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ON SYMMETRIES OF PQ-HYPERELLIPTIC RIEMANN SURFACES *

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Abstract

A symmetry of a Riemann surface X is an antiholomorphic involution ϕ. The species of ϕ is the integer εk, where k is the number of connected components in the set $\mathrm{Fix}(\phi)$ of fixed points of ϕ and $\varepsilon = -1$ if $X \setminus \mathrm{Fix}(\phi)$ is connected and $\varepsilon = 1$ otherwise. A compact Riemann surface X of genus $g > 1$ is said to be p-hyperelliptic if it admits a conformal involution ρ, called a p-hyperelliptic involution, for which X/ρ is an orbifold of genus p. Symmetries of p-hyperelliptic Riemann surfaces has been studied by Klein for $p = 0$ and by Bujalance and Costa for $p > 0$. Here we study the species of symmetries of so called pq-hyperelliptic surface defined as a Riemann surface which is p- and q-hyperelliptic simultaneously.

keywords *: p-hyperelliptic Riemann surface, automorphisms of Riemann surface, fixed points of automorphism, symmetry*

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1. Introduction

A symmetry of a Riemann surface X is an antiholomorphic involution ϕ. It is known that projective complex algebraic curves bijectively and functorially correspond to compact Riemann surfaces. Under this correspondence the fact that a surface X is symmetric means that the corresponding curve can be defined over the reals numbers. Furthermore the non-conjugate, in the group of all automorphisms of X, symmetries correspond to non-isomorphic, over the reals numbers, real curves called real forms. Finally if X has genus g then the set $\mathrm{Fix}(\phi)$ of fixed points of ϕ consists of k disjoint Jordan curves called ovals, where by the classical Harnack Theorem [4], k varies between 0 and $g+1$. This set is homeomorphic to a smooth projective real model of the corresponding curve. Let ε be the *separability character* of ϕ defined as $\varepsilon = -1$ if $X \setminus \mathrm{Fix}(\phi)$ is connected and $\varepsilon = 1$ otherwise. A conjugate in $\mathrm{Aut}^{\pm}X$ of ϕ is also a symmetry with the same k and ε. We define the *species* $\mathrm{sp}(\phi)$ of the real form represented by ϕ to be the integer εk.

A compact Riemann surface X of genus $g \geq 2$ is said to be *p-hyperelliptic* if X admits a conformal involution ρ, called a *p-hyperelliptic involution*, such that X/ρ is an orbifold of genus p. This notion has been introduced by H. Farkas and I. Kra in [3] where they also proved that for $g > 4p+1$, p-hyperelliptic involution is unique and central in the full automorphism group of X. In particular cases $p=0$ and $p=1$, X are called *hyperelliptic* and *elliptic-hyperelliptic* Rieman sufaces respectively. Let ϕ be a symmetry commuting with p-hyperelliptic involution ρ. F.Klein in [5] studied the species of the symmetries ϕ and $\phi\rho$ in the hyperelliptic case. E.Bujalance and A.Costa [2] found all possible species of such pair in the general case $p \geq 0$. In particular they determined the symmetry types of any p-hyperelliptic Riemann surface of genus $g > 4p+1$. We show that their results can be applied for g in range $3p+1 < g \leq 4p+1$, since in this case any symmetry and p-hyperelliptic involution commute if g is even while for odd g, except $g = 3p+2$ and $p \equiv 1$ (4), X always admits some conformal involution commuting with any symmetry. Moreover, for any g in this range, there exists a Riemann surface of genus g admitting exactly two p-hyperelliptic involutions whose product is $(g-2p)$-hyperelliptic involution. We present an argumentation providing more detailed results concerning symmetry types of such surface. In particular we obtain the symmetry types of any p-hyperelliptic surface of genus g in range $4p-2 \leq g \leq 4p+1$.

