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ON OUTGOING SOLUTIONS FOR A SYSTEM OF TIME–HARMONIC ELASTIC WAVE IN THE EXTERIOR OF A STAR–SHAPED DOMAIN

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Abstract

In this work we consider the propagation of time–harmonic elastic waves outside of a star–shaped domain with a "linear velocity at the boundary". We describe a new approach to investigate results of existence and uniqueness for this exterior problem. To this end, we used a method similar to the one discussed in [11, 12] which has its genesis in [13] and relies on a stationary approach of resonances. The fundamental step of our approach is to reduce the unbounded nature of the problem to a bounded domain introducing an auxiliary boundary condition of Dirichlet type. In particular, we find a large region in the complex plane which is "free" of resonances.

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1. INTRODUCTION AND MOTIVATION

In this note, we use a recent technique to study existence and uniqueness of outgoing solutions for elastic wave equations with boundary conditions that involve the derivative in time of the dynamic. Additionally, we characterize resonances (or scattering frequencies) for this problem. These complex numbers play an important role when we try to obtain further properties of the solutions of the evolution problem.

It is not rare finding in the literature studies of evolution equations with boundary conditions that involve the time derivative of the dynamics, as for example:

$$\beta\frac{\partial u}{\partial n}+\gamma u+u_t=0. \tag{1.1}$$

Such conditions arise in the applications and are of great importance in diverse fields, such as control theory and boundary value problems of partial differential equations (PDE's). Significant mathematical results have already been obtained in this topics, see [3, 4, 6, 25] and [16, 21]. These references also include semigroup techniques and a extensive bibliographic list.

In this context, boundary value problems for the system of elastic waves or the acoustic wave equation with prescribed linear velocity at the boundary play an important role when modelling several interesting physical phenomena occurring in applied science, for instance, those modelling the dynamical vibrations of higher dimensional system of thermoelasticity with a linear boundary feedback, those involving the stabilization of a linear hyperbolic equation with time dependent coefficients or those involving the asymptotic behavior in linear viscoelastic plates (see, [32] and references contained therein).

The study and analysis of equations with dynamical boundary conditions, from a rigorous mathematical point of view, was started around 1960, with the work of J.L. Lions who explored some important models and studied, among other things, the existence of weak solutions by means of variational methods. Since then, these methods have been studied by several authors, for instance [15, 17, 25, 29], and the references therein, where different physical, mathematical and mechanical problems are treated.

In this context, we study in the exterior region $\Omega=\mathbf{R}^3\backslash\overline{\mathcal{O}}$, the system of elastic waves:

$$(1.2)\qquad \begin{cases} \mathbf{u}_{tt} - b^2\mathbf{\Delta u} - (a^2 - b^2)\nabla(\nabla \bullet \mathbf{u}) = e^{i\sigma t}\mathbf{h}(x) \text{ in } \Omega \times \mathbf{R}, \\ \mathbf{u}(x, 0) = \mathbf{f}_0(x) \ \text{ in } \Omega, \\ \mathbf{u}_t(x, 0) = \mathbf{f}_1(x) \text{ in } \Omega \end{cases}$$

subject to a linear condition on $\partial\Omega$ of the form:

$$(1.3)\qquad \mathbf{T}_\eta \mathbf{u}(x, t) + d(x)(\mathbf{m} \bullet \eta) + (\mathbf{m} \bullet \eta)\mathbf{u}_t(x, t) = \mathbf{0} \text{ on } \partial\Omega \times \mathbf{R}\ ,$$

where $\mathbf{T}_\eta$ is the so-called stress-traction vector calculated on surface element:

$$\mathbf{T}_\eta \mathbf{u} = 2b^2 \frac{\partial \mathbf{u}}{\partial \eta} + (a^2 - 2b^2)(\nabla \bullet \mathbf{u})\eta + b^2 \eta \times (\nabla \times \mathbf{u}) \text{ on } \partial\Omega \times \mathbf{R}.$$

We assume that the boundary $\partial\Omega$ of $\mathcal{O}$ is smooth, say of class C^2, $\mathcal{O}$ is an open bounded and connected subset of $\mathbf{R}^3$ which is star-shaped with respect to a point $x_0 = (x_0^1, x_0^2, x_0^3) \in \mathcal{O}$. Let us denote by $\eta = \eta(x)$ the unit normal vector at $x \in \partial\Omega$ directed towards the interior of $\mathcal{O}$. Also, the dot $\bullet$ means the usual inner product in $\mathbf{R}^3$, $\times$ is the usual cross product in $\mathbf{R}^3$, the constants "a" and "b" are given in terms of Lam constants μ and λ: $a^2 = \lambda + 2\mu$, $b^2 = \mu$ with $\mu > 0$ and $\lambda + 2\mu > 0$, σ is a complex number (the frequency) and $\mathbf{h}(x)$ is a given field on Ω, $i = \sqrt{-1}$. Moreover, $\mathbf{u}(x, t) = (u^1(x, t),\ u^2(x, t),\ u^3(x, t))$ is the displacement at the time t and location x in $\mathbf{R}^3$ scattered by $\mathcal{O}$, $\mathbf{u}_t(x, t) = (u_t^1(x, t), u_t^2(x, t), u_t^3(x, t))$, ∇ is the gradient, $\mathbf{u}_{tt}(x, t) = (u_{tt}^1(x, t), u_{tt}^2(x, t), u_{tt}^3(x, t))$, $\nabla \bullet \mathbf{u}$ denotes the (spatial) divergence of the displacement vector $\mathbf{u}$ and $\mathbf{\Delta u} = (\triangle u^1, \triangle u^2, \triangle u^3)$, where $\triangle$ is the usual Laplacian operator. Finally, $\mathbf{f} = (\mathbf{f}_0, \mathbf{f}_1)$ is the initial value for this exterior initial boundary value problem.

We will be interested in time-harmonic solutions and describe an approach to investigate existence and uniqueness of outgoing solutions. In particular, we present an alternative approach to the problem of existence of resonances for this model.

In the literature, resonances are sometime named scattering frequencies, complex singularities, poles of the scattering matrix, etc. Such complex numbers play an important role when we try to obtain further properties of the solutions of the evolution problem or in the so-called inverse problem: Suppose that we know the distribution of such resonances in the complex plane, then the question is whether or not we can recover information about the geometry of the obstacle $\mathcal{O}$ such as its volume, surface area of $\partial\Omega$, etc (see for instance [24, 27, 34, 36] and references therein). Further contributions to the subject where given in [5, 7, 9, 26, 27, 30, 35, 37, 38, 39].

The essence of our method follows the framework developed in [11, 12], which can be briefly described as follows. First, we reduce the unbounded nature of the problem to a bounded domain introducing an auxiliary boundary condition of Dirichlet type. Next, combining uniqueness and existence of solution in the whole space and in a bounded domain, we reduce the problem to a Fredholm type equation the which depends analytically of a parameter. Finally, we use uniqueness theorem to obtain the invertibility for this equation.

In our opinion, the most important qualitative feature from this method is that it combines both simplicity and flexibility. Indeed, as it is observed in [11, 12], the method may be used in a variety of problems, for example, elastic resonators [18] and crack plane problems [2], among others.

The results of this note are in the spirit of those in [11, 12], which have its genesis in [13] and reliy on a stationary approach of resonances.

We present our main results using the strategy described in this introduction. For this purpose, consider time-harmonic waves $\mathbf{u}(x,t)$ of the system (1.2–1.3) which are outgoing:

$$\mathbf{u}(x,t) = e^{i\sigma t}\ \mathbf{v}(x),\ (x,t) \in \Omega \times \mathbf{R} \tag{1.4}$$

Then, we see that the vector field $\mathbf{v}(x)$ in (1.4) must obey the following model:

$$\begin{cases} b^2\, \boldsymbol{\Delta}\mathbf{v} + (a^2 - b^2)\, \nabla(\nabla \bullet \mathbf{v}) + \sigma^2 \mathbf{v} = -\, \mathbf{h} \text{ in } \Omega, \\ \mathbf{T}_\eta \mathbf{v} + (d(x) + i\sigma)\, (\mathbf{m} \bullet \eta)\, \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega \\ \mathbf{v}(x;\sigma) \text{ is outgoing.} \end{cases} \tag{1.5}$$

Remark 1. The outgoing condition for the elastic wave means that $\mathbf{v}(x;\sigma)$ in (1.5) is the $\left[L^2(\Omega)\right]^3$–solution if $\Im(\sigma) > 0$ and the analytic continuation of an $\left[L^2(\Omega)\right]^3$–solution in the region $\Im(\sigma) > 0$ if $\Im(\sigma) \le 0$, where $\Im(\sigma)$ denote the imaginary part of σ.

