality and estimate (5.6) to obtain that

$$\|v_n(t)\|^2_{L^2(\Omega)} = \int_\Omega |v_n(x,t)|^2 \, dx = \int_\Omega \frac{1}{\lambda_n^2} |u_n(x,t+t_n)|^2 \, dx$$

$$\leq C_1 \int_\Omega \frac{1}{\lambda_n^2} |\nabla u_n(x,t+t_n)|^2 \, dx = C_1 \int_\Omega |\nabla v_n(x,t)|^2 \, dx$$

$\leq C_2 \int_\Omega | \triangle v_n(x,t)|^2 \, dx \leq C for 0 \leq t \leq T$ and $n \in \mathbf{N}$.

That is, there exists a constant $C > 0$ such that

$$\int_\Omega |v_n(x,t)|^2 \, dx \leq C \tag{5.7}$$

for $0 \leq t \leq T$ and $n \in \mathbf{N}$.

Now, from (5.6) and (5.7) we conclude that the sequence (v_n) is such that

$$(\, v_n)_{n \in \mathbf{N}} \quad \text{is bounded in } \; W^{1,\infty}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H_0^2(\Omega)) \tag{5.8}$$

Now, we claim that

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \rho(x, u_{n_t}(t+tn)) = 0 \; \text{ em } \; L^1([0,T] \times \Omega). \tag{5.9}$$

where we have used the notation that $u_{n_t} = (u_n)_t$.

In order to prove (5.9) we observe that

$$\int_0^T \int_\Omega |\rho(x, u_{n_t}(x,t+t_n))| \, dxds = \int_t^{t+T} \int_\Omega |\rho(x, u_{n_t})| \, dxds$$

$$\leq \int_t^{t+T} \int_{\Omega_1} K_2 a(x) [\, |u_{n_t}|^{r+1} + |u_{n_t}| \,] \, dxds +$$

$$\int_t^{t+T}\int_{\Omega_2} K_4 a(x)[\,|u_{n_t}|^{p+1}+|u_{n_t}|\,]\,dxds$$

due to hypothesis (iii) on the function $\rho(x,s)$, where Ω_1 and Ω_2 were defined in the proof of Lemma 4.8.

We need estimate the last two integrals in according each case for the number p.

Case 1: $0 \le p \le 2$ *if*N=3 *or* $0 \le p < \infty$ if $N = 1$ or $N = 2$.

Proceeding as in Lemma 4.8 we obtain that any solution u of (1.1) satisfies

$$\int_t^{t+T}\int_\Omega |\rho(x,u_t)|\,dxds \le C\left\{\Big[\,E(t){-}E(t{+}T)\Big]^{\frac{1}{p+2}}+\Big[\,E(t){-}E(t{+}T)\Big]^{\frac{p+1}{p+2}}\right\}.$$

In the above estimate we used the fact that $a(x) \in L^\infty(\Omega)$ and the energy identity (3.1).

Using the definition of $D_1(t)$ given in the Proposition 5.1 we obtain that

$$\int_{t_n}^{t_n+T}\int_\Omega |\rho(x,u_{n_t})|\,dxds \le C\Big[\,D_1(t_n)+D_1(t_n)\Big].$$

for $N > 1$. If $N = 1$ the estimate is the same that for $N = 2$.

Thus, using the definition of $I_n(t_n)$ given in (5.3), we conclude that

$$\text{(5.10)}\qquad \frac{1}{\lambda_n}\int_{t_n}^{t_n+T}\int_\Omega |\rho(x,u_{n_t})|\,dxds \le C\Big[\frac{1}{\lambda_n}D_1(t_n)\Big] \le C\ I_n(t_n).$$

for $N > 1$. If $N = 1$ the estimate is the same that for $N = 2$.

Hence, combining (5.4) and (5.10) we obtain that

$$\frac{1}{\lambda_n}\int_{t_n}^{t_n+T}\int_\Omega |\rho(x,u_{n_t})|\,dxds \to 0$$

as $n \to \infty$.

That is,

$$\frac{1}{\lambda_n}\int_0^T\int_\Omega |\rho(x,u_{n_t}(x,t+t_n))|\,dxds \to 0$$

as $n \to \infty$.

Therefore, for this case, (5.9) holds.

Case 2: $-1 < p < 0$.

In this case, we work in a similar way like in the proof of Lemma 4.8. But, here is more easy because we have only $\rho(x,u_t)$ in the term under the integral sign, instead of $\rho(x,u_t)[|u|+|\nabla u|]$ like in the estimates for I_1 and I_2 in the proof of Lemma 4.8. Then, using the definition of $D_2(t)$ given in the Proposition 5.1, we have that

$$\int_t^{t+T}\int_\Omega |\rho(x,u_t)|\,dxds \le C\Big\{\Big[\,E(t)-E(t+T)\Big]^{\frac{p+1}{p+2}}+$$

$$\Big[\,E(t)-E(t+T)\Big]\Big\} \le C\Big[\,D_2(t)+D_2(t)^2\Big]$$

for each solution u of (1.1).

