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THE COMPLEX LINEAR REPRESENTATIONS OF $GL(2,k)$, k A FINITE FIELD

LUISA ABURTO [*]
ROBERTO JOHNSON [†‡]
JOSÉ PANTOJA [§¶]
P. UNIVERSIDAD CATÓLICA DE VALPARAÍSO, CHILE

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Abstract

Let k be a finite field of odd characteristic, and let G be the group of all invertible 2×2 matrices over k. We construct the irreducible complex linear representations of the group G.The constructions lean on the method of induction from subgroups and on the theory of characters. To accomplish this goal, the basic facts from the theory of representations and characters of finite groups are presented. Furthermore, we describe the structure of G that we need, and the theory of representations of some subgroups of G that we use. As a final result, we obtain the theory of the irreducible representations of G,by describing either the irreducible representations of , or the irreducible characters of the group G.

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1. Preliminaries.

The Theory of Group Representations is important for several reasons. Besides its intrinsic interest and the knowledge of the structure of the groups that provides, it has numerous applications to other branches of mathematics (Geometry, Probabilities, Theory of Numbers, ...). Also it is important in different problems in Physics, Chemistry, Neurophysiology (See, for example [SA1]).

To study this theory, one can consider first its "basic" constituents, namely the irreducible representations and from there consider the more "complex" representations (Another approach it is to study "big" natural representations and decompose them into its elementary parts, which is call the method of the harmonic analysis). The first point of view requires the knowledge of *all* irreducible representations. This is far from trivial and it is an open question for many groups.

Among the most interesting groups are the General Linear Groups defined over a field F. In particular, it is of interest the case when F is a finite field, a p-adic field or a real field. The theory over these three types of fields has shown that many times the results and method used in one case can be transpose to the other cases.

We are going to deal with the group $G = GL_2(k)$ k a finite field, of odd characteristic. Specifically, we will classify the irreducible complex linear representation of the group G or its irreducible characters. These are defined once has the representation, but on the other hand, they can be constructed independently from the representation and are enough to give a complete classification of the theory. To perform this task, we will consider first the basic facts about the theory of representation of finite groups, that we will need. Secondly, we will describe the structure of G, then, representations or characters of certain subgroups will be constructed. Finally we will determine the irreducible representations or irreducible characters of G.

The irreducible representations are classify in series of representations. In this case, the Principal and the Discrete series. The *Principal Series* consists of the representations that can be obtained by the process of induction from the subgroup of the upper triangular matrices. The rest of the irreducible representations define the *Discrete Series*. The Principal Series is suitable to geometric constructions, i.e., the space of the representation is of form $\mathbf{C}^X$, where $GL_2(k)$ acts on the set X.The treatment given in this note is more or less the standard one, although it will be more direct than others presentations, (see for example [PS], [G], [T]).

The *Discrete Series* is not suitable to simple geometric constructions. In this case "meta-geometric" methods can be employed (like the one that use Weil Representations to obtain the series (see for example [SA2]). On the other hand, we can describe the character of the series to describe these series of representations. This will be the approach in this monograph.

1.1. Linear Representations of Finite Groups.

In what follows, and unless explicitly stated, G will denote a finite group and $\mathbf{C}$ the field of complex numbers. If V is a $\mathbf{C}-$vector space, then $Aut_{\mathbf{C}}(V)$ denotes the group of all linear automorphisms of V. As usual, $\mathbf{C}[G]$ is the group algebra of G, i.e. the algebra of formal finite sums $\sum\limits_{g\in G} a_g g$, where $a_g \in C$

Definition 1. *A complex linear representation of G, is a pair (V,ρ) where V is a complex vector space and ρ is a group homomorphism from G to $Aut_{\mathbf{C}}(V)$. We will write sometimes simply ρ or even V for the representation (V,ρ).*

The space V is the space of the representation, and the dimension of the space V, $\dim_{\mathbf{C}}(V)$, is the degree *or dimension of the representation.*

Definition 2. *Let G be a group and $\mathbf{C}^G$ be the $\mathbf{C}-$vector space of all functions from G to $\mathbf{C}$. Let us consider the canonical basis $\mathcal{C} = \{\delta_g \mid g \in G\}$ where δ_g is the characteristic function, that δ_g has the value 1 in g and 0 elsewhere. We define for each $g \in G$, $\rho_g : \mathbf{C}^G \longrightarrow \mathbf{C}^G$ by $\rho_g(\delta_{g'}) = \delta_{gg'} \quad (g' \in G)$. This defines a representation of G which is called the left regular representation of G. We note that the degree of ρ is the order of G.*

More generally, let us assume that G acts on a finite set X by $x \longrightarrow g{\cdot}x$. Consider the $\mathbf{C}-$vector space $\mathbf{C}^X$ with the standard basis given by the characteristic functions $\{\delta_x \mid x \in X\}$ If $g \in G$, we define the linear transformation $\rho_g : \mathbf{C}^X \longrightarrow \mathbf{C}^X$ by $\rho_g(\delta_x) = \delta_{g\cdot x}$. The representation defined above is called natural representation of G associated to the geometric space (X,G). The degree of ρ is $|X|$.

Remark 3. *We should note that a representation (V,ρ) of G is just a linear action of G on the vector space V, $v \longmapsto g\cdot v = \rho_g(v)$.*

Then V becomes a $\mathbf{C}[G]$-module using the above action.

Remark 4. *We leave to the reader verify that the set* $\mathbf{C}^G = \{f : G \longrightarrow \mathbf{C}\}$ *is a* $\mathbf{C}$*-algebra with the usual structure of a* C*-vector space and the convolution product of functions,* $f{*}g$ *given by* $(f{*}g)(x) = \sum_{y\in G} f(y)g(y^{-1}x)$*. In addition, the* $\mathbf{C}$*-algebras* $\mathbf{C}[G]$ *and* $\mathbf{C}^G$ *are isomorphic.*

Definition 5. *Let* (V,ρ) *and* (W,σ) *be two representations of* G*. Then a linear transformation* $\phi : V \longrightarrow W$ *is an homomorphism of representations (a* G*-homomorphism between* V *and* W*, or an intertwining between* V *and* W*) if for any* g *in* G *we have that* $\sigma_g \circ \phi = \phi \circ \rho_g$*. As usual, a monomorphism (respectively epimorphism or isomorphism) of representations is an injective homomorphism (respectively epijective or bijective) of representations. When a representation* (V,ρ) *is isomorphic to a representation* (W,σ) *we will write* $(V,\rho) \cong (W,\sigma)$ *or simply* $\rho \cong \sigma$ *The space of all intertwinings between* V *and* W *will be denoted by* $\mathrm{Hom}_G(V,W)$*.*