Hencenfort, we will refer to (1.5) as the system of time-harmonic elastic waves with prescribed linear type velocity at the boundary. Generally speaking a resonance is a complex number σ for which the system (1.5) with $\mathbf{h} \equiv \mathbf{0}$ has a nontrivial solution $\mathbf{v}$. For a general review about system of elastic wave equations we refer to [19, 20, 22, 23]. We shall use standard notation: For any vector $\mathbf{v} = (v^1, v^2, v^3)$ with $v^j \in \mathbf{C}$, $\overline{\mathbf{v}}$ means the conjugate of $\mathbf{v}$, that is $\overline{\mathbf{v}}(x) = (\overline{v}^1(x), \overline{v}^3(x), \overline{v}^3(x))$, the norm of a vector $\mathbf{v}(x)$ is given by

$$\|\mathbf{v}(x)\| = \left(\sum_{j=1}^{3} \left|v^j(x)\right|^2\right)^{1/2}.$$

Given a vector function $\mathbf{f} : \mathbf{R}^3 \to \mathbf{C}^3$, $\mathbf{f}(x) = (f^1(x),\ f^2(x),\ f^3(x))$, the support of $\mathbf{f}$ is given by supp $\mathbf{f} = \cap_{j=1}^{3}$supp f^j where supp f^j denotes the support of the function f^j. Also, the function χ_D denotes the characteristic function of D. Moreover, $C_0^\infty(\Omega)$ denote the space of all C^∞ functions defined on Ω, with compact support in Ω. On the other hand, $H^s(\Omega)$ and $H^r(\partial\Omega)$ denote the usual Sobolev spaces of order s and r on Ω and $\partial\Omega$ respectively, and $H^{-s}(\Omega)$ and $H^{-r}(\partial\Omega)$ their corresponding dual spaces. If E is a vector space then we write $[E]^3 = \oplus_{i=1}^{3} E$ and the norm of a vector $\mathbf{v}$ belonging to $[E]^3$ will be denoted by $\|\cdot\|_{[E]^3}$. Given a positive number R, B_R denotes the ball centered at zero and radius R. Also, we denote by $\partial\overline{B}_R = \{x \in \mathbf{R}^3 : \|x\| = R\}$, where $\|x\|^2 = \sum\limits_{j=1}^{3} (x^j)^2$ whenever $x = (x^1, x^2, x^3)$ and by $[L_R^2(\mathbf{R}^3)]^3$ the space $[L_R^2(\mathbf{R}^3)]^3 = \{\mathbf{v} \in [L^2(\mathbf{R}^3)]^3 : \mathbf{v} = \mathbf{0}$, if $\|x\| \geq R\}$.

Without loss of generality we can assume that $x^0 = (x_0^1, x_0^2, x_0^3) \in \mathcal{O}$. Finally, $\eta = \eta(x)$ will always denote the unit normal vector pointing the exterior of the set where we are considering the equations.

The remaining part of this work is organized as follows. In Section 2 we state and prove our main result concerning the existence and uniqueness of outgoing solutions of (1.2) and (1.3). In Section 3 we consider a second problem, which concerns the existence of resonances associated to the system (1.2) and (1.3). Finally, in the last section, we given an Appendix with some technical results.

2. THE MAIN RESULT

In this section we shall establish the existence and uniqueness of outgoing solutions to system of the elastic waves with prescribed linear velocity on the boundary(1.5).

We recall some lemmas (see for instance [12, 22]) in the whole space $\mathbf{R}^3$:

Lemma 2.1. Let $\sigma \in \mathbf{C}$ with $\Im(\sigma) > 0$ and take $\mathbf{v} \in [H^2(\mathbf{R}^3)]^3$ an outgoing solution of the system

$$b^2\, \mathbf{\Delta}\, \mathbf{v}(x) + (a^2 - b^2)\, \nabla\, (\nabla \bullet \mathbf{v}(x)) + \sigma^2\, \mathbf{v}(x) = \mathbf{0},\ x \in \mathbf{R}^3 \tag{2.1}$$

for $a^2 > \frac{4}{3}b^2 > 0$. Then, we have that:

$$\lim_{R\to\infty} \int_{\|x\|=R} \overline{\mathbf{v}} \bullet \mathbf{T}_\eta \mathbf{v} d\Gamma = 0,$$

where $\overline{\mathbf{v}}$ means the conjugate of $\mathbf{v}$, that is $\overline{\mathbf{v}} = (\overline{v}^1, \overline{v}^2, \overline{v}^3)$.

Lemma 2.2. Let $\sigma \in \mathbf{C}$ with $\Im(\sigma) > 0$. Then, for any $\mathbf{g} \in [L^2_R(R^3)]^3$, the system

$$(2.2) \quad b^2 \,\boldsymbol{\Delta}\, \mathbf{v}(x) + (a^2 - b^2)\, \nabla\, (\nabla \bullet \mathbf{v}(x)) + \sigma^2\, \mathbf{v}(x) = \mathbf{g}(x),\ x \in \mathbf{R}^3$$

admits an outgoing solution $\mathbf{v} \in [H^2(\mathbf{R}^3)]^3$ which depend analytically on σ and $\mathbf{v} = \mathbf{A}(\sigma)\mathbf{g}$, where

$$\mathbf{A}(\sigma) : [L^2_R(\mathbf{R}^3)]^3 \to [H^2(\mathbf{R}^3)]^3$$

is a linear continuous operator. In particular, if $\mathbf{v}_1$ and $\mathbf{v}_2$ are two outgoing solutions of (2.2), then $\mathbf{v}_1(x) = \mathbf{v}_2(x)$ for any $x \in \mathbf{R}^3$.

Let Ω be as in Section 1 and $d = d(x)$ be a nonnegative real-valued function on $\partial\Omega$ such that $d \in C(\partial\Omega)$, with $M = \max_{x\in\partial\Omega} d(x)$ and ε an arbitrary positive real number. Let $P(M;\varepsilon) \subset \mathbf{C}$ the set defined by

$$(2.3) \qquad P(M;\varepsilon) = \{\sigma \in \mathbf{C} : \Im(\sigma) > M + \varepsilon\}.$$

We fix $R > 0$ and let $\overline{\Omega}_R$ the domain given by $\overline{\Omega}_R = \{x \in \Omega : \|x\| \le R\} \cup \partial\Omega$.

The following theorem is the main result of this work.

Theorem 2.3. Let $\sigma \in P(M;\varepsilon)$. Then, for any $\mathbf{h} \in [L^2(\Omega)]^3$ with support contained in Ω_R, the system of elastic waves with prescribed linear boundary velocity (1.5) has a unique solution $\mathbf{v} \in [H^2(\Omega)]^3$. Furthermore, $\mathbf{v} = \mathbf{v}(x,\sigma)$ can be extended in a meromorphic way to the whole complex plane except for a countable number of poles (resonances) in $\mathbf{C}\backslash P(M;\varepsilon)$.