Furthermore, we study the species of symmetries of so called *pq-hyperelliptic* Riemann surface defined as a Riemann surface which is p- and q-hyerelliptic simultaneously. In [7] we proved that for $q > p$, the genus g of such surface is bounded by $2q-1 \leq g \leq 2p+2q+1$ and for any g in this range, there exists a Riemann surface of genus g admitting commuting p- and q-hyperelliptic involutions δ and ρ whose product is t-hyperelliptic involution if and only if $t = (g-p-q+2k)$ for some integer k in range $0 \leq k \leq (2p+2q+1-g)/4$. Moreover, we justified that p- and q-hyperelliptic involutions of a Riemann surface of genus $g > 3q+1$ are central and unique in the full automorphism group and so they commute with any symmetry ϕ. We study the possible species of symmetries $\phi, \phi\rho, \phi\delta$ and $\phi\rho\delta$ in the case when the product $\delta\rho$ is $(g-p-q)$-hyperelliptic involution. In particular we determine the symmetry types of any pq-hyperelliptic Riemann surface of genus g in range $2p+2q-2 \leq g \leq 2p+2q+1$.

2. Preliminaries

We shall approach the problem using Riemann uniformization theorem by which each compact Riemann surface X of genus $g \geq 2$ can be represented as the orbit space of the hyperbolic plane $\mathcal{H}$ under the action of some Fuchsian surface group Γ. Furthermore a group of automorphisms (including possibly anticonformal automorphisms) of a surface $X = \mathcal{H}/\Gamma$ can be represented as Λ/Γ for an NEC group Λ containing Γ as a normal subgroup. An NEC group is a discrete subgroup of the group of isometries $\mathcal{G}$ of $\mathcal{H}$ with compact quotient space, including those reversing orientation. Let $\mathcal{G}^+$ denote a subgroup of $\mathcal{G}$ consisting of orientation-preserving isometries. Then an NEC group is called a *Fuchsian group* if it is contained in $\mathcal{G}^+$ and a *proper NEC group* otherwise. Macbeath and Wilkie associated to every NEC group a signature which determines its algebraic and geometric structure. It has the form

$$(g; \pm; [m_1, \ldots, m_r]; \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\}) \tag{2.1}$$

The numbers $m_i \geq 2$ are called the *proper periods*, the brackets $(n_{i1}, \ldots, n_{is_i})$ the *period cycles*, the numbers $n_{ij} \geq 2$ the *link periods* and $g \geq 0$ is said to be the *orbit genus* of Λ. The orbit space $\mathcal{H}/\Lambda$ is a surface having k boundary components, orientable or not according to the sign being $+$ or $-$ and having topological genus g.

NEC groups with the signatures $(g; \pm; [-]; \{(-), \ldots, (-)\})$ are called *surface NEC groups.* A Fuchsian group can be regarded as an NEC group with the signature

$$(g;+;[m_1,\dots,m_r];\{-\}). \tag{2.2}$$

If Λ is a proper NEC group with the signature (2.1) then its *canonical Fuchsian subgroup* $\Lambda^+ = \Lambda \cap \mathcal{G}^+$ has the signature

$$(\gamma;+;[m_1,m_1,\dots,m_r,m_r,n_{11},\dots n_{1s_1},\dots,n_{k1},\dots,n_{ks_k}];\{-\}), \tag{2.3}$$

where $\gamma = \alpha g + k - 1$ and $\alpha = 2$ if the sign is $+$ and $\alpha = 1$ otherwise. The group with the signature (2.1) has a presentation given by generators:

(i)	$x_i, i = 1,\dots,r,$	(elliptic generators)
(ii)	$c_{ij}, i = 1,\dots,k; j = 0,\dots s_i,$	(reflection generators)
(iii)	$e_i, i = 1,\dots,k,$	(boundary generators)
(iv)	$a_i, b_i, i = 1,\dots g$ if the sign is $+$,	(hyperbolic generators)
	$d_i, i = 1,\dots g$ if the sign is $-$,	(glide reflection generators)

and relations

(1) $x_i^{m_i} = 1, i = 1,\dots,r,$
(2) $c_{is_i} = e_i^{-1} c_{i0} e_i, i = 1,\dots,k,$
(3) $c_{ij-1}^2 = c_{ij}^2 = (c_{ij-1}c_{ij})^{n_{ij}} = 1, i = 1,\dots,k; j = 1,\dots,s_i,$
(4) $x_1\dots x_r e_1\dots e_k a_1 b_1 a_1^{-1} b_1^{-1}\dots a_g b_g a_g^{-1} b_g^{-1} = 1$, if the sign is $+$,
$x_1\dots x_r e_1\dots e_k d_i^2\dots d_g^2 = 1$, if the sign is $-$.