Remark 6. *A representation of dimension one of* G *is simply an homomorphism* σ *of* G *on the multiplicative group,* $\mathbf{C}^\times$*, of* $\mathbf{C}$*. In these notes, this representation will be named* linear character *of* G*. In particular,* the trivial character 1 *is just the constant function* 1*.*

Proposition 7. *(One dimensional representations of* G*) Let* $\psi : G \longrightarrow \mathbf{C}^\times$ *be an homomorphism. Then there exist an homomorphism* $\overline{\psi} : G/G' \longrightarrow \mathbf{C}^\times$ *where* G' *is the derived subgroup of* G*, such that* $\overline{\psi} \circ \pi = \psi$*,* π *the natural epimorphism from* G *onto* G/G'*. Conversely, any homomorphism from* G/G' *to* $\mathbf{C}^\times$ *produces an homomorphism from* G *to* $\mathbf{C}^\times$ *by composition with* π*.*

Proof. Follows from the standard properties of the derived subgroups, applying the Fundamental Theorem of the Homomorphism of groups. □

Example 8.

1. *Let* G *be a cyclic group generated by* g_0*, then* $\rho : G \longrightarrow \mathbf{C}^\times$ *defined by* $\rho(g_0^s) = \exp(s)$ *gives a one dimensional representation of* G*. In particular, one obtains* $q-1$ *linear characters of the multiplicative group* $k^\times$ *of* k *the finite field of* q *elements.*

2. *We observe that the derived subgroup of the General Linear Group* $GL_2(k)$*, is the Special Linear Group,* $SL_2(k)$*, consisting of all matrices of determinant* 1*. Since* $GL_2(k)/SL_2(k)$ *is isomorphic to* $k^\times$*, the characters of this last group parametrize the one dimensional representations of* $GL_2(k)$*: given a one dimensional representation* ρ *of*

$GL_2(k)$ there is a unique linear character α of $k^\times$ such that $\rho = \alpha \circ \det$, i.e., the following diagram is commutative:

$$\begin{array}{ccc} G & \xrightarrow{\rho} & \mathbf{C}^\times \\ \det \downarrow & \nearrow \alpha & \\ k^\times & & \end{array}$$

Definition 9. *Let $\rho : G \longrightarrow \mathrm{Aut}_{\mathbf{C}}(V)$ be a representation of G and let W be a subspace of V. Then W is a stable subspace of V if $\rho_g(W) \subset W$ for all $g \in G$.*

We note that a stable subspace W of V, gives a new representation σ of G: the space of this representation is W and the action $\sigma_g(w) = \rho_g(w)$.

Definition 10. *The representation (W, σ) is a subrepresentation of (V, ρ). We will write also (W, ρ^W) for (W, σ).*

We observe that (V, ρ) is an (improper) subrepresentation of (V, ρ) and $\{0\}$ is a stable subspace that produces a trivial subrepresentation of (V, ρ).

Definition 11. *A representation (V, ρ) of a group G is irreducible if and only if it has no non trivial proper subrepresentations.*

Proposition 12. *Any irreducible representation of a finite group G has degree less or equal than $|G|$, the order of G.*

Proof. Let (V, ρ) be an irreducible representation of G. Let $v_0 \in V$. The set $\{\rho_g v_0 \mid g \in G\}$ is a subset of V that generates V since V is irreducible. It follows that $\dim_{\mathbf{C}} V \leq |G|$. □

1.1.1. Direct Sum of Subrepresentations

Let (W, ρ^W) be a subrepresentation of (V, ρ). Then there exists a complement subspace W' of W in V, stable under ρ (i.e., $V = W \bigoplus W'$ where W and W' are stable). We will say that ρ is the direct sum of ρ^W and $\rho^{W'}$ and we will write $\rho = \rho^W \bigoplus \rho^{W'}$. It is clear that $\dim_{\mathbf{C}} \rho = \dim_{\mathbf{C}} \rho^W + \dim_{\mathbf{C}} \rho^{W'}$. (See [Se] for more details). The direct sum of n copies of ρ will be denoted by $n\rho$.

We observe that (V, ρ) is irreducible if and only if ρ cannot be decompose as a direct sum of two subrepresentation ρ' and ρ'' of lower dimension.

As a consequence, any finite dimensional representation of G is a direct sum of a finite number of irreducible representations (Maschke's Theorem) ([Cu-Re]).

It can be proved that the number of irreducible representations of G, up to isomorphisms, is l the number of conjugacy classes of G (See for example [Se]). We can restate then the previous statement: let $\rho_1, \dots, \rho_l$ be representatives of the different isomorphism classes of the irreducible representations of G. Then if ρ is a representation of G, there exist integers $n_i \geq 0$ $(i = 1, ..., l)$ such that $\rho \cong \bigoplus_{i=1}^{l} n_i \rho_i$

It should be noted that the n_i are unique.

In the case when ρ is the regular representation, the numbers n_i are equal to the dimensions of the ρ_i, i.e., $\rho \cong \bigoplus_{i=1}^{l} \dim(\rho_i)\, \rho_i$. As a consequence, we obtain the completitude criterion: $\sum_{i=1}^{l} (\dim \rho_i)^2 = |G|$

By the above, if G is an abelian group, then all the irreducible representations of G are one dimensional and the number of them is the order of G.

In particular, if K is a finite field of q elements, then the number of linear characters of K^+ and $K^\times$ are, respectively, q and $q - 1$.

Remark 13. *We observe that a fundamental problem in representation theory is to determine the isomorphism classes of the irreducible representations of G. We will denote by $\widehat{G}$ the set of the above classes. The set $\widehat{G}$ is naturally a group only if G is commutative, and in this case G and $\widehat{G}$ are isomorphic.*

The following is an important result about irreducibility of representations:

Lemma 14. *(Schur's Lemma). Let (V_1, ρ^1) and (V_2, ρ^2) be irreducible representations of G. Let f be an homomorphism of representations of V_1 into V_2. Then:*

1. If ρ^1 and ρ^2 are not isomorphic, then $f = 0$
2. If $V_1 = V_2$ and $\rho^1 = \rho^2$ then f is a homothecy (i.e. a scalar multiple of the identity).