Proof. We first prove uniqueness: Suppose we have two outgoing solutions $\mathbf{v}_1$ and $\mathbf{v}_2$. Let $\mathbf{w}$ be the difference $\mathbf{w} = \mathbf{v}_1 - \mathbf{v}_2$. Thus, $\mathbf{w}$ satisfies (1.5) with $\mathbf{h} = \mathbf{0}$. Let R be a positive real number such that $\partial\overline{B}_R$ is contained in Ω. Now, we use the Betti-Green formula (see for instance [14] or [22, 23]) to obtain

$$(2.4) \qquad \int_{\Omega_R} \overline{\mathbf{w}} \bullet \widetilde{\boldsymbol{\Delta}}\ \mathbf{w}\ dx + \int_{\Omega_R} e(\overline{\mathbf{w}}, \mathbf{w})\ dx = \int_{\partial\Omega_R} \overline{\mathbf{w}} \bullet \mathbf{T}_\eta \mathbf{w}\ d\Gamma,$$

where $\widetilde{\mathbf{\Delta}}\ \mathbf{w} = b^2\, \mathbf{\Delta}\, \mathbf{w} + (a^2 - b^2)\, \nabla(\nabla \bullet \mathbf{w})$ and

$$e(\overline{\mathbf{w}}, \mathbf{w}) = \frac{3a^2 - 4b^2}{3} |\nabla \bullet \mathbf{w}|^2 + \frac{b^2}{2} \sum_{p \neq q} \left| \frac{\partial w_p}{\partial x_q} + \frac{\partial w_q}{\partial x_p} \right|^2 + \frac{b^2}{3} \sum_{p,q=1}^{3} \left| \frac{\partial w_p}{\partial x_p} - \frac{\partial w_q}{\partial x_q} \right|^2.$$

Since $\widetilde{\mathbf{\Delta}}\ \mathbf{w} = -\sigma^2\ \mathbf{w}$ in $\Omega_R \subseteq \Omega$ and $\partial \overline{\Omega}_R = \partial \overline{B}_R \cup \partial\Omega$ then it follows from (2.4) that

$$-\sigma^2 \int_{\Omega_R} ||\mathbf{w}||^2 dx + \int_{\Omega_R} e(\overline{\mathbf{w}}, \mathbf{w})\ dx = \int_{||x||=R} \overline{\mathbf{w}} \bullet \mathbf{T}_\eta \mathbf{w}\ d\Gamma + \int_{\partial\Omega} \overline{\mathbf{w}} \bullet \mathbf{T}_\eta \mathbf{w}\ d\Gamma. \tag{2.5}$$

In addition, using (2.5) together with Lemma 2.1, the boundary condition, and passing to the limit as $R \to \infty$. We have

$$-\sigma^2 \int_{\Omega} ||\mathbf{w}||^2 dx + \int_{\Omega} e(\overline{\mathbf{w}}, \mathbf{w})\ dx = -\int_{\partial\Omega} (d(x) + i\ \sigma)\ (\mathbf{m} \bullet \eta)\ ||\mathbf{w}||^2 d\Gamma. \tag{2.6}$$

Consequently,

$$\int_{\Omega} e(\overline{\mathbf{w}}, \mathbf{w}) dx = \sigma^2 \int_{\Omega} ||\mathbf{w}||^2 dx - \int_{\partial\Omega} d(x)(\mathbf{m} \bullet \eta) ||\mathbf{w}||^2 d\Gamma - i\sigma \int_{\partial\Omega} (\mathbf{m} \bullet \eta) ||\mathbf{w}||^2 d\Gamma$$
$$= \mathrm{I}_1 + I_2, \tag{2.7}$$

with

$$I_1 = [\ \Re^2(\sigma) - \Im^2(\sigma) + 2i\Re(\sigma)\Im(\sigma)\ ] \int_{\Omega} ||\mathbf{w}||^2 dx$$

and

$$I_2 = -\int_{\partial\Omega} d(x)(\mathbf{m} \bullet \eta)\ ||\mathbf{w}||^2 d\Gamma + [\Im(\sigma) - i\Re(\sigma)] \int_{\partial\Omega} (\mathbf{m} \bullet \eta)\ ||\mathbf{w}||^2 d\Gamma,$$

where $\Re(\sigma)$ denote the real part of σ.

Now, by taking the imaginary part in (2.7), it follows that

$$0 = 2\Re(\sigma)\Im(\sigma)\int_\Omega ||\mathbf{w}||^2 dx - \Re(\sigma)\int_{\partial\Omega} (\mathbf{m}\bullet\eta)\ ||\mathbf{w}||^2 d\Gamma\ . \tag{2.8}$$

Also, the real part of (2.7) gives us that

$$\int_\Omega e(\overline{\mathbf{w}}, \mathbf{w})dx =$$

$$(\Re^2(\sigma) - \Im^2(\sigma))\int_\Omega ||\mathbf{w}||^2 dx + \int_{\partial\Omega} [\Im(\sigma) - d(x)](\mathbf{m}\bullet\eta)\ ||\mathbf{w}||^2 d\Gamma\ . \tag{2.9}$$

There are two possibilities:

(a) If $\Re(\sigma) = 0$, then from (2) we obtain that

$$\int_\Omega e(\overline{\mathbf{w}}, \mathbf{w})dx =$$

$$-\Im^2(\sigma)\int_\Omega ||\mathbf{w}||^2 dx + \int_{\partial\Omega} [\Im(\sigma) - d(x)](\mathbf{m}\bullet\eta)||\mathbf{w}||^2 d\Gamma. \tag{2.10}$$

Taking into account that $\mathbf{m}\bullet\eta$ <0 on $\partial\Omega$ and $M = \max_{x\in\partial\Omega} d(x) \geq d(x)$ with $M < \varepsilon + M < \Im(\sigma)$, $\Im(\sigma) - d(x) \geq 0$ on $\partial\Omega$, it follows that

$$\int_{\partial\Omega} [\Im(\sigma) - d(x)](\mathbf{m}\bullet\eta)||\mathbf{w}||^2 d\Gamma \leq 0.$$

On the other hand, we know that $\int_\Omega e(\overline{\mathbf{w}}, \mathbf{w})dx \geq 0$ then (2) yields to $\mathbf{w} = \mathbf{0}$ a.e. on Ω.

(b) If $\Re(\sigma) \neq 0$, then from (2.8) we obtain that

$$0 = 2\Im(\sigma)\int_\Omega ||\mathbf{w}||^2 dx - \int_{\partial\Omega} (\mathbf{m}\bullet\eta)\ ||\mathbf{w}||^2 d\Gamma.$$

Thus

$$2\Im(\sigma)\int_\Omega ||\mathbf{w}||^2 dx = \int_{\partial\Omega} (\mathbf{m}\bullet\eta)\ ||\mathbf{w}||^2 d\Gamma.$$

Now, $\sigma \in P(M;\varepsilon)$ and $\mathbf{m}\bullet\eta$ <0 on $\partial\Omega$, which implies $\mathbf{w} = \mathbf{0}$ a.e. on Ω.

Next, we prove existence: To do this we introduce the following space

$$\left[H^1_{\partial\Omega}(\Omega_R)\right]^3 = \left\{\mathbf{u} \in \left[H^1(\Omega_R)\right]^3 : \mathbf{u} = \mathbf{0} \text{ on } \partial\overline{B}_R\right\}.$$

Due to technical reasons we prefer to divide the proof into several lemmas.

Lemma 2.4. Let $\widetilde{\mathbf{g}} \in [H^{1/2}(\partial\Omega)]^3$ and $\sigma \in P(M;\varepsilon)$ (given by (2.3)). Then, the problem

$$\begin{cases} b^2\,\mathbf{\Delta}\,\mathbf{w} + (a^2-b^2)\,\nabla(\nabla\bullet\mathbf{w}) = \mathbf{0} & \text{in } \Omega_R, \\ \mathbf{T}_\eta\mathbf{w} + (d(x)+i\sigma)\,(\mathbf{m}\bullet\eta)\,\mathbf{w} = \widetilde{\mathbf{g}} & \text{on } \partial\Omega, \\ \mathbf{w} = \mathbf{0} & \text{on } \partial\overline{B}_R \end{cases} \tag{2.11}$$

has a unique solution $\mathbf{w} \in [H^2(\Omega_R)]^3 \cap \left[H^1_{\partial\Omega}(\Omega_R)\right]^3$. The existence and uniqueness of solution to (2.11) can be established by an standard argument. For the sake of completeness we present a proof in the Appendix.

Let $R>0$, $R_0>0$ be such that $B_{R_0} \subseteq \mathcal{O}$ and $\partial\Omega \subseteq B_R$. We choose $\zeta = \zeta(x) \in C_0^\infty(\mathbf{R}^3)$ satisfying the following conditions

(A) $\partial\Omega \subset \operatorname{supp}\zeta \subset B_R/B_{R_0}$,

(B) $\zeta \equiv 1$ in a neighborhood of $\partial\Omega$.