Any system of generators of an NEC group satisfying the above relations will be called *a canonical system* of generators.

Every NEC group has a fundamental region, whose hyperbolic area is given by

$$\mu(\Lambda) = 2\pi(\alpha g + k - 2 + \sum_{i=1}^{r}(1 - 1/m_i) + 1/2\sum_{i=1}^{k}\sum_{i=1}^{s_i}(1 - 1/n_{ij})), \tag{2.4}$$

where α is defined as in (2.3). It is known that an abstract group with the presentation given by the generators $(i) - (iv)$ and the relations $(1) - (4)$ can be realized as an NEC group with the signature (2.1) if and only if the right-hand side of (2.4) is positive. If Γ is a subgroup of finite index in an NEC group Λ then it is an NEC group itself and the Riemann-Hurwitz formula says that

$$[\Lambda : \Gamma] = \mu(\Gamma)/\mu(\Lambda) \tag{2.5}$$

We shall use the following theorem of Macbetath [6] on the number of fixed points [6].

Theorem 2.1. *Let $X = \mathcal{H}/\Gamma$ be a Riemann surface with the group of conformal automorphisms $G = \Lambda/\Gamma$ and let $x_1, ..., x_r$ be elliptic canonical generators of the Fuchsian group Λ with periods $m_1, ..., m_r$ respectively. Let $\theta : \Lambda \to G$ be the canonical epimorphism and for $1 \neq h \in G$ let $\varepsilon_i(h)$ be 1 or 0 according as h is or is not conjugate to a power of $\theta(x_i)$. Then the number $F(h)$ of points of X fixed by h is given by the formula*

$$F(h) = |N_G(\langle h\rangle)| \sum_{i=1}^{r} \varepsilon_i(h)/m_i. \tag{2.6}$$

3. Symmetry types of pq-hyperelliptic Riemann surfaces

Let $X = \mathcal{H}/\Gamma$ be a pq-hyperelliptic Riemann surface of genus $g > 3q + 1$ for some $q > p$. By Theorem 3.7 in [7], p- and q-involutions of X are central and unique in the full automorphism group and so their product is t-hyperelliptic involution, where the possible values of t are given in the next

Lemma 3.1. *For any integers g, p, q such that $0 \leq p \leq q$, $2q \leq g \leq 2p + 2q + 1$ and $g > 1$, there exists a Riemann surface of genus g admitting commuting p- and q-involutions whose product is a t-involution if and only if $t = g - p - q + 2k$ for some integer k in range $0 \leq k \leq (2p + 2q + 1 - g)/4$.*

Proof. By Theorem 3.4 in [7], such surface exists if and only if t is a nonnegative integer with $(g+1)/2-(p+1) \leq t \leq (g+1)/2$ for which $p+q+t-g$ is even and nonnegative. Thus $t = g - p - q + 2k$ for some integer k. If l denotes an integer such that $(2p+2q+1)-4(l+1) < g \leq (2p+2q+1)-4l$ then $k \leq l$ and so $0 \leq k \leq (2p + 2q + 1 - g)/4$.

In particular for any p, q, g such that $2 \leq p < q < 2p$ and $g > 3q + 1$, there exists a pq-hyperelliptic Riemann surface of genus g with central p and q-involutions whose product is a $(g - p - q)$-involution. The next theorem determines the symmetry types of such surface.

Theorem 3.2. *Let X be a symmetric Riemann surface of genus g admitting p- and q-hyperelliptic involutions δ and ρ such that $\rho\delta$ is a $(g-p-q)$-hyperelliptic involution for some integers p, q, g such that $p < q < 2p$, $3q + 1 < g \leq 2p + 2q + 1$, and let ϕ be a symmetry of X. Then ρ, δ and ϕ pairwise commute and the possible species of symmetries $\phi, \phi\rho, \phi\delta$ and $\phi\rho\delta$ are:*

(i) If $g \equiv 0\ (2)$:
$(0,-1,-1,-1),(-1,0,-1,-1),(-1,-1,0,-1),(-1,-1,-1,0)$.