Proof. See, for example [Se]. □

Let $\mathrm{Hom}_G(\rho^1, \rho^2)$ denotes the set of all the homomorphism between (V_1, ρ^1) and (V_2, ρ^2). Then, Schur's Lemma says that:

1. If (V_1, ρ^1) and (V_2, ρ^2) are non-isomorphic and irreducible, then $\mathrm{Hom}_G(\rho^1, \rho^2) = 0$.
2. If $(V_1, \rho^1) = (V_2, \rho^2)$ is irreducible, then $\mathrm{Hom}_G(\rho^1, \rho^2) \simeq \mathbf{C}$

1.2. The Character of a Representation.

Let (V, ρ) be a representation of finite dimension n of a group G. If we choose a basis of V, then $\mathrm{Aut}_{\mathbf{C}}(V) \simeq GL_n(\mathbf{C})$. Let us denote by $[\rho_g]$ the matrix associated to ρ_g under the above isomorphism.

Definition 15. *The character χ_ρ of the representation (V, ρ) of dimension n, is the function $\chi_\rho : G \longrightarrow \mathbf{C}$ such that:* $\chi_\rho(g) = tr\,[\rho_g]\,(g \in G)$

Since $tr(AB) = tr(BA)$ for all matrices $A, B \in M_n\,(C)$, the above function is well defined.

The character χ_ρ is a class function, i.e., is a function that is constant over the conjugacy classes of G. Also, we have $\chi_{\rho_1 \bigoplus \rho_2} = \chi_{\rho_1} \bigoplus \chi_{\rho_2}$

It should be noted that when the dimension of ρ is one, then $\chi_\rho = \rho$.

There is a natural scalar product on the complex vector space C^G for which

$$\langle \chi_\rho, \chi_{\rho'} \rangle_G = \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g) \overline{\chi_{\rho'}(g)}$$

where, $\overline{z}$ stands for the complex conjugate of z.

Among the important properties of the characters of the group we can mention:

1. If χ_ρ is the character of an irreducible representation of G, then $\langle \chi_\rho, \chi_\rho \rangle_G = 1$. (i.e., χ_ρ has "norm one").
2. If χ_ρ and $\chi_{\rho'}$ are the irreducible characters of two non isomorphic representations of G , then $\langle \chi_\rho, \chi_{\rho'} \rangle_G = 0$.
3. If $\rho \cong \bigoplus_{i=1}^{l} n_i \rho_i$ then $\chi_\rho = \sum_{i=1}^{l} n_i \chi_{\rho_i}$.

4. If $\rho \cong \bigoplus_{i=1}^{l} n_i\rho_i$ then $n_i = \langle \chi_\rho, \chi_{\rho_i} \rangle_G$.

5. ρ is irreducible if and only if $\langle \chi_\rho, \chi_\rho \rangle_G = 1$.

Notation 16. *We will write also,* $\langle \rho, \rho' \rangle_G$ *for* $\langle \chi_\rho, \chi_{\rho'} \rangle_G$.

1.3. Induced Representations

Let H be a subgroup of a finite group G, and let (U, σ) be a representation of H. We will construct in a natural way, a representation (V, ρ) of G starting from (U, σ). The representation (V, ρ) will be called the induced representation of (U, σ) from H to G. It will be denoted by $\mathrm{Ind}_H^G \sigma$.

Let $V = \{f : G \longrightarrow U | f(hg) = \sigma_h f(g),\ h \in H, g \in G\}$. Then G acts on V by right translations,

$$[\rho_g(f)]\,(g') = f(g'g) \qquad\qquad (g, g' \in G;\ f \in V)$$

It follows that (V, ρ) is a representation of G.

We have just associated to a given representation σ of H a representation ρ of G.

On the other hand, starting from a representation (W, τ) of G, we can produce a representation of H, $(W, \tau\mid_H)$ which it will be denote by $\mathrm{Res}_H^G \tau$ (In fact we are constructing a functor from the category of the representations of G to the category of the representations of H).

1.3.1. Frobenius Reciprocity

We will see that induction and restriction are adjoint functors. This property is often known as Frobenius Reciprocity and also in classical terms as the Universal Property of the induced representation.

We note first that the representation (U, σ) can be immersed in (V, ρ) by i:u$\longmapsto (f_u : G \longrightarrow U)$

where $f_u(g) = 0$ if $g \notin H$ and $f_u(g) = \sigma_g(u)$ if $g \in H$.

Proposition 17. *Let* (W, τ) *be a representation of* G. *For each* H*-homomorphism* ϕ *from* (U, σ) *to* (W, τ) *there exist a unique* G*-homomorphism* Ψ *from*

$\mathrm{Ind}_H^G \sigma$ *to* (W,τ) *such that* $\Psi \circ i = \phi$, *i.e. the following diagram is commutative*

$$\begin{array}{ccc} \mathrm{Ind}_H^G \sigma & \Psi\longrightarrow & (W,\tau) \\ i\ \uparrow & \phi \nearrow & \\ (U,\sigma) & & \end{array}$$

As an immediate consequence, we have an isomorphism (Frobenius Reciprocity):

$$\mathrm{Hom}_G(\mathrm{Ind}_H^G \sigma, \tau) \simeq \mathrm{Hom}_H(\sigma, \mathrm{Res}_H^G \tau)$$

We will describe now, the character of the induced representation in terms of the inducing character.

Let χ_ρ be the character of $\rho = \mathrm{Ind}_H^G \sigma$ and let χ_σ be the character of σ. Then

$$\chi_\rho(g) = \mathrm{Ind}_H^G \chi_\sigma(g) = \frac{1}{|H|} \sum_{\substack{x \in G \\ x^{-1}gx \in H}} \chi_\sigma(x^{-1}gx)$$

In terms of characters, Frobenius Reciprocity says:

$$\left\langle \mathrm{Ind}_H^G \chi_\sigma, \chi_\tau \right\rangle_G = \left\langle \chi_\sigma, \mathrm{Res}_H^G \chi_\tau \right\rangle_H$$

Let H and K be subgroups of G. If $g \in G$ then the set $HgK = \{hgk/h \in H, k \in K\}$ is called a double coset respect to the subgroups H and K, and the element g is called a representative of the double coset. The set of all (H,K)-double cosets is a partition of G. A complete set of representatives of all (H,K)-double coset is denoted by $H\backslash G/K$. For $s \in H\backslash G/K$ we set $H_s = sHs^{-1} \cap K$.