Let us introduce the following function

$$\mathbf{v}(x) = \mathbf{v}_0(x) + \zeta(x)\ \tilde{\mathbf{u}}(x), \quad x \in \mathbf{R}^3, \tag{2.12}$$

where $\tilde{\mathbf{u}}$ is the Caldern extension (see [31], theorem 5.3.1) to $\mathbf{R}^3$ of the solution $\mathbf{w} \in [H^2(\Omega_R)]^3 \cap \left[H^1_{\partial\Omega}(\Omega_R)\right]^3$ of system (2.11) with

$$\widetilde{\mathbf{g}} = -\mathbf{T}_\eta\mathbf{v}_0 - (d(x)+i\sigma)\,(\mathbf{m}\bullet\eta)\,\mathbf{v}_0 \in [H^{1/2}(\partial\Omega)]^3, \tag{2.13}$$

where $\mathbf{v}_0 = \mathbf{v}_0(x)$ satisfies (see Lemma 2.2) the system

$$b^2\,\mathbf{\Delta}\mathbf{v}_0(x) + (a^2-b^2)\,\nabla(\nabla\bullet\mathbf{v}_0(x)) + \sigma^2\,\mathbf{v}_0(x) = \mathbf{g}(x),\ x \in \mathbf{R}^3$$

with $\mathbf{g} = \mathbf{f}_0 = \chi_{\Omega_R}\mathbf{f}$ for a given element $\mathbf{f} \in [L^2(\Omega)]^3$. Additionally, we require that $\mathbf{v}_0$ be outgoing.

Clearly from (2.12) we obtain

$$\mathbf{T}_\eta\mathbf{v} + (d(x)+i\sigma)\,(\mathbf{m}\bullet\mathbf{n})\,\mathbf{v} = \mathbf{0} \text{ on } \partial\Omega.$$

Furthermore, the property (A) implies that $\mathbf{v} = \mathbf{v}_0$ on $\mathbf{R}^3/\overline{B}_R$. Since $\mathbf{v}_0$ is outgoing, so is $\mathbf{v}$. Consequently, for any $\mathbf{h} \in [L^2(\Omega)]^3$ with $\operatorname{supp}\mathbf{h} \subseteq \Omega_R$ and $\sigma \in P(M;\varepsilon)$, we deduce that $\mathbf{v}$, given by (2.12), will solve the system (1.5) if and only if,

$$\begin{aligned} -\mathbf{h} &= b^2\,\mathbf{\Delta}\mathbf{v} + (a^2-b^2)\,\nabla(\nabla\bullet\mathbf{v}) + \sigma^2\,\mathbf{v} \\ &= \mathbf{f}_0 + b^2\,\mathbf{\Delta}(\zeta(x)\tilde{\mathbf{u}}) + (a^2-b^2)\,\nabla[\nabla\bullet(\zeta(x)\tilde{\mathbf{u}})\ ] + \sigma^2\,\zeta(x)\tilde{\mathbf{u}}. \end{aligned} \tag{2.14}$$

Observe that due to the choice (A) and the fact that $\mathbf{f}_0 = \chi_{\Omega_R}\mathbf{f}$, we have that (2.14) holds in the region $\mathbf{R}^3 \backslash B_R$. In fact, the supp $\mathbf{h} \subseteq \Omega_R$. So, we deduce that $\mathbf{v}$ given by (2.12) will be solution of (1.5) if and only if, the identity

$$(2.15) \quad -\mathbf{h} = \mathbf{f} + b^2\, \mathbf{\Delta}(\zeta(x)\mathbf{w}) + (a^2 - b^2)\ \nabla\ (\nabla \bullet \mathbf{w}\ \zeta(x)) + \sigma^2\zeta(x)\mathbf{w}.$$

holds for any $x \in \Omega_R$, because $\widetilde{\mathbf{u}} = \mathbf{w}$ on Ω_R.

Now we use the well known vector identity

$$(2.16) \qquad \nabla \times\ (\nabla \times\ \mathbf{F}) = -\mathbf{\Delta}\ \mathbf{F} +\ \nabla\ (\nabla \bullet \mathbf{F}),$$

where $\mathbf{F} = (F_1, F_2, F_3)$ is a field, together with the fact that $\mathbf{w}$ solves

$$b^2\, \mathbf{\Delta}\ \mathbf{w}(x) + (a^2 - b^2)\, \nabla(\nabla \bullet \mathbf{w}(x)) = \mathbf{0},\ x \in \Omega_R$$

to rewrite (2.15) in the form

$$(2.17) \qquad -\mathbf{h} = \mathbf{f} + \mathbf{G}_\zeta(\sigma)\mathbf{w},$$

where $\mathbf{G}_\zeta(\sigma) : [H^2(\Omega_R)]^3 \cap \left[H^1_{\partial\Omega}(\Omega_R)\right]^3 \to [H^1(\Omega_R)]^3$ is given by

$$(2.18) \qquad \begin{aligned} \mathbf{G}_\zeta(\sigma)\mathbf{w} &= (a^2 + b^2)[(\nabla\ \zeta \bullet \nabla)\ \mathbf{w}] + [b^2\triangle\zeta + \sigma^2\zeta]\ \mathbf{w} + \\ &+ (a^2 - b^2)[(\mathbf{w} \bullet \nabla)\ \nabla\ \zeta + \nabla\zeta \times\ (\nabla \times\ \mathbf{w}) + \nabla\ \zeta(\ \nabla \bullet \mathbf{w})]. \end{aligned}$$

Finally, we want to rewrite the operator relation (2.17) as a composition of operators $L(\sigma)$, Λ_η, $Q_R(\sigma)$ and $\mathbf{A}_M(\sigma)$. Consider the operators in the diagram:

$$\begin{array}{ccccc} \left[L^2(\Omega_R)\right]^3 & \xrightarrow{A_M(\sigma)} & \left[H^2(R^3)\right]^3 & \xrightarrow{Q_R} & \left[H^2(\Omega)\right]^3 \\ \uparrow & & B_\varsigma(\sigma) \hookleftarrow & & \downarrow\Lambda_\eta \\ \left[H^1(\Omega_R)\right]^3 & \xleftarrow{G_\varsigma(\sigma)} & \left[H^2(\Omega_R)\right]^3 \cap \left[H^1_{\partial\Omega}(\Omega_R)\right]^3 & \xleftarrow{L(\sigma)} & \left[H^{\frac{1}{2}}(\partial\Omega)\right]^3 \end{array}$$

(2.19)

Where $L(\sigma)$ is the solution operator associated to the system (2.11) with $\widetilde{\mathbf{g}}$ as in (2.13), that is, $L(\sigma)\widetilde{\mathbf{g}} = \mathbf{w}$, Λ_η (the trace) is defined as $\Lambda_\eta\mathbf{v}_0 = \widetilde{\mathbf{g}}$,

Q_R is the restriction operator, $Q_R(\mathbf{v}_0) = \mathbf{v}_{0|\Omega_R}$ and $\mathbf{A}_M(\sigma) = \mathbf{A}(\sigma)M$, where the operator $M : [L^2(\mathbf{R}^3)]^3 \mapsto [L^2(\Omega_R)]^3$ is given by

$$(M\ \mathbf{k})(x) = \chi_{\Omega_R}\mathbf{k}(x) \quad \text{for } \mathbf{k} \in \left[L^2(\mathbf{R}^3)\right]^3$$

and $\mathbf{A}(\sigma)$ is the solution operator of the system

$$b^2\Delta\mathbf{v}_0 + (a^2 - b^2)\nabla(\nabla \bullet \mathbf{v}_0) + \sigma^2\mathbf{v}_0 = \mathbf{f}_0 \text{ in } \mathbf{R}^3,$$

that is, $\mathbf{A}(\sigma)\mathbf{f}_0 = \mathbf{v}_0$. Clearly all the above operators are linear and continuous.

Remark 2. One can easily check that the operators $L(\sigma)$ and Λ_η depend analytically on σ. In fact, the function $\mathbf{v}_0$ has this property and $\widetilde{\mathbf{g}}$ depends intrinsically of $\mathbf{v}_0$ ($\mathbf{v}_0$ is the solution of the system given above).

Now, we can rewrite (2.17) in the form

$$-\,\mathbf{h} = \mathbf{f} + B_\zeta(\sigma)\mathbf{f} \tag{2.20}$$

where

$$B_\zeta(\sigma) = G_\zeta(\sigma)L(\sigma)\Lambda_\eta\, Q_R\, \mathbf{A}_M(\sigma) \tag{2.21}$$

Lemma 2.5. With the above considerations (and the assumptions of Theorem 2.3) we have

1) The set $\{B_\zeta(\sigma)\}$ is a family of compact operators from $[L^2(\Omega_R)]^3$ into itself.