(ii) If $g \equiv 1\ (2)$ *and* $g \neq 2p+2q+1$:
$(0,0,0,0),(-1,-1,-1,-1)$,
$(0,-2,-2,-2),(-2,0,-2,-2),(-2,-2,0,-2),(-2,-2,-2,0)$,
$(-2a,-2a,0,0),(0,0,-2a,-2a),\ \ 1 \leq a \leq (g+1-2q)/2$,
$(-2b,0,-2b,0),(0,-2b,0,-2b),\ \ 1 \leq b \leq (g+1-2p)/2$,
$(-2c,0,0,-2c),(0,-2c,-2c,0)\ \ 1 \leq c \leq (2p+2q+1-g)/2$,
$(+d,0,0,0),(0,+d,0,0),(0,0,+d,0),(0,0,0,+d)$,
where $d=2$ *or* $d=4$ *according to* $pq \equiv 0\ (2)$ *or* $pq \equiv 1\ (2)$.

(iii) If $g = 2p+2q+1$:
$(+(2q+2),0,+(2q+2),0),(0,+(2q+2),0,+(2q+2))$,
$(+(2p+2),+(2p+2),0,0),(0,0,+(2p+2),+(2p+2))$ *and those listed in (ii) except* $(-2c,0,0,-2c),(0,-2c,-2c,0)$.

In particular this theorem determines the symmetry types of any pq-hyperelliptic Riemann surface of genus $g \geq 2p+2q-2$.

Proof. Let $X = \mathcal{H}/\Gamma$ be a Riemann surface defined in the theorem and let $t = g-p-q$. Then there exist Fuchsian groups Γ_p, Γ_q and Γ_t admitting Γ as a subgroup of index 2 such that $\langle\delta\rangle \simeq \Gamma_p/\Gamma$, $\langle\rho\rangle \simeq \Gamma_q/\Gamma$ and $\langle\rho\delta\rangle \simeq \Gamma_t/\Gamma$. By the Hurwitz Riemann formula, $\sigma(\Gamma_j) = (j;+;[2,\overset{2g+2-4j}{\ldots},2])$ for $j=p,q,t$ and so j-hyperelliptic involution admits $2g+2-4j$ fixed points. By Theorem 3.7 [7], p- and q-hyperelliptic involutions of X are unique and central in the full automorphism group. Thus ρ, δ and ϕ generate the group $G = Z_2 \oplus Z_2 \oplus Z_2$ which is isomorphic to Λ/Γ for an NEC group Λ with a signature

$$(g';\pm;[2,\overset{r}{\ldots},2];\{(2,\overset{r_1}{\ldots},2),\ldots,(2,\overset{r_s}{\ldots},2)\}),$$

where g', r, r_i are nonnegative integers for which $\mu(\Lambda)$ given by (2.4) is positive. Let Λ^+ be the canonical Fuchsian subgroup of Λ. Then $G^+ = \Lambda^+/\Gamma$ is a subgroup of G generated by ρ and δ. By Theorem 2.1 and the Hutwitz-Riemann formula, Λ^+ has the signature $(0;+;[2,\overset{g+3}{\ldots},2])$. Thus by (2.3), $g+3 = 2r + \sum_{i=1}^{s} r_i$ and $0 = \alpha g' + s - 1$, where $\alpha = 2$ or 1 according to the sign in $\sigma(\Lambda)$ being $+$ or $-$. So there are only two possible signatures of Λ:

$$\tau_1 = (1;-;[2,\overset{(g+3)/2}{\ldots},2];\{-\}) \text{ or } \tau_2 = (0;+;[2,\overset{(g+3-r_1)/2}{\ldots},2];\{(2,\overset{r_1}{\ldots},2)\}).$$

Let ϕ_1, ϕ_2, ϕ_3 and ϕ_4 denote the symmetries $\phi, \phi\rho, \phi\delta$ and $\phi\rho\delta$ respectively and let S be the sequence of species $(sp(\phi_1), sp(\phi_2), sp(\phi_3), sp(\phi_4))$. For