We consider now, a representation (W,τ) of H, then the subgroup H_s has a natural representation (W,τ^s) defined by

$$\tau^s(x) = \tau(s^{-1}xs) \qquad\qquad x \in H_s.$$

The next two Theorems, due to Mackey, give information about the restriction of an induced representation and about the intertwining of two induced representations.

Theorem 18. *(Mackey's Restriction Theorem):*

$$\mathrm{Res}_K^G \mathrm{Ind}_H^G \tau \simeq \bigoplus_{s \in H\backslash G/K} \mathrm{Ind}_{H_s}^K \tau^s$$

Theorem 19. *(Mackey's Intertwining Theorem) Let H and K be subgroups of G and (U,σ) a representation of K and (W,τ) a representation of H Then:*

$$\operatorname{Hom}_G(\operatorname{Ind}_K^G \sigma, \operatorname{Ind}_H^G \tau) \simeq \bigoplus_{s\in H\backslash G/K} \operatorname{Hom}_{sHs^{-1}\cap K}(\sigma, \tau^s)$$

Proof. This last theorem is a consequence of the previous Mackey's theorem and of Frobenius Reciprocity:

By Frobenius Reciprocity, we have
$\operatorname{Hom}_G(\operatorname{Ind}_K^G \sigma, \operatorname{Ind}_H^G \tau) \simeq \operatorname{Hom}_K(\sigma, \operatorname{Res}_K^G \operatorname{Ind}_H^G \tau)$.

Now, by Mackey's theorem,

$$\operatorname{Hom}_K(\sigma, \operatorname{Res}_K^G \operatorname{Ind}_H^G \tau) \simeq \operatorname{Hom}_K(\sigma, \bigoplus_{s\in H\backslash G/K} \operatorname{Ind}_{H_s}^K \tau^s)$$

but

$$\operatorname{Hom}_K(\sigma, \bigoplus_{s\in H\backslash G/K} \operatorname{Ind}_{H_s}^K \tau^s) \simeq \bigoplus_{s\in H\backslash G/K} \operatorname{Hom}_K(\sigma, \operatorname{Ind}_{H_s}^K \tau^s)$$

applying Mackey's theorem one more time, we have

$$\bigoplus_{s\in H\backslash G/K} \operatorname{Hom}_K(\sigma, \operatorname{Ind}_{H_s}^K \tau^s) \simeq \bigoplus_{s\in H\backslash G/K} \operatorname{Hom}_{H_s}(\sigma, \tau^s)$$

from here the result follows. □

Remark 20. *We can state in terms of characters the last two theorems:*

1. $\chi_\rho \mid_K = \displaystyle\sum_{s\in H\backslash G/K} \chi_{\operatorname{Ind}_{H_s}^K \tau^s}$.
2. $\left\langle \operatorname{Ind}_K^G \chi_\sigma, \operatorname{Ind}_H^G \chi_\tau \right\rangle_G = \displaystyle\sum_{s\in H\backslash G/K} \langle \chi_\sigma, \chi_{\tau^s} \rangle_{H_s}$.

2. The General Linear Group over a Finite Field k.

In what follows k will denote a finite field of $q = p^n$ elements, $p \neq 2$. As usual $k^\times$ denotes the group of non zero elements of k. Also k^+ stand for the additive group of k and G for the General Linear group over k.

Let $\omega_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\omega_{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and let $K = k(\sqrt{\varepsilon})$ be the (unique) quadratic extension of k (ε a non square of k).

2.1. Some Subgroups of G

We will consider the following subgroups of G :

1. The (standard) Borel subgroup of G

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \ / \ a, d \in k^{\times}, b \in k \right\}$$

2. The (standard) Unipotent subgroup of G

$$U = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \ / \ b \in k \right\}$$

3. The Isotropic torus

$$T_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \ / \ a, d \in k^{\times} \right\}$$

4. The Anisotropic torus

$$T_{-1} = \left\{ \begin{pmatrix} a & b\varepsilon \\ b & a \end{pmatrix} \ / \ a^2 - b^2\varepsilon \neq 0 \right\}$$

5. The Affine subgroup of G

$$Q = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \ / \ a \in k^{\times}, b \in k \right\}$$

6. The Center of G

$$Z = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \ / \ a \in k^{\times} \right\}$$

7. The Multiplicative field subgroup of G

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \ / \ a \in k^{\times} \right\}$$

8. The Special Linear group over k

$$SL_2(k) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ / \ ad - bc = 1 \right\}$$

Furthermore, we will consider the subgroups $N_G(H)$, the normalizers of the subgroup H of G.

Proposition 21. The following diagrams give the inclusion relation between the above subgroups.

$$\begin{tikzcd}
 & G & & \\
 & B \arrow[u] & & \\
T_1 \arrow[ur] & & Q \arrow[ul] & SL_2(k) \arrow[uulll, phantom] \arrow[uull] \\
Z \arrow[u] & A \arrow[uu] \arrow[ul] \arrow[ur] & U \arrow[u] \arrow[ur] & \\
 & \{1\} \arrow[ul] \arrow[u] \arrow[ur] & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & G & \\
N_G(T_{-1}) \arrow[ur] & & N_G(T_1) \arrow[ul] \\
T_{-1} \arrow[u] & & T_1 \arrow[u] \\
 & Z \arrow[ul] \arrow[ur] & \\
 & \{1\} \arrow[u] &
\end{tikzcd}$$

Proposition 22. The order for the above groups are:

i) $|G| = (q-1)^2\, q\, (q+1)$

ii) $|B| = (q-1)^2\, q$

iii) $|U| = q$

iv) $|T_1| = (q-1)^2$

v) $|T_{-1}| = q^2 - 1$

vi) $|Z| = |A| = q - 1.$

vii) $|Q| = q(q-1)$

viii) $|SL_2(k)| = (q-1)q(q+1)$

Furthermore, the next proposition gives others properties of these groups:

Proposition 23. The subgroups under consideration satisfy:

i) $U \trianglelefteq B$; $Z \trianglelefteq B$; $B/U \simeq T_1$; $B/Z \simeq Q$

ii) $U \times T_1 = B$; $Z \times Q = B$; $Q = UA$ (semidirect products).

iii) If H' denotes the commutator subgroup of H, then $U = B' = Q'$; $G' = SL_2(k)$.

iv) $U \simeq k^+$; $T_1 \simeq k^\times \times k^\times$; $Z \simeq A \simeq k^\times$.