2) The homogeneous equation $\mathbf{f} + B_\zeta(\sigma)\mathbf{f} = 0$ has only the trivial solution.

Proof.

1) Taking $\widetilde{\mathbf{g}}$ as in (2.13), we have that each $B_\zeta(\sigma)$ is a compact operator. This follows from the fact that the embedding $[H^1(\Omega_R)]^3 \to [L^2(\Omega_R)]^3$ is compact.

2) Let $\mathbf{f} \in [L^2(\Omega_R)]^3$ be such that $\mathbf{f} + B_\zeta(\sigma)\mathbf{f} = \mathbf{0}$. Then, the function $\mathbf{v}$ is a solution of the system

$$\begin{cases} b^2\,\boldsymbol{\Delta}\,\mathbf{v} + (a^2 - b^2)\,\nabla(\nabla \bullet \mathbf{v}) + \sigma^2\,\mathbf{v} = \mathbf{0} \text{ in } \Omega, \\ \mathbf{T}_\eta \mathbf{v} + (d(x) + i\sigma)\,(\mathbf{m} \bullet \eta)\,\mathbf{v} = \mathbf{0} \text{ on } \partial\Omega \\ \mathbf{v}(x,\sigma) \text{ is outgoing.} \end{cases} \tag{2.22}$$

Due to the uniqueness result we have already proven, we obtain that $\mathbf{v} = \mathbf{0}$ on Ω. In particular, from (2.12) it follows that $-\zeta(x)\widetilde{\mathbf{u}} = \mathbf{v}_0$ on Ω. In particular $\mathbf{v}_0 = \mathbf{0}$ in $\mathbf{R}^3 \backslash B_R$. Since supp $\zeta \subseteq B_R \backslash B_{R_0}$ then we get that $\mathbf{v}_0 = \mathbf{0}$ in $\partial \overline{B}_R$. Let us consider the auxiliary function

$$\mathbf{z}(x) = \psi(x)\mathbf{v}_0(x) + (1 - \psi(x))\widetilde{\mathbf{u}}(x) \tag{2.23}$$

where

$$\psi(x) = \begin{cases} 1 \text{ if } x \in \overline{\mathcal{O}} \\ 0 \text{ if } x \in \Omega_R \cup \partial \overline{B}_R. \end{cases}$$

Note that $\mathbf{z} \in \left[H^2(B_R)\right]^3$. Furthermore,

$$b^2\,\boldsymbol{\Delta}\,\mathbf{z} + (a^2 - b^2)\,\nabla(\nabla \bullet \mathbf{z}) = -\sigma^2 \psi(x)\mathbf{v}_0 \text{ on } B_R.$$

Note also that $\mathbf{z} = \mathbf{0}$ on $\partial \overline{B}_R$, because $\mathbf{v}_0 = \widetilde{\mathbf{u}} = \mathbf{0}$ on $\partial \overline{B}_R$. Now, the Betti-Green formula on B_R yields to

$$\int_{B_R} \overline{\mathbf{z}} \bullet \widetilde{\Delta} \mathbf{z}\; dx + \int_{B_R} e(\overline{\mathbf{z}}, \mathbf{z}) dx = -\int_{\partial \overline{B}_R} \overline{\mathbf{z}} \bullet \mathbf{T}_\eta \mathbf{z}\, d\Gamma = 0$$

which implies that

$$\int_{B_R} e(\overline{\mathbf{z}}, \mathbf{z})\; dx = \sigma^2 \int_{B_R} \psi(x) ||\mathbf{v}_0||^2\, dx. \tag{2.24}$$

From (2.24) we deduce that

$$\int_{B_R} e(\overline{\mathbf{z}}, \mathbf{z}) dx = \left[(\Re^2(\sigma) - \Im^2(\sigma)\right] \int_{B_R} \psi(x) ||\mathbf{v}_0||^2\, dx \tag{2.25}$$

and

$$0 = 2\,\Re(\,\sigma)\,\Im(\sigma) \int_{B_R} \psi(x) ||\mathbf{v}_0||^2\, dx. \tag{2.26}$$

Since $\sigma \in P(M;\varepsilon)$, if $\Re(\,\sigma) \neq 0$, then we conclude from (2.26) that $\mathbf{v}_0 = \mathbf{0}$ a.e. in $\overline{\mathcal{O}}$. Now if $\Re(\,\sigma) = 0$, then (2.25), together with the fact that $\int_{B_R} e(\overline{\mathbf{z}}, \mathbf{z}) dx \geq 0$ imply that $\mathbf{v}_0 = \mathbf{0}$ a.e. in $\overline{\mathcal{O}}$.

In conclusion, for any $\sigma \in P(M;\varepsilon)$, the function $\mathbf{z} = \mathbf{z}(x)$ given by (2.23) belongs to $\left[H^2(B_R)\right]^3$ and solves

$$\begin{cases} \widetilde{\Delta}\mathbf{z} = b^2\Delta\mathbf{z} + (a^2 - b^2)\nabla(\nabla \bullet \mathbf{z}) = \mathbf{0} \text{ in } B_R, \\ \mathbf{z} = \mathbf{0} \text{ on } \partial\overline{B}_R. \end{cases} \tag{2.27}$$

A well known result from elliptic theory (see for instance [33]) implies that the solution of (2.27) is identically zero in B_R, that is $\widetilde{\mathbf{u}} = \mathbf{0}$ in Ω_R. This, together with $-\zeta(x)\widetilde{\mathbf{u}}(x) = \mathbf{v}_0(x)$ for $x \in \Omega_R$ allow us to deduce that

$$\mathbf{0} = b^2\Delta\mathbf{v}_0 + (a^2 - b^2)\nabla(\nabla \bullet \mathbf{v}_0) + \sigma^2\mathbf{v}_0 = \mathbf{f}(x), \quad x \in \Omega_R$$

i.e. $\mathbf{f} = \mathbf{0}$ in Ω_R. Using Lemmas 2.4 and 2.5 we conclude the proof of existence of Theorem 2.3. In fact, using the Fredholm Theory, it follows that the equation

$$\mathbf{f} + B_\zeta(\sigma)\mathbf{f} = -\mathbf{h}$$

is uniquely solvable. The remaining part of Theorem 2.3 (that is, the meromorphic extension) will be proven in the next section.

3. MEROMORPHIC EXTENSION

This section is devoted to study the extension of the solution of the boundary problem (1.5) to all complex numbers σ except for some countable number of complex singularities called resonances. Our approach borrows some ideas on the subject presented in [11], [12] and [9]. We follow the same notations as in the previous sections.

The following theorem is classic and is given to Steinberg's; for a more general versions, see [40].

Theorem 3.1 (Steinberg's theorem)

If $\{T(\sigma)\}$ is an analytic family of compact operators for σ, then either $I + T(\sigma)$ is nowhere invertible or else $[I + T(\sigma)]^{-1}$ is meromorphic in σ.

Now, we emphasize that the solution $\mathbf{v}_0$ of the system

$$b^2\,\Delta\mathbf{v}_0 + (a^2 - b^2)\nabla(\nabla \bullet \mathbf{v}_0) + \sigma^2\mathbf{v}_0 = \mathbf{f} \tag{3.1}$$

depends analytically of $\sigma \in P(M;\varepsilon)$. So, evidently all the operators considered in (2.19), have this property. By Lemma 2.5 the family $\{B_\xi(\sigma)\}$ consist of compact operators from $\left[L^2(\Omega_R)\right]^3$ into itself.

Theorem 3.2. The inverse operators $[I + B_\xi(\sigma)]^{-1}$ have an analytic extension from the set $P(M;\varepsilon)$ given by (2.3) to all the complex plane except for a countable set of poles, called resonant frequencies. Furthermore, σ is a resonant frequency of the operator $[I + B_\xi(\sigma)]^{-1}$ if and only if, the operator equation $\mathbf{f} + B_\xi(\sigma)\mathbf{f} = 0$ has nonzero solutions.