$i = 1, 2, 3, 4$, let Λ_i denote an NEC group such that $\phi_i \cong \Lambda_i/\Gamma$. By the Hurwitz-Riemann formula, Λ_i has one of the following signatures

$$((g+1-k_i)/2; +; [-]; \{(-) \overset{k_i}{\ldots}, (-)\}) \text{ or } (g+1-k_i; -; [-]; \{(-) \overset{k_i}{\ldots}, (-)\}).$$

The number k_i of empty period cycles and the sign in $\sigma(\Lambda_i)$ determine the species of ϕ_i. We shall find them using theorems of section 2 in [1]. If $\sigma(\Lambda) = \tau_1$ then g is odd and $S = (0, 0, 0, 0)$. So assume that $\sigma(\Lambda) = \tau_2$. Let $\theta : \Lambda \to G$ be the canonical epimorphism and let $x_1, \ldots, x_r, e, c_0, \ldots, c_{r_1}$ denote the canonical generators of Λ. First suppose that $r_1 = 0$. Then $r = (g+3)/2$ and so g is odd. Let $l \in \{1, 2, 3, 4\}$ be an integer such that $\theta(c_0) = \phi_l$. Then $\mathrm{sp}(\phi_i) = 0$ for $i \neq l$ and $k_l = 4$ or 2 according to $\theta(e)$ is or is not the identity. By Theorem 2.1, θ maps $(g+1)/2 - j$ of elliptic generators onto j-hyperelliptic involution for $j = p, q, t$ and so $\theta(e) = \theta(x_r) \overset{r}{\ldots} \theta(x_1)$ is identity only if both integers p and q are odd. Since any nonorientable word does not belong to Λ_l, it follows that $\mathrm{sp}(\phi_l) = +4$ or $+2$ according to pq being odd or even.

Next assume that $r_1 \neq 0$. For any pair (l, m) of indices from the set $\{1, 2, 3, 4\}$, let $\Lambda_{l,m}$ denote $\theta^{-1}(\langle \phi_l, \phi_m \rangle)$ and suppose that $\phi_l \phi_m$ is $j(l, m)$-hyperelliptic involution for some $j(l, m) \in \{p, q, t\}$. The epimorphism θ cannot transform all the canonical reflections of Λ onto the same symmetry ϕ_l since otherwise $\sigma(\Lambda_l)$ would have nonempty period-cycle. First suppose that every canonical reflection belongs to $\Lambda_{l,m}$ for some fixed pair (l, m). Since Γ is a surface group, it follows that $\theta(c_0) = \theta(c_{2i})$ and $\theta(c_{2i-1}) = \theta(c_0)\phi_l\phi_m$ for $i = 1, \ldots, [r_1/2]$. Thus the relation $\theta(c_{r_1}) = \theta(e)^{-1}\theta(c_0)\theta(e)$ implies that r_1 is even, which needs odd g ones again. By Theorem 2.3.3 in [1], $k_l = k_m = r_1$ and $k_i = 0$ for $i \neq l, m$. Since every period of the period-cycle in $\sigma(\Lambda)$ provides two proper periods in the signature of $\Gamma_{j(l,m)}$, it follows that r_1 does not exceed $F_{j(l,m)}/2$. If $r_1 < F_{j(l,m)}/2$ then there exists an elliptic generator $x_k \in \Lambda$ such that $\theta(x_k) = \phi_l\phi_m$. So $x_k c_0$ and $x_k c_1$ are nonorientable words such that one of them belongs to Λ_l while the other one to Λ_m and consequently $\mathrm{sp}(\phi_l) = \mathrm{sp}(\phi_m) = -r_1$. Now assume that $r_1 = F_{j(l,m)}/2$. If $g \neq 2p + 2q + 1$ then the sets F_p, F_q and F_t are nonempty and so there exist two elliptic generators of Λ, say x_k and x_n, such that $\theta(x_k)$ and $\theta(x_n)$ are two different involutions from the set $\{\rho, \delta, \rho\delta\} \setminus \{\phi_l\phi_m\}$. Since $\theta(x_k)\theta(x_n) = \phi_l\phi_m$, it follows that $x_k x_n c_0$ and $x_k x_n c_1$ are two nonorientable words such that one of them belongs to Λ_l and the other one to Λ_m. Consequently we obtain the same sequence S of species as before. If $g = 2p + 2q + 1$ then $F_t = 0$ and so neither Λ_l nor Λ_m does not admit any nonorientable word and consequently $\mathrm{sp}(\phi_l) = \mathrm{sp}(\phi_m) = +r_1$.