Proposition 24. (Bruhat Decomposition). The number of (B, B)-double cosets of G is two, i.e. $B\backslash G/B = \{B, B\omega_1 B\}$, and so G=B$\cup B\omega_1 B$.

Therefore $\{1, \omega_1\} = B\backslash G/B$.(See for example [S]).

2.2. Conjugacy Classes and Double Cosets.

2.2.1. Conjugacy Classes of G.

For the eigenvalues of $g \in G$ we have the following cases:

1. The eigenvalues of g belong to k. In this case, the possible Jordan forms of g are: $\mathrm{c}_1(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, $c_2(a) = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$, $c_3(a,b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $(a \neq b)$

2. The eigenvalues of g do not belong to k. In this case, g is irreducible as a matrix (the characteristic polynomial of g is irreducible over k). Then g is conjugate to a matrix $\mathrm{c}_4(\alpha) = \begin{pmatrix} 0 & -N\alpha \\ 1 & Tr\alpha \end{pmatrix}$

   where N, Tr denote (respectively) the norm and the trace of the quadratic extension $K = k(\alpha)$, α being one of the eigenvalues of g.

The following lemma presents the number of the different conjugacy classes of G,

Lemma 25. a)There are $(q-1)$ conjugacy classes of type $c_1(a)$ and $(q-1)$ of type $c_2(a)$.

b)There are $\frac{1}{2}(q-1)(q-2)$ conjugacy classes of type $c_3(a,b)$ $(a \neq b)$, and $\frac{1}{2}(q^2-q)$ of type $c_4(\alpha)$.

2.2.2. Double Cosets of G.

The following lemma describes the normalizers in G of the isotropic and anisotropic tori.

Lemma 26. a) $N_G(T_1) = T_1 \times \{1, \omega_1\}$
b)$N_G(T_{-1}) = T_1 \times \{1, \omega_{-1}\}$

Proof. The result follows from the Bruhat decomposition. □

The next lemma gives the number of different double cosets of G,

Lemma 27. i)The number of (T_{-1}, T_{-1})-double cosets of G is q.
ii)The number of (ZU, T_{-1})-double cosets of G is $q-1$.
iii)The number of (ZU, ZU)-double cosets of G is $2(q-1)$.

Proof. Let H, L be subgroups of G. If H acts on the left cosets $gL, g \in G$ by left multiplication, then the orbit of gL under H is $orb_H(gL) = \{tgL \mid t \in H\}$. On the other hand, the stabilizer $\mathrm{Stab}_H(gL)$ of gL in H, is $H \cap gLg^{-1}$.

We have then that

$$\begin{aligned} |HgL| &= |orb_H(gL)|\,|L| \\ &= \frac{|H|}{|\mathrm{Stab}_H(gL)|}\,|L| \\ &= \frac{|H|}{|H \cap gLg^{-1}|}\,|L| \\ &= \frac{|H|\,|L|}{|H \cap gLg^{-1}|}. \; (*) \end{aligned}$$

i) If $g \in N_G(T_{-1})$; i.e.if $g \in T_{-1} \times \{1, \omega_{-1}\}$ where

$$\omega_{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then we have the double cosets $T_{-1}1T_{-1}$ and $T_{-1}\omega_{-1}T_{-1}$, which have order $|T_{-1}|$ using $(*)$. On the other hand, if $g \notin N_G(T_{-1})$, then $T_{-1} \cap gT_{-1}g^{-1} = k^{\times} I_2 = Z$ (I_2 the identity matrix). Now, since $|GL_2(k)| = |\cup T_{-1}gT_{-1}| = 2\,|T_{-1}| + (q-2)\,|T_{-1}g_0T_{-1}|$, where $g_0 \notin N_G(T_{-1})$, we have that the number of (T_{-1}, T_{-1})-double cosets of G is q.

ii)Since $T_{-1} \cap gZUg^{-1} = Z$ for any $g \in GL_2(k)$, then each (ZU, T_{-1})-double coset has the same order. Given that $|GL_2(k)| = (q-1)\,|ZUgT_{-1}|$

where $g \in GL_2(k)$, we have that the number of (ZU, T_{-1})-double cosets is $q-1$.

iii)Let $g \in N_G(ZU)$,i.e., let g be an element of the Borel subgroup. Set $g = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ then $ZUgZU = ZU \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} ZU$. So we have $q-1$ cosets in this case.

On the contrary, if $g \notin B$, we observe that $ZU \cap gZUg^{-1} = Z$, so that $|GL_2(k)| = (q-1)\,|ZU| + (q-1)\,|ZUgZU|$ .and we have $q-1$ cosets of this type. It follows that the number of (ZU, ZU)-double cosets is $2(q-1)$. $\square$

2.3. Representations of Z, A, U, T_1.

The irreducible representations of these groups are of dimension one, since they are abelian groups. These representations are given by linear characters and the number of them is equal to the order of the respective group.

1. Since $Z \simeq k^{\times}$, each $\chi \in \widehat{Z}$ is defined by $\chi \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \chi(a)$ (this last χ is a character of $k^{\times}$).
2. In this case $\nu \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} = \nu(\alpha)$, for $\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \in A$, this last ν denotes a character of $k^{\times}$.
3. The linear characters of U are given by $\Psi \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \Psi(b)$, where this last Ψ is a character of k^{+}.
4. Let μ_1, μ_2 be elements of $\widehat{k^{\times}}$. Then the linear characters of T_1 are given by $(\mu_1, \mu_2) \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \mu_1(a)\,\mu_2(d)$.

Remark 28. Self-duality of k^{+}

Let us fix a non-trivial character Ψ of k^{+}. If $a \in k^{+}$, we define Ψ^a as $\Psi^a(u) = \Psi(au)$ Then $\widehat{k^{+}} = \{\Psi^a / a \in k^{+}\}$.

2.4. Representations of Q.

The number of conjugacy classes of Q is q, so that the number of irreducible representations of Q is q.