Proof. We use the theorem 3.1 to conclude that either (a) The operators $[I + B_\xi(\sigma)]^{-1}$ are never invertible for $\sigma \in \mathbf{C}$ or (b) There is $\sigma_0 \in \mathbf{C}$ such that the operator $[I + B_\xi(\sigma_0)]^{-1}$ is invertible. From Theorem 2.3 we have the existence and uniqueness of the solution for system (1.5) for all $\sigma \in \mathbf{C}$ such that $\sigma \in P(M;\varepsilon)$. The equivalence between problem (2.20) discussed in the previous section with our original system says that we are in case (b). In this case, Steinberg's theorem also establishes that the operator $[I + B_\xi(\sigma)]^{-1}$ is defined analytically in the whole complex plane $\mathbf{C}$, except for a countable number of poles.

4. APPENDIX

Before beginning the proof of Lemma 2.4, we remark that the regularity of the solution of the system (2.11) is related to the regularity of the solution of a auxiliary boundary value problem.

In fact, we began by recalling that $[H^1_{\partial\Omega}(\Omega_R)]^3 = \left\{\mathbf{u} \in [H^1(\Omega_R)]^3 : \mathbf{u} = \mathbf{0} \text{ on } \partial B_R\right\}$. In what follows, we fix $q \in (0,1]$, and set $s \in (0, \frac{q}{2})$.

Lemma A1.

Suppose $\mathbf{f} \in [L^2(\Omega_R)]^3$, $\mathbf{h} \in [H^s(\partial\Omega)]^3$and $\mathbf{u}$ a weak solution of the following elliptic problem:

$$\left\{\begin{array}{l} -b^2\mathbf{\Delta u} - (a^2 - b^2)\nabla(\nabla\cdot\mathbf{u}) = \mathbf{f} \quad \text{in} \quad \Omega_R, \\ 2b^2\dfrac{\partial\mathbf{u}}{\partial\eta} + (a^2-b^2)\eta(\nabla\cdot\mathbf{u}) + d(x)(\mathbf{m}\cdot\eta)\mathbf{u} = \mathbf{h} \quad \text{on} \quad \partial\Omega, \\ \mathbf{u} = \mathbf{0} \quad \text{on} \quad \partial\overline{B}_R. \end{array}\right. \tag{3.2}$$

Then $\mathbf{u} \in [H^p(\Omega_R)]^3 \cap [H^1_{\partial\Omega}(\Omega_R)]^3$ with $p = \frac{3}{2} + s > \frac{3}{2}$.

Remark 3. In this part we repeated and we adapted, only for convenience of the reader, a proof given in ([29], p.p., 292-295). In this work they also prove the existence of weak solutions.

Proof: Let $\eta = (\eta_1, \eta_2, \eta_3)$ denote the unit normal on $\partial\Omega_R = \partial\Omega \cup \partial B_R$ directed to exterior of Ω_R and consider the following system

$$\sum_{j=1}^{3} [(a^2 - b^2)\eta_i \eta_j + 2\delta_{ij} b^2]\zeta_j = h_i, \quad i = 1, 2, 3, \tag{3.3}$$

where δ_{ij} denote the Kronecker symbol, i.e;

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

It is easy to see that the system has a solution $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \left[H^1(\partial\Omega_R)\right]^3$ for $\mathbf{h} \in [H^s(\partial\Omega_R)]^3$. By a trace theorem (see, for instance [28], p. 39, Theorem 8.3), there exists $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in [H^p(\Omega_R)]^3 \cap \left[H^1_{\partial\Omega}(\Omega_R)\right]^3$ with

$$\frac{\partial \varphi}{\partial \mathbf{n}} = \zeta, \;\; \varphi = \mathbf{0} \text{ on } \partial\Omega_R.$$

Let $\{\tau^1(x), \tau^2(x)\}$ be a tangential vector field such that $\{\mathbf{n}(x), \tau^1(x), \tau^2(x)\}$ forms an orthonormal basis in $\mathbf{R}^3$ for almost all $x \in \partial\Omega_R$. Hence, there exist $\gamma^{k,j}$ $(j = 1, 2, 3;\ k = 1, 2)$ depending on $\{\eta(x), \tau^1(x), \tau^2(x)\}$ such that

$$\begin{aligned} \frac{\partial \varphi_j}{\partial x_j} &= \eta_j \frac{\partial \varphi_j}{\partial \eta} + \gamma^{1,j} \frac{\partial \varphi_j}{\partial \tau^1} + \gamma^{2,j} \frac{\partial \varphi_j}{\partial \tau^2} \\ &= \eta_j \frac{\partial \varphi_j}{\partial \eta} \\ &= \eta_j \zeta_j \;\text{ on }\; \partial\Omega_R, \;\; j = 1, 2, 3. \end{aligned}$$

Therefore, it follows from (3.3) that

$$2b^2 \frac{\partial \varphi}{\partial \eta} + (a^2 - b^2)\eta(\nabla \cdot \varphi) + d(x)(\mathbf{m} \cdot \eta)\varphi = \mathbf{h} \text{ on } \partial\Omega.$$

Put $\psi = \mathbf{u} - \varphi$, then ψ satisfies

$$\begin{cases} -b^2 \mathbf{\Delta}\psi - (a^2 - b^2)\nabla(\nabla \cdot \psi) = \mathbf{F} \;\text{ in }\; \Omega_R, \\ 2b^2 \dfrac{\partial \psi}{\partial \eta} + (a^2 - b^2)\eta(\nabla \cdot \psi) + d(x)(\mathbf{m} \cdot \eta)\psi = \mathbf{0} \text{ on } \;\partial\Omega, \\ \psi = \mathbf{0} \quad\text{on}\;\; \partial\overline{B}_R, \end{cases} \tag{3.4}$$

where $\mathbf{F} = \mathbf{f} - b^2\mathbf{\Delta}\varphi - (a^2 - b^2)\nabla(\nabla \cdot \varphi) \in \left[H^{p-2}(\Omega_R)\right]^3$. Thus, problem (3.2) is equivalent to (3.4). By classical varational methods (see, e.g.,[28]), for every $\mathbf{F} \in \left[(H^1(\Omega_R))'\right]^3$, the problem (3.4) has a unique weak solution $\psi \in \left[H^1_{\partial\Omega}(\Omega_R)\right]^3$ in the sense of distribution

$$\int_{\Omega_R} (b^2\nabla\psi{\cdot}\nabla\phi + (a^2-b^2)(\nabla\cdot\psi)(\nabla\cdot\phi)dx + \int_{\partial\Omega} d(x)(\mathbf{m}\cdot\eta)\psi\cdot\phi d\Gamma$$
$$= \int_{\Omega_R} \mathbf{F}\cdot\phi\, dx, \tag{3.5}$$

for all $\phi \in \left[H^1_{\partial\Omega}(\Omega_R)\right]^3$. Moreover, by the classical Nirenberg's traslation method (see, eg., [1], p.107, Lemma 9.2) or ([28], p. 124), we prove that, if $\mathbf{F} \in \left[L^2(\Omega_R)\right]^3$, then $\psi\in \left[H^2(\Omega_R)\right]^3 \cap \left[H^1_{\partial\Omega}(\Omega_R)\right]^3$. Since the regularity is local property, it suffices to prove that, for any $x \in \overline{\Omega}_R$, there exists a neighborhood $O(x)$ such that $\psi \in \left[H^2(O(x)\cap\Omega_R)\right]^3$. We only consider the case $x \in \partial\Omega$ since the case $x \in \Omega$ is easier (see, Lemma 9.2 of [1], p.107). In fact, for simplicity, we may as well assume that $x = 0 \in \partial\Omega$ and the boundary is flat with the normal oriented in the direction x_3 since the general case can be transformed into the special case by a mapping of class C^2. Therefore, there exists a hemisphere $O_\epsilon = \{x : |x| < \epsilon,\ x_3 > 0\}$ such that $O_\epsilon \subset \Omega_R$ and $\partial\Omega_{2O} = \left\{x \in \overline{O}_\epsilon : x_3 = 0\right\} \subset \partial\Omega$. Let $0 < \epsilon' < \epsilon$ and $\epsilon" = \frac{1}{2}(\epsilon' + \epsilon)$ and let ξ denote a real function which is infinitely differentiable on $\mathbf{R}^3$ and $\xi = 1$ on $O_{\epsilon'}$ and $\xi = 0$ outside $O_{\epsilon"}$. Note that ξ need not vanish on the flat part $\partial\Omega_{2O}$ of the boundary of O_ϵ. By (3), we have for any $\phi \in [C_0^\infty(O_\epsilon \cup \partial\Omega_{2O})]^3$

$$\int_{O\epsilon} (b^2\nabla\psi{\cdot}\nabla\phi + (a^2-b^2)(\nabla\cdot\psi)(\nabla\cdot\phi)dx + \int_{\partial\Omega_{2O}} d(x)(\mathbf{m}\cdot\eta)\psi\cdot\phi d\Gamma$$
$$= \int_{O_\epsilon} \mathbf{F}\cdot\phi\, dx. \tag{3.6}$$

define the bilinear form $B(\psi,\phi)$ by

$$B(\psi,\phi) = \int_{O\epsilon} (b^2\nabla\psi{\cdot}\nabla\phi + (a^2-b^2)(\nabla\cdot\psi)(\nabla\cdot\phi)dx + \int_{\partial\Omega_{2O}} d(x)(\mathbf{m}\cdot\eta)\psi\cdot\phi d\Gamma. \tag{3.7}$$