Now suppose that for every pair (l,m) of indices from $\{1,2,3,4\}$, there exists a canonical reflection not belonging to $\Lambda_{l,m}$. Since the periods in a period-cycle can be cyclically reordered we can assume that there exist α,β in range $0 \le \alpha < \beta < r_1$ such that $c_{\alpha+1},\dots,c_\beta \notin \Lambda_{l,m}$ and $c_0,\dots,c_\alpha,c_{\beta+1},\dots,c_{r_1} \in \Lambda_{l,m}$. Since $\theta(c_i) \neq \theta(c_{i+1})$ for every $i = 0,\dots,r_1-1$, it follows that every period in the period-cycle but $n_{0\alpha+1}$ and $n_{0\beta+1}$ provides proper periods in the signature of $\Gamma_{j(l,m)}$ while the exceptional periods $n_{0\alpha+1}$ and $n_{0\beta+1}$ provide the proper periods in the signature of $\Gamma_{j(a,b)}$ for some $a \in \{l,m\}$ and $b \notin \{l,m\}$. Repeating above argumentation for the pair (a,b) we obtain that r_1-2 periods of period-cycle provide the proper periods in the signature of $\Gamma_{j(a,b)}$ which implies that $r_1-2=1$ or $r_1-2=2$. Since $r_1 = g+3-2r$, it follows that g is even in the first case and odd in the second one. If $r_1 = 3$ then there exists $i \in \{1,2,3,4\}$ such that none of canonical reflections does not belong to Λ_i and so $\mathrm{sp}(\phi_i) = 0$. For $k \neq i$, there exists nonorientable word in Λ_k expressible as a composition of elliptic generators and a reflection and so $\mathrm{sp}(\phi_k) = -1$. Next assume that $r_1 = 4$. If there exists $i \in \{1,2,3,4\}$ such that $\theta(c_t) \neq \phi_i$ for $t = 0,\dots,r_1$ then $\mathrm{sp}(\phi_i) = 0$, $\mathrm{sp}(\phi_k) = -2$ for $k \neq i$ and otherwise $S = (-1,-1,-1,-1)$.

Finally for any sequence S listed in the theorem, there exists an NEC group Λ and an epimorphism $\theta : \Lambda \to Z_2 \oplus Z_2 \oplus Z_2$ such that $X = \mathcal{H}/\ker\theta$ is a pq-hyperelliptic Riemann surface with symmetries $\phi, \phi\rho, \rho\delta$ $\phi\rho\delta$ having species S.

By Theorem 3.7 in [7] and Lemma 3.1, for any pq-hyperelliptic Riemann surface of genus g in range $2p+2q-2 \le g \le 2p+2q+1$, the product of p- and q-involutions is $(g-p-q)$-involution and so this theorem determines the symmetry types of such surface.

4. On symmetric p-hyperelliptic Riemann surfaces

Let ϕ and ρ be a symmetry and p-hyperelliptic involution of a Riemann surface X of genus $g > 1$. E. Bujalance and A. Costa in [2] determined the possible species of the pair of symmetries ϕ and $\phi\rho$ in the case when ϕ and ρ commute. In particular, they determined the symmetry types of any p-hyperelliptic Riemann surface of genus $g > 4p+1$. The next lemma shows that their results can be applied for some lower genera either.