Therefore, using again the definition of $I_n(t)$ it follows that

$$\frac{1}{\lambda_n}\int_{t_n}^{t_n+T}\int_\Omega |\rho(x,u_{n_t})|\,dxds$$

$$\le C\Big[\,\frac{1}{\lambda_n}D_2(t_n)+\frac{1}{\lambda_n}D_2(t_n)^2\Big] \le C\Big[I_n(t_n)+\lambda_n I_n(t_n)\Big].$$

In this point we observe that the sequence (λ_n) is bounded. In fact, Poincaré inequality implies that

$$\lambda_n = \left(\int_{t_n}^{t_n+T}\|u_n(s)\|^2_{L^2(\Omega)}+\|\nabla u_n(s)\|^2_{L^2(\Omega)}\,ds\right)^{\frac{1}{2}}$$

$$\le C\left(\int_{t_n}^{t_n+T}\|\nabla u_n(s)\|^2_{L^2(\Omega)}\,ds\right)^{\frac{1}{2}} \le C\left(\int_{t_n}^{t_n+T}\|\triangle u_n(s)\|^2_{L^2(\Omega)}\,ds\right)^{\frac{1}{2}}$$

$$\le CE(u_n(0)) \le CR,$$

because the initial data for all solutions we are considering belong to the ball $B(0,R)$ for some $R>0$ fixed, that is, $E(0)\le R$.

Then, the property (5.4) for $I_n(t_n)$ implies that the claimed property (5.9) is valid for this case, too.

Now, finely, we can pass the limit of $(v_n(t))_{n\in\mathbf{N}}$. We note that from (5.8) and Aubin-Lions Theorem we conclude that there exist a function $v(t)$ and a subsequence v_n of v_n such that

$$v_n(t)\rightharpoonup v(t)\ \text{ weak star in }\ W^{1,\infty}(0,T;L^2(\Omega))\cap L^\infty(0,T;H_0^2(\Omega))$$

and

$$v_n(t)\to v(t)\ \text{ strong in }\ H_0^1((0,T)\times\Omega). \tag{5.11}$$

Thus, the function $v(t)$ satisfies:

i) $v\in W^{1,\infty}(0,T;L^2(\Omega))\cap L^\infty(0,T;H_0^2(\Omega))$;

ii) $v_{tt} + \triangle^2 v - b(\int_\Omega |\nabla v|^2 dx) \triangle u = 0$ em $(0,T)\times\Omega$ (due to (5.9)) ;

iii) $\displaystyle\int_0^T \int_\omega |v_t|^2 \, dxds = 0$ (due to (5.3), (5.4) and (5.11) ;

iv) $\displaystyle\int_0^T \Big[\|v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)}\Big] \, ds = 1$ (due to (5.5) and (5.11).

From items (ii), (iii) and the Unique Continuation Property (see Kim [9]) it follows that $v \equiv 0$ in $(0,T)\times\Omega$.

This fact contradicts the above item (iv). Then, the Proposition 5.2 is valid.

6. Proof of the Theorem of Stabilization

From the propositions above, it follows that

$$E(t) \le C\Big\{\tilde{D}_i(t)^2 + \int_t^{t+T}\int_\omega |u_t|^2 \, dxds\Big\}$$

for all $t \ge 0$ with $i = 1$ if $0 \le p \le 2$ and $i = 2$ if $-1 < p < 0$. The constant C is independent of the solution u and of t, but it depends of the radius of the ball where the initial data is located.

Using the hypothesis that $a(x) \ge a_0 > 0$ on ω, we obtain that

$$\int_t^{t+T}\int_\omega |u_t|^2 dxds \le \frac{1}{a_0}\int_t^{t+T}\int_\omega a(x)|u_t|^2 dxds.$$

Now, using the same technique used to prove Lemma 4.8(ver [2]), we prove that $\int_t^{t+T}\int_\omega |u_t|^2 dxds$ is also bounded by the same kind of differences of energy.

Thus, we obtain that the energy $E(t)$ satisfies

$$E(t) \le C\Big\{[E(t) - E(t+T)] + [E(t) - E(t+T)]^{\frac{2}{p+2}} + [E(t) - E(t+T)]^{\frac{2(p+1)}{p+2}}\Big\}$$

if $0 \le p \le 2$ and

$$E(t) \le C\Big\{[E(t) - E(t+T)] + [E(t) - E(t+T)]^{\frac{2(p+1)}{p+2}} + [E(t) - E(t+T)]^{\frac{4}{4-p}}\Big\}$$

if $-1 < p < 0$.

Then,

$$\sup_{t\leq s\leq t+T} E(s)^{\frac{1}{K}} \leq C[E(t)-E(t+T)]$$

where

$$\begin{cases} K = min\left\{\frac{2}{p+2}, \frac{2(p+1)}{p+2}\right\} = \frac{2}{p+2} \text{if} 0 \leq p \leq 2 \ \text{ and} \\ N > 2 \ (\ 0 \leq p < \infty \ \text{ if } \ N = 1 \text{ or } 2) \\ K = min\left\{\frac{2(p+1)}{p+2}, \frac{4}{4-p}\right\} = \frac{2(p+1)}{p+2} \text{if} -1 < p < 0 \end{cases}$$

Therefore, we obtained that $E(t)$ satisfies an inequality similar to (3.3).
Then, Nakao's Lemma implies the conclusion of the Theorem of Stabilization.

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