Then we have

(3.8) $$|B(\psi,\phi)| \leq ||\mathbf{F}|| \cdot ||\phi||, \text{ for all } \phi \in [C_0^\infty(O_\epsilon \cup \partial\Omega_{2O})]^3.$$

For a real number, we define the difference quotients δ_ϱ^i, $i = 1, 2, 3$ by
(*a*) $\delta_\varrho^1 \mathbf{u} = \varrho^{-1}[\mathbf{u}(x_1+\varrho, x_2, x_3) - \mathbf{u}(x_1, x_2, x_3)]$,
(*b*) $\delta_\varrho^2 \mathbf{u} = \varrho^{-1}[\mathbf{u}(x_1, x_2+\varrho, x_3) - \mathbf{u}(x_1, x_2, x_3)]$,
(*c*) $\delta_\varrho^3 \mathbf{u} = \varrho^{-1}[\mathbf{u}(x_1, x_2, x_3+\varrho) - \mathbf{u}(x_1, x_2, x_3)]$.
We now want to estimate the difference quotients $\delta_\varrho^i(\xi\psi)$ for $i = 1, 2$. Since

$$\begin{aligned}
B(\delta_\varrho^i(\xi\psi),\phi) &= \int_{O_\epsilon} (b^2 \nabla \delta_\varrho^i(\xi\psi) \cdot \nabla\phi + (a^2-b^2)(\nabla \cdot \delta_\varrho^i(\xi\psi))(\nabla\cdot\phi) dx \\
&\quad + \int_{\partial\Omega_{2O}} d(x)(\mathbf{m}\cdot\eta)\delta_\varrho^{\mathbf{i}}(\xi\psi)\cdot\phi d\Gamma \\
&= \int_{O_\epsilon} (b^2 \delta_\varrho^i(\nabla(\xi\psi)) \cdot \nabla\phi + (a^2-b^2)\delta_\varrho^i(\nabla\cdot(\xi\psi))(\nabla\cdot\phi)) dx \\
&\quad + \int_{\partial\Omega_{2O}} d(x)(\mathbf{m}\cdot\eta)\delta_\varrho^{\mathbf{i}}(\xi\psi)\cdot\phi d\Gamma \\
&= \int_{O\epsilon} (b^2 \delta_\varrho^i(\psi_j\nabla\xi + \xi\nabla\psi_j) \cdot \nabla\phi + (a^2-b^2)\delta_\varrho^i(\nabla\cdot(\xi\psi))(\nabla\cdot\phi)) dx \\
&\quad + \int_{\partial\Omega_{2O}} d(x)(\mathbf{m}\cdot\eta)\delta_\varrho^{\mathbf{i}}(\xi\psi)\cdot\phi d\Gamma
\end{aligned}$$

$$\begin{aligned}
= \;& b^2 \int_{O_\epsilon} (\delta_\varrho^i(\psi_j\nabla\xi)\cdot\nabla\phi_j + \xi\nabla\psi_j\cdot\delta_{-\varrho}^i(\nabla\phi_j)) dx \\
& +(a^2-b^2)\int_{O\in} (\delta_\varrho^i(\nabla\xi\cdot\psi)(\nabla\cdot\phi) + \xi\nabla\cdot\psi)\delta_{-\varrho}^i(\nabla\cdot\phi) dx \\
& + \int_{\partial\Omega_{2O}} d(x)(\mathbf{m}\cdot\eta)\xi\psi\cdot\delta_{-\varrho}^{\mathbf{i}}(\phi) d\Gamma
\end{aligned}$$

So,

$$\begin{aligned}
B(\delta_\varrho^i(\xi\psi),\phi) &= b^2 \int_{O\in} (\delta_\varrho^i(\psi_j\nabla\xi)\cdot\nabla\phi_j + \nabla\psi_j\cdot\nabla(\xi\delta_{-\varrho}^i\phi_j) - \nabla\psi_j\cdot\nabla(\xi\delta_{-\varrho}^i\phi_j)) dx \\
&\quad +(a^2-b^2)\int_{O\in} (\delta_\varrho^i(\nabla\xi\cdot\psi)(\nabla\cdot\phi) + \nabla\cdot\psi\nabla\cdot(\xi\delta_{-\varrho}^i\phi))\; dx \\
&\quad -(a^2-b^2)\int_{O\epsilon} \nabla\cdot\psi\nabla\xi\cdot\delta_{-\varrho}^i\phi \; dx \\
&\quad + \int_{\partial\Omega_{2O}} d(x)(\mathbf{m}\cdot\eta)\psi\cdot(\xi\delta_{-\varrho}^{\mathbf{i}}\phi) d\Gamma.
\end{aligned}$$

That is,

$$B(\delta^i_\varrho(\xi\psi),\phi) = B(\psi,\xi\delta^{\mathbf{i}}_{-\varrho}\phi) + b^2\int_{O_\epsilon}(\delta^i_\varrho(\psi_j\nabla\xi)\cdot\nabla\phi_j - \nabla\psi_j\cdot\nabla(\xi\delta^i_{-\varrho}\phi_j))dx$$
$$+(a^2-b^2)\int_{O_\epsilon}(\delta^i_\varrho(\nabla\xi\cdot\psi)(\nabla\cdot\phi) - \nabla\cdot\psi\nabla\xi\cdot\delta^i_{-\varrho}\phi)dx \tag{3.9}$$

It therefore follows from (3.9) that

$$|B(\delta^i_\varrho(\xi\psi),\phi)| \leq ||\mathbf{F}||\ ||\xi\delta^i_{-\varrho}\phi|| + C||\phi||_{[H^1(O_\epsilon)]^3}||\psi||_{[H^1(O_\epsilon)]^3}$$
$$\leq C(||\mathbf{F}||\ + ||\psi||_{[H^1(O_\epsilon)]^3})||\phi||_{[H^1(O_\epsilon)]^3}. \tag{3.10}$$

Let $\left[H^1_{\partial\Omega_{2O}}(O_\epsilon)\right]^3$ be the completation of $[C_0^\infty(O_\epsilon\cup\partial\Omega_{2O})]^3$ in $\left[H^1(O_\epsilon)\right]^3$. Then by a density argument, we obtain for any $\phi\in\left[H^1_{\partial\Omega_{2O}}(O_\epsilon)\right]^3$ the estimated

$$|B(\delta^i_\varrho(\xi\psi),\phi)| \leq C(||\mathbf{F}||\ + ||\psi||_{[H^1(O_\epsilon)]^3})||\phi||_{[H^1(O_\epsilon)]^3}.$$

Since $\delta^i_\varrho(\xi\psi)\in\left[H^1_{\partial\Omega_{2O}}(O_\epsilon)\right]^3$ if ϵ is small enough, we deduce that

$$|B(\delta^i_\varrho(\xi\psi),\delta^i_\varrho(\xi\psi))| \leq C(||\mathbf{F}||\ + ||\psi||_{[H^1(O_\epsilon)]^3})||\delta^i_\varrho(\xi\psi)||_{[H^1(O_\epsilon)]^3}. \tag{3.11}$$

On the other hand, it is clear that

$$|B(\delta^i_\varrho(\xi\psi),\delta^i_\varrho(\xi\psi))| \geq C||\delta^i_\varrho(\xi\psi)||_{[H^1(O^\epsilon)]^3}.$$

Hence it follows from (3.11) that

$$||\delta^i_\varrho(\xi\psi)||_{[H^1(O_\epsilon)]^3} \leq C(||\mathbf{F}||\ + ||\psi||_{[H^1(O_\epsilon)]^3}).$$