Lemma 4.1. *Let X be a symmetric p-hyperelliptic Riemann surface of genus g in range $3p+1 < g \le 4p+1$. If g is even then any symmetry ϕ and p-hyperelliptic involution ρ of X commute. If ϕ and ρ do not commute for* *some odd g, then $(\phi\rho)^2$ is $(g-2p+2k)$-hyperelliptic involution for some integer k in range $0 \le k \le (4p+1-g)/4$ and $(\phi\rho)^2$ is central in the full automorphism group of X except $g = 3p+2$ and $p \equiv 1\,(4)$.*

Proof. Let $X = \mathcal{H}/\Gamma$ be a p-hyperelliptic Riemann surface of genus $g > 3p+1$ and suppose that ϕ is a symmetry not commuting with p-hyperelliptic involution ρ. Then $\rho' = \phi\rho\phi$ is another p-hyperelliptic involution of X. By Theorem 3.2 in [8], every two p-involutions of X commute. Thus ϕ and ρ generate the dihedral group G of order 8 which can be identified with Λ/Γ for some NEC group Λ. Let Λ^+ be the canonical subgroup of Λ. Then Λ^+/Γ is isomorphic to the group $Z_2 \oplus Z_2$ generated by ρ and ρ'. By Lemma 3.1, the product $\rho\rho'$ is $(g-2p+2k)$-hyperelliptic involution, for some integer k in range $0 \le k \le (4p+1-g)/4$. Thus by Theorem 2.1 and the Hurwitz-Riemann formula, $\sigma(\Lambda^+) = (k; +; [2, \overset{g+3-4k}{\ldots}, 3])$ and consequently by (2.3), $\sigma(\Lambda) = (\gamma; \pm; [2, \overset{r}{\ldots}, 2]; \{(2, \overset{r_i}{\ldots}, 2)_{i=1,\ldots,s}, (-), \overset{u}{\ldots}, (-)\})$, for some integers r, r_i, s, u such that $\alpha\gamma + s + u = k$ and $2r + \sum_{i=1}^{s} r_i = g + 3 - 4k$. The canonical epimorphism $\theta : \Lambda \to G$ maps the canonical reflections of Λ onto ϕ or $\rho\phi\rho$. Since Γ is a surface Fuchsian group, it follows that $\theta(c_{ij-1}) \ne \theta(c_{ij})$ for $1 \le i \le s$, $1 \le j \le r_i$. Furthermore $\theta(c_{ir_i}) = \theta(e_i)^{-1}\theta(c_{i0})\theta(e_i)^{-1}$, which implies that r_i is even for $i = 1, \ldots, s$ and consequently $g = 2r + \sum_{i=1}^{s} r_i + 4k - 3$ is odd. Thus any symmetry of p-hyperelliptic surface of even genus $g > 3p+1$ commutes with p-hyperelliptic involution. Finally by Theorem 3.2 and Proposition 3.5 in [8], for any $g > 3p+1$ except $g = 3p+2$ and $p \equiv 1\,(4)$, X can admit at most two p-involutions which means that $\rho\rho'$ is central in the full automorphism group of X.

So if a symmetry ϕ and p-hyperelliptic involution ρ of a Riemann surface X of genus $g > 3p+1$ do not commute then X is t-hyperelliptic, where $t = g - 2p + 2k$ for some k in range $0 \le k \le (4p+1-g)/4$. Furthermore, except the case when $g = 3p+2$ and $p \equiv 1\,(4)$, ϕ is commuting with a t-hyperelliptic involution of X and consequently we can determine the possible species of ϕ using results of Bujalance and Costa.

For any $p > 0$ and g in range $3p+1 < g \le 4p+1$, there exists a Riemann surface admitting two p-involutions whose product is $(g-2p)$-involution. The next theorem determines the symmetry types of such surface.

Theorem 4.2. *Let X be a symmetric Riemann surface of genus $g > 3p+1$, except $g = 3p+2$ and $p \equiv 1$ (4), admitting two p-hyperelliptic involutions whose product is $(g-2p)$-hyperelliptic involution and let ϕ be a symmetry* *of X. Then for even g,* $\mathrm{sp}(\phi) = 0$ *or* -1. *If g is odd then* $\mathrm{sp}(\phi)$ *is one of integers* $0, -1, -2a, +d$, *where $d = 2$ or $d = 4$ according to $p \equiv 0$ (2) or $p \equiv 1$ (2) or $d = 2p + 2$ for $g = 4p + 1$ and a is positive integer not exceeding $(g+1-2p)/2$ if ϕ commutes with p-involutions and not exceeding $(4p + 1 - g)/2$ otherwise.*