It is easy to verify that Q', the derived subgroup of Q, is equal to U. Since $\left[Q:Q'\right]=[Q:U]=q-1$, then the number of linear characters of Q is $q-1$. To describe these irreducible representations we use the fact that $Q=UA$, so that $\widehat{Q}=\widehat{UA}$. Let χ be an element of $\widehat{A}$. We "lift" χ to a linear character $\widetilde{\chi}$ of UA through $\widetilde{\chi}(ua)=\chi(a)$ $(U\lhd Q)$. Since the number of characters of A is $q-1$, this gives all the linear characters of Q.

We describe now the irreducible representation of Q of dimension greater than one. To this end, let us consider a non trivial character Ψ of U, (see 3) after 2.3), and let $\pi=\operatorname{Ind}_U^Q\Psi$. We will prove that this representation of dimension $q-1$ (notice that $\dim\pi=[Q:U]=q-1$) is irreducible. In fact,

$$\begin{aligned}
\operatorname{Hom}_Q(\pi,\pi) &\simeq \bigoplus_{x\in U/Q/U} \operatorname{Hom}_{xUx^{-1}\cap U(\Psi,\Psi^x)},\\
&\simeq \bigoplus_{x\in Q/U} \operatorname{Hom}_{xUx^{-1}\cap U(\Psi,\Psi^x)},\ U\lhd Q\\
&\simeq \bigoplus_{a\in A} \operatorname{Hom}_{aUa^{-1}\cap U}(\Psi,\Psi^a),\ Q/U\simeq A\\
&\simeq \bigoplus_{a\in A} \operatorname{Hom}_U(\Psi,\Psi^a)
\end{aligned}$$

Finally, we observe that, if $a=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, we have $\operatorname{Hom}_U(\Psi,\Psi)\simeq\mathbf{C}$, and if $a\neq\begin{pmatrix}1&0\\0&1\end{pmatrix}$, then $\operatorname{Hom}_U(\Psi,\Psi^a)=\{0\}$, from which we get that $\dim_{\mathbf{C}}(\operatorname{Hom}_Q(\pi,\pi))=1$ and π is irreducible.

2.5. Representations of B.

Using the fact that $B=T_1U$ and UB, we can define a character of B by "lifting" a character of T_1 to B, in the same way as we did above with Q, i.e., $\mu\begin{pmatrix}a&b\\0&d\end{pmatrix}=\mu_1(a)\,\mu_2(d)\,,(\mu_1,\mu_2\in\widehat{k^\times})$. We produce in this form $(q-1)^2$ characters. On the other hand $\left[B:B'\right]=[B:U]=\frac{(q-1)^2q}{q}=(q-1)^2\,,$

so that the above characters are all the linear characters of B.

We will construct $q-1$ non-isomorphic irreducible representations of dimension $q-1$. Given that $(q-1)^2 \cdot 1 + (q-1)(q-1)^2 = q(q-1)^2 = |B|$, we will obtain all the irreducible representations of B. We will construct these representations as follows: let χ be any of the $q-1$ characters of Z, we extend χ to $\widetilde{\chi}$, a character of $B = ZQ$ by $\widetilde{\chi}(zt) = \chi(z)$. Furthermore let us consider the natural proyection $p : B \longrightarrow Q$ and the representation $\pi = \mathrm{Ind}_U^Q \Psi$. Then $\widetilde{\pi} = \pi \circ p$ defines an irreducible representation of B of dimension $q-1$. Finally we define the representations $\widetilde{\chi}\widetilde{\pi}$, $(\chi \in \widehat{Z})$, by $\widetilde{\chi}\widetilde{\pi}(r) = \widetilde{\chi}(r)\,\widetilde{\pi}(r)$. This provides the $q-1$ irreducible representations which we state before.

3. Principal Series of $GL_2(k)$

A representation π of G is a principal series representation if it can be obtained by induction to G from a character μ of B.

Let $b5 = (b5_1, b5_2)$ be a character of B, and let $\widehat{\mu} = \mathrm{Ind}_B^G \mu$. Then

Lemma 29.

1) $\dim V_{\widehat{\mu}} = [G : B] \dim \mu = [G : B] = q + 1$

2) $\mathrm{Hom}_G(\mu, \mu) = [\mathrm{Hom}_B((\mu_1, \mu_2), (\mu_1, \mu_2))] \oplus [\mathrm{Hom}_{T_1}((\mu_1, \mu_2), (\mu_2, \mu_1))]$

Proof. We will prove 2), since 1) is clear.

$$\mathrm{Hom}_G(\widehat{\mu}, \widehat{\mu}) = \mathrm{Hom}_B\left((\mu_1, \mu_2), \mathrm{Res}_B^G\left(\mathrm{Ind}_B^G(\mu_1, \mu_2)\right)\right)$$

$$= \mathrm{Hom}_B\left((\mu_1, \mu_2), \mathrm{Ind}\,_{B\cap B}^{B}(\mu_1, \mu_2) \oplus \mathrm{Ind}\,_{B\cap\omega_1 B\omega_1^{-1}}^{B}(\mu_1, \mu_2)^{\omega_1}\right)$$

$$= [\mathrm{Hom}_B((\mu_1, \mu_2), (\mu_1, \mu_2))] \oplus \left[\mathrm{Hom}_B\left((\mu_1, \mu_2), \mathrm{Ind}\,_{T_1}^{B}(\mu_1, \mu_2)^{\omega_1}\right)\right]$$

$$= [\mathrm{Hom}_B((\mu_1, \mu_2), (\mu_1, \mu_2))] \oplus [\mathrm{Hom}_{T_1}((\mu_1, \mu_2), (\mu_2, \mu_1))] \;\square$$

Corollary 30 1) If $\mu_1 \neq \mu_2$, then $\widehat{\mu}$ is irreducible.

2) If $\mu_1 = \mu_2$, then $\widehat{\mu}$ is the sum of two irreducible representations of dimension 1 and q, respectively. This last representation is called the Steinberg representation associated to the character α.

Proof. 1)Since $(\mu_1, \mu_2) \not\simeq (\mu_2, \mu_1)$ as characters of T_1 we have that $\mathrm{Hom}_{T_1}((\mu_1, \mu_2), (\mu_2, \mu_1)) = 0$. Then, from Lemma 29, the space $\mathrm{Hom}_G(\widehat{\mu}, \widehat{\mu})$ is one dimensional. So, the result follows.