Since $\xi = 1$ on $O_{\epsilon'}$, by Theorem 3.16 of ([1], p.45), we deduce that $\dfrac{\partial\psi_i}{\partial x_j}\in\left[H^1(O_{\epsilon'})\right]^3$ for all $i=1,2,3$, $j=1,2$. It remains to show that $\dfrac{\partial\psi_i}{\partial x_3}\in\left[H^1(O_{\epsilon'})\right]^3$. To do this we have to distinguish the components ψ_i for $i=1,2$ and for $i=3$. In what concerns $i=1,2$, we have

$$-b^2\frac{\partial^2\psi_i}{\partial x_3^2} = b^2\Delta'\psi_i + (a^2-b^2)\frac{\partial}{\partial x_i}(\nabla\cdot\psi) + F_i \in \left[L^2(O_{\epsilon'})\right]^3,$$

while

$$-a^2\frac{\partial^2\psi_3}{\partial x_3^2} = b^2\Delta'\psi_3 + (a^2-b^2)\frac{\partial}{\partial x_3}(\frac{\partial\psi_1}{\partial x_1}+\frac{\partial\psi_2}{\partial x2}) + F_3 \in \left[L^2(O_{\epsilon'})\right]^3,$$

where $\Delta' = \dfrac{\partial^2\psi_1}{\partial x_1^2}+\dfrac{\partial^2\psi_2}{\partial x_2^2}$. By interpolation (see, e.g., [28], p.29, Theorem 6.2), for $\mathbf{F} \in \left[H^{p-2}(\Omega_R)\right]^3$ we have $\psi \in [H^p(\Omega_R)]^3 \cap \left[H^1_{\partial\Omega}(\Omega_R)\right]^3$, and then $\mathbf{u} \in [H^p(\Omega_R)]^3 \cap \left[H^1_{\partial\Omega}(\Omega_R)\right]^3$. This complete the proof. Now, with the above information we have

Proof of Lemma 2.4: Let $\widetilde{\mathbf{g}} \in [H^{1/2}(\partial\Omega)]^3$ and $\sigma \in P(M;\varepsilon)$ (given by (2.3)). Then, the problem

$$\text{(3.12)}\qquad \begin{cases} b^2\,\boldsymbol{\Delta}\,\mathbf{w}(x) + (a^2-b^2)\,\nabla(\nabla\bullet\,\mathbf{w}(x)) = \mathbf{0} & \text{in } \Omega_R, \\ \mathbf{T}_\eta\mathbf{w} + (d(x)+i\sigma)\,(\mathbf{m}\bullet\eta)\,\mathbf{w} = \widetilde{\mathbf{g}} & \text{on } \partial\Omega, \\ \mathbf{w}(x) = \mathbf{0} & \text{on } \partial\overline{B}_R \end{cases}$$

has a unique solution $\mathbf{w} \in [H^2(\Omega_R)]^3 \cap \left[H^1_{\partial\Omega}(\Omega_R)\right]^3$.

Proof: Let $\mathbf{w} = \mathbf{w}(\mathbf{h})$ a solution of (3.2) with $\mathbf{f} = \mathbf{0}$ and $\mathbf{h} \in [H^\eta(\partial\Omega)]^3$ arbitrary. Then by Lemma A1 we have $\mathbf{w} \in [H^2(\Omega_R)]^3 \cap \left[H^1_{\partial\Omega}(\Omega_R)\right]^3$ with $\eta = \frac{1}{2}$. Now, if in particular the data $\widetilde{\mathbf{g}}$ in (3.2) is the form

$$\text{(3.13)}\qquad \widetilde{\mathbf{g}}{=}i\sigma\,(\mathbf{m}\bullet\eta)\;\;\mathbf{w} + b^2\;\eta\times(\nabla\times\mathbf{w}) + \;\mathbf{h} \in [H^{1/2}(\partial\Omega)]^3,$$

then the existence of one solution to (3.12) follows of the existence of the solution of the problem (3.2). Let us consider the operator

$$C(\sigma) : [H^{1/2}(\partial\Omega)]^3 \longrightarrow [H^{1/2}(\partial\Omega)]^3$$

given by

$$C(\sigma)\mathbf{w} = i\sigma\,(\mathbf{m}\bullet\eta)\;\;\mathbf{w} + b^2\;\eta\times(\nabla\times\mathbf{w}) + \;\mathbf{h}$$

By Lemma A1, we have

$$||C(\sigma)\mathbf{w}||_{\left[H^{1/2}(\partial\Omega)\right]^3} \le C||\mathbf{w}||_{[H^2(\Omega)_R]^3} \le C_1||\mathbf{h}||_{[L^2(\partial\Omega)]^3}.$$

Hence

$$C(\sigma) : [H^{1/2}(\partial\Omega)]^3 \longrightarrow \left[L^2(\partial\Omega)\right]^3$$

is continuous. By compactness of the Sobolev inmersion of L^2 in $H^{1/2}$, the operator

$$C(\sigma) : [H^{1/2}(\partial\Omega)]^3 \longrightarrow [H^{1/2}(\partial\Omega)]^3$$

is compact. By the analytic Fredholm theorem, except for an at most countable set of σ's, the operator $C(\sigma)+I$ is invertible. Thus, given $\widetilde{\mathbf{g}} \in [H^{1/2}(\partial\Omega)]^3$, there exist $\mathbf{h} \in [H^{1/2}(\partial\Omega)]^3$ such that (3.13) is holds.

Now, uniqueness is obtained by taking the difference of two solutions $\mathbf{w}_1$ and $\mathbf{w}_2$ of (3.12). In fact, $\mathbf{w} = \mathbf{w}_1 - \mathbf{w}_2$ satisfies (3.12) with $\widetilde{\mathbf{g}} = \mathbf{0}$. Now, thanks to Betti-Green's formula over Ω_R we have, in particular,

$$\int_{\Omega_R} e(\overline{\mathbf{w}}, \mathbf{w})\ dx = \int_{\partial\Omega_R} \overline{\mathbf{w}} \bullet \mathbf{T}_\eta \mathbf{w}\ d\Gamma. \tag{3.14}$$

Taking the real part of (3.14) give us that

$$\int_{\Omega_R} e(\overline{\mathbf{w}}, \mathbf{w}) dx = \int_{\partial\Omega} [\Im(\sigma) - d(x)](\mathbf{m} \bullet \eta)\ ||\mathbf{w}||^2 d\Gamma. \tag{3.15}$$

So

$$\int_{\Omega_R} e(\overline{\mathbf{w}}, \mathbf{w}) dx = \int_{\partial\Omega} [(\Im(\sigma) - d(x))(\mathbf{m} \bullet \eta) - \varepsilon\delta]\ \|\mathbf{w}\|^2\ d\Gamma + \varepsilon\delta \int_{\partial\boldsymbol{\Omega}} ||\mathbf{w}||^2 d\Gamma, \tag{3.16}$$

where δ =max$\{\mathbf{m}(x) \bullet \eta(x) : x \in \partial\Omega\} < 0$. Moreover, we know that $\sigma \in P(M;\varepsilon)$, this implies that $(\Im(\sigma) - d(x))\mathbf{m} \bullet \eta - \varepsilon\delta \leq 0$ in $\partial\Omega$. As,

$$\int_{\Omega_R} e(\overline{\mathbf{w}}, \mathbf{w}) dx \geq 0$$

then, (3.5) yields to $\mathbf{w} = \mathbf{0}$ a.e. on $\partial\Omega$. Now, $\mathbf{w} = \mathbf{0}$ a.e. on ∂B_R. Then,

$$\mathbf{w} = \mathbf{0} \text{ a.e. on } \partial\overline{\Omega}_R.$$

Thus, we see that $\mathbf{w}$ solves

$$\begin{cases} b^2\, \boldsymbol{\Delta}\, \mathbf{w}(x) + (a^2 - b^2)\, \nabla(\nabla \bullet\ \mathbf{w}(x)) = \mathbf{0} & \text{in } \Omega_R, \\ \mathbf{w}(x) = \mathbf{0} & \text{on } \partial\overline{\Omega}_R. \end{cases}$$

Which finally implies $\mathbf{w} = \mathbf{0}$ a.e. on Ω_R. This completes the proof of Lemma 2.4.

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