Proof. Let ϕ be a symmetry of a p-hyperelliptic Riemann surface X defined in theorem. If ϕ commutes with p-involutions of X then we can find $\mathrm{sp}(\phi)$ by repeating the argumentation from the proof of Theorem 3.2 for $q = p$. In particular, using the previous Lemma we obtain that $\mathrm{sp}(\phi) = 0$ or -1 if g is even. So suppose that ϕ does not commute with some p-involution ρ of X and let $\rho' = \phi\rho\phi$. Then by the proof of Lemma 4.1, the involutions ϕ and ρ generate the dihedral group G of order 8 which can be identified with Λ/Γ for some NEC group Λ with one of signatures $\tau_1 = (1; -; [\overset{(g+3)/2}{\ldots}, 2]; \{-\})$ or $\tau_2 = (0; +; [2, \overset{r}{\ldots}, 2]; \{(2, \overset{r_1}{\ldots}, 2)\})$, where $2r + r_1 = g + 3$. Let $\theta : \Lambda \to G$ be the canonical epimorphism. Then $\Lambda' = \theta^{-1}(\langle \phi, \rho\phi\rho \rangle)$ and $\Gamma_{\rho\rho'} = \theta^{-1}(\rho\rho')$ are normal subgroups of Λ of indices 2 and 4 respectively. If $\sigma(\Lambda) = \tau_1$ then Λ' has not any period cycle and consequently $\mathrm{sp}(\phi) = \mathrm{sp}(\rho\phi\rho) = 0$. So assume that $\sigma(\Lambda) = \tau_2$. For any $1 \neq h \in G$, let s_h denote the number of elliptic generators x_i of Λ such that $\theta(x_i) = h$. Then by Theorem 2.2.4 in [1], the number of periods in the signature of $\Gamma_{\rho\rho'}$ is equal to $4s_{\rho\rho'} + 2r_1$. On the other hand such number is equal to $2g + 2 - 4(g - 2p) = 8p + 2 - 2g$ and so $s_{\rho\rho'} = (4p + 1 - g - r_1)/2$. Thus $s_{\phi\rho\phi} + s_\rho = r - s_{\rho\rho'} = g + 1 - 2p$ is even which means that $s_{\phi\rho\phi}$ and s_ρ have the same parities and consequently the relation $\theta(x_1 \ldots x_r e_1) = 1$ implies that $\theta(e_1) = 1$ or $\rho\rho'$. So $e_1 \in \Lambda'$ and by Theorem 2.3.2 in [1], Λ' admits two period-cycles of the form $(2, \overset{r_1}{\ldots}, 2)$. Since every period in the period cycle of Λ' provides one proper period in the signature of $\Gamma_{\rho\rho'}$, it follows that $r_1 \leq F(\rho\rho')/2 = 4p + 1 - g$. For $g = 4p + 1$, $F(\rho\rho') = 0$, which means that there is no proper periods nor link periods in the signature of Λ' and consequently $\mathrm{sp}(\phi) = \mathrm{sp}(\rho\phi\rho) = +1$ or $+2$. For $g \neq 4p + 1$, $\mathrm{sp}(\phi) = \mathrm{sp}(\rho\phi\rho) = -1$ or -2 if $r_1 = 0$ and $\mathrm{sp}(\phi) = \mathrm{sp}(\rho\phi\rho) = \pm r_1$ otherwise, where the sign is $-$ for $r_1 < 4p + 1 - g$ and the sign is $+$ for $r_1 = 4p + 1 - g$.

Corollary 4.3. *Let ϕ be a symmetry of a Riemann surface of genus g in range $4p - 2 \leq g \leq 4p + 1$ admitting two p-hyperelliptic involutions. Then the possible species of ϕ are given in the table, where $d = 2$ or $d = 4$ according to p is or is not even.*

g	sp(ϕ)	Conditions
$4p+1$	$0, \pm 1, \pm 2, -2a, +(2p+2), +d$	$1 \leq a \leq p+1, p > 0$
$4p$	$0, -1$	$p > 1$
$4p-1$	$0, -1, \pm 2, -2a, +d$	$1 \leq a \leq p, p > 2$
$4p-2$	$0, -1$	$p > 3$

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