2) In this case, and using the above Lemma, we have that $dim\,\mathrm{Hom}_G(\widehat{\mu}, \widehat{\mu}) = 1 + 1 = 2$, which implies that $\widehat{\mu}$ is the sum of two irreducible representations. Let now $\alpha \in \widehat{k^\times}$, then $\alpha \circ \det = \chi^1_\alpha$ is a character of G. Using Frobenius Reciprocity, we get

$$\left\langle \widehat{\mu}, \chi^1_\alpha \right\rangle = \begin{cases} 1 & \text{si } \mu_1 = \alpha \\ 0 & \text{si } \mu_1 \neq \alpha \end{cases}$$

Thus

$$\begin{aligned} \langle \widehat{\mu} - \chi^1_\alpha, \widehat{\mu} - \chi^1_\alpha \rangle &= \langle \widehat{\mu}, \widehat{\mu} \rangle - 2 \langle \chi^1_\alpha, \widehat{\mu} \rangle + \langle \chi^1_\alpha, \chi^1_\alpha \rangle \\ &= 2 - 2 \cdot 1 + 1 = 1 \end{aligned}$$

and

$$\widehat{\mu} = \chi^1_\alpha \oplus St^q_\alpha$$

this last summand is the irreducible q-dimensional representation called the Steinberg representation asociated to the character α. $\square$

Remark 31. We have obtained, via induction from characters of B, the following:

1. $\dfrac{(q-1)(q-2)}{2}$ irreducible representations $\widehat{\mu} = \mathrm{Ind}^G_B \, \mu$, $\quad b5 = (b5_1, b5_2)$, $b5_1 \neq b5_2$, of dimension $q + 1$.
2. $q - 1$ irreducible representations St^q_α of dimension q.
3. $q - 1$ irreducible representations χ^1_α of dimension 1.

4. Discrete Series of $GL(2, k)$.

Any irreducible representation (V, ρ) of $GL_2(k)$ is a discrete series representation if and only if (V, ρ) is not an induced representation of any character from the Borel subgroup.

We will construct the characters of the Discrete Series. These characters will be described as differences of induced characters from some subgroups of $GL_2(k)$. We will follow [A]. To this end, let Γ_{-1} be the set of characters

ρ on T_{-1} such that $\rho \neq \rho^{\omega_{-1}}$, i.e., $\rho(x) \neq \rho(\omega_{-1}^{-1} x \omega_{-1})$ for $x \in T_{-1}$. We fix a non trivial character φ of k^+.

Theorem 32. Let $\rho \in \Gamma_{-1}$ and let λ be the character of ZU, defined by

$\lambda(z \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}) = \rho(z)\varphi(b)$ for $z \in Z$ and $b \in k$. Then:

1. The character, $\chi_\rho = Ind_{ZU}^G \lambda - Ind_{T_{-1}}^G \rho$ is the character of an irreducible representation of $GL(2,k)$ of dimension $q-1$.
2. Furthermore, $\chi_\rho = \chi_{\rho'}$, if and only if ρ' is either ρ or $\rho^{\omega_{-1}}$.

Proof. We show first that χ_ρ is an irreducible character of G, by showing that $\langle \chi_\rho, \chi_\rho \rangle_G = 1$.

$\langle \chi_\rho, \chi_\rho \rangle_G = \left\langle \mathrm{Ind}_{T_{-1}}^G \rho, \mathrm{Ind}_{T_{-1}}^G \rho \right\rangle_G - 2 \left\langle \mathrm{Ind}_{T_{-1}}^G \rho, \mathrm{Ind}_{ZU}^G \lambda \right\rangle_G +$
$\left\langle \mathrm{Ind}_{ZU}^G \lambda, \mathrm{Ind}_{ZU}^G \lambda \right\rangle_G$

We need to compute the above last three summands. In first place, let us apply Mackey's Intertwining Theorem, (Theorem 19.), to get

$$\left\langle \mathrm{Ind}_{T_{-1}}^G \rho, \mathrm{Ind}_{T_{-1}}^G \rho \right\rangle_G = \textstyle\sum_{h \in T_{-1} \backslash G / T_{-1}} \left\langle \rho^h, \mathrm{Res}\, \rho \right\rangle_{T_{-1,h}},$$

where $T_{-1,h} = hT_{-1}h^{-1} \cap T_{-1}$, and h is a representative for the (T_{-1}, T_{-1})-double cosets of G.

By Lemma 27 part (i) if $h \in N_G(T_{-1})$ then there are two (T_{-1}, T_{-1})-double cosets, whose representatives can be taken to be 1 and ω_{-1}. Given that $\rho \neq \rho^{\omega_{-1}}$ and $T_{-1,h} = T_{-1}$ $(h \in N_G(T_{-1}))$, we have

$$\sum_{h \in T_{-1} \backslash G / T_{-1}} \left\langle \rho^h, \mathrm{Res}\, \rho \right\rangle_{T_{-1,h}} = 1, \ \ h \in N_G(T_{-1}).$$

On the other hand, if $h \notin N_G(T_{-1})$ then $T_{-1,h} = Z$, therefore, $\left\langle \rho^h, \mathrm{Res}\, \rho \right\rangle_{T_{-1,h}} = 1$ for each $h \notin N_G(T_{-1})$. Applying once more time the mentioned Lemma, we get,

$$\left\langle \mathrm{Ind}_{T_{-1}}^G \rho, \mathrm{Ind}_{T_{-1}}^G \rho \right\rangle_G = q - 2 + 1 = q - 1$$

Again, Mackey's Theorem allows us to compute the second above summand.

$$\left\langle \operatorname{Ind}_{T_{-1}}^G \rho, \operatorname{Ind}_{ZU}^G \lambda \right\rangle_G = \sum_{h\in ZU\backslash G/T_{-1}} \left\langle \rho^h, \operatorname{Res}\mu \right\rangle_{T_{-1,h}}$$

Since $T_{-1,h} = Z$ for each $h \in ZU\backslash G/T_{-1}$ and $\rho^h(x) = \rho(h^{-1}xh) = \rho(x) = \lambda(x)$ for $x \in Z$, by Lemma 27 part (ii) we have

$$\left\langle \operatorname{Ind}_{T_{-1}}^G \rho, \operatorname{Ind}_{ZU}^G \lambda \right\rangle_G = q-1$$

Finally, regarding the third summand, similar arguments apply to get

$$\left\langle \operatorname{Ind}_{ZU}^G \lambda, \operatorname{Ind}_{ZU}^G \lambda \right\rangle_G = \sum_{h\in ZU\backslash G/ZU} \left\langle \lambda^h, \operatorname{Res}\lambda \right\rangle_{ZU_h}$$

In this case, Lemma 27 part (iii) says that we have $q-1$ (ZU, ZU)-double cosets with representatives $\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$, $t \in k^\times$, that lie in $N_G(ZU)$. Since

$$\lambda^{\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}} \begin{pmatrix} a & ab \\ 0 & a \end{pmatrix} = \lambda \begin{pmatrix} a & tab \\ 0 & a \end{pmatrix} = \rho \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \lambda(tb)$$

and

$$\lambda \begin{pmatrix} a & ab \\ 0 & a \end{pmatrix} = \rho \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \lambda(b),$$

then, $h = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the only contribution into the sum. i.e.,

$$\sum_{h\in ZU\backslash G/ZU} \left\langle \lambda^h, \operatorname{Res}\lambda \right\rangle_{ZU_h} = 1, \qquad h \in N_G(ZU).$$

On the other hand, if $h \notin N_G(ZU)$, then $(ZU)_h = hZUh^{-1} = Z$ and $\lambda^h = \lambda$ on Z.

Therefore, we have

$$\left\langle \operatorname{Ind}_{ZU}^G \lambda, \operatorname{Ind}_{ZU}^G \lambda \right\rangle_G = 1 + q - 1 = q.$$

Gathering the above information, we obtain $\langle \chi_\rho, \chi_\rho \rangle_G = (q-1) - 2(q-1) + q = 1$.

Thus, χ_ρ is an irreducible character of G. Furthermore, the dimension of ρ is $q-1$:

$$\begin{aligned}
\chi_\rho(1) &= \operatorname{Ind}_{ZU}^G \lambda(1) - \operatorname{Ind}_{T_{-1}}^G \rho(1) \\
&= [G:ZU] - [G:T_{-1}] \\
&= \frac{q(q+1)(q-1)^2}{q(q-1)} - \frac{q(q-1)(q^2-1)}{q^2-1} \\
&= (q-1)(q+1-q) \\
&= q-1
\end{aligned}$$

To prove part 2 of the theorem, we perform the following computations:

$$\begin{aligned}
\langle \chi_\rho, \chi_{\rho^{\omega_{-1}}} \rangle_G &= \left\langle \operatorname{Ind}_{T_{-1}}^G \rho, \operatorname{Ind}_{T_{-1}}^G \rho^{\omega_{-1}} \right\rangle_G - \left\langle \operatorname{Ind}_{ZU}^G \lambda, \operatorname{Ind}_{T_{-1}}^G \rho^{\omega_{-1}} \right\rangle_G \\
&\quad - \left\langle \operatorname{Ind}_{T_{-1}}^G \rho, \operatorname{Ind}_{ZU}^G \lambda \right\rangle_G + \left\langle \operatorname{Ind}_{ZU}^G \lambda, \operatorname{Ind}_{ZU}^G \lambda \right\rangle_G \\
&= q-1-(q-1)-(q-1)+q \\
&= 1
\end{aligned}$$

which implies that

$$\chi_\rho = \chi_{\rho^{\omega_{-1}}}.$$

Now, let $\rho^1 \not\simeq \rho^{\omega_{-1}}$ and $\rho^1 \not\simeq \rho$; then

$$\begin{aligned}
\left\langle \chi_\rho, \chi_{\rho^1} \right\rangle_G &= \left\langle \operatorname{Ind}_{T_{-1}}^G \rho^1, \operatorname{Ind}_{T_{-1}}^G \rho^1 \right\rangle_G - \left\langle \operatorname{Ind}_{ZU}^G \lambda, \operatorname{Ind}_{T_{-1}}^G \rho \right\rangle_G \\
&\quad - \left\langle \operatorname{Ind}_{T_{-1}}^G \rho^1, \operatorname{Ind}_{ZU}^G \lambda \right\rangle_G + \left\langle \operatorname{Ind}_{ZU}^G \lambda, \operatorname{Ind}_{ZU}^G \lambda \right\rangle \\
&= q-2-(q-1)-(q-1)+q \\
&= 0
\end{aligned}$$

We have proved then: $\chi_\rho = \chi_{\rho^1}$ if and only if, either $\rho^1 \simeq \rho$ or $\rho^1 \simeq \rho^{\omega_{-1}}$. $\square$

5. Summary

We list below the irreducible characters of the group $G = Gl_2(k)$, where k is a finite field of odd characteristic:

1. There are $\dfrac{(q-1)\,(q-2)}{2}$ characters χ_μ corresponding to the representations $\widehat{\mu} = \operatorname{Ind}_B^G(\mu_1, \mu_2)$, (with $\mu_1 \neq \mu_2$), of dimension $q+1$. (See section 3).

2. There are $q-1$ characters χ_α^q corresponding to the representations St_α^q, of dimension q. (See section 3).

3. There are $q-1$ characters χ^1_α of dimension 1. (See section 3).

4. There are $\dfrac{q^2-q}{2}$ characters χ_ρ belonging to the Discrete Series of dimension $q-1$. (See section 4)

Thus, we have found q^2-1 distint irreducible characters of G. Since the number of conjugacy classes of G is q^2-1, we have obtained all the characters of G. We write down now the character table of G.

	Z	U	T_1	T_{-1}
χ^1_α	$\alpha \circ \det$	$\alpha \circ \det$	$\alpha \circ \det$	$\alpha \circ \det$
χ^q_α	$q(\alpha \circ \det)$	0	$\alpha \circ \det$	$-\alpha \circ \det$
χ_μ	$(q+1)\mu$	μ	$\mu + \mu^{\omega_1}$	0
χ_ρ	$(q-1)\rho$	$-\rho$	0	$-(\rho + \rho^{\omega_{-1}})$

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Luisa Aburto-Hageman
Instituto de Matemáticas
Pontificia Universidad Católica de Valparaíso
Casilla 4059
Valparaíso
CHILE
email: laburto@ucv.cl

Roberto Johnson
Instituto de Matemáticas
Pontificia Universidad Católica de Valparaíso
Casilla 4059
Valparaíso
CHILE
email: rjohnson@ucv.cl

and

José Pantoja
Instituto de Matemáticas
Pontificia Universidad Católica de Valparaíso
Casilla 4059
Valparaíso
CHILE
email: jpantoja@ucv.cl