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ON OPERATOR IDEALS DEFINED BY A REFLEXIVE ORLICZ SEQUENCE SPACE

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Abstract

Classical theory of tensornorms and operator ideals studies mainly those defined by means of sequence spaces ℓ_p. Considering Orlicz sequence spaces as natural generalization of ℓ_p spaces, in a previous paper [12] an Orlicz sequence space was used to define a tensornorm, and characterize minimal and maximal operator ideals associated, by using local techniques. Now, in this paper we give a new characterization of the maximal operator ideal to continue our analysis of some coincidences among such operator ideals. Finally we prove some new metric properties of tensornorm mentioned above.

Key words : *Maximal operator ideals. Ultraproducts of spaces, Orlicz spaces.*

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1. Introduction

Using ideas from ℓ_p spaces, Saphar introduced a tensornorm g_p (see [18]) and given the great relationship between tensornorms and operator ideals, minimal and maximal operator ideals associated to the tensornorm in the sense of Defant and Floret were studied in the classical theory of tensornorm and operator ideals.

Since Orlicz spaces ℓ_M are natural generalizations of ℓ_p spaces, in [12] we introduced a tensornorm g_M^c by means of an Orlicz space ℓ_M and using some aspects of local theory we characterized the minimal and maximal operator ideals associated to g_M^c. Now in this paper our aim is to give another characterization of maximal operator ideal associated to this tensornorm to study the coincidence between components of the two operator ideals which in turn enables us to prove some metric properties of g_M^c and its dual.

Notation is standard. We will always consider Banach spaces over the real field, since we shall use results in the theory of Banach lattices. The canonical inclusion map from Banach space E into the bidual E'' will be denoted by J_E. In general if E is a subspace of F, the inclusion of E into F is denoted by $I_{E,F}$. The set of finite dimensional subspaces of a normed space E will be denoted by $FIN(E)$.

We recall the more relevant aspects on Banach lattices (we refer the reader to [1]). A Banach lattice E is order complete or Dedekind complete if every order bounded set in E has a least upper bound in E, and it is order continuous if every order convergent filter is norm convergent. Every dual Banach sequence lattice E' is order complete and all reflexive spaces are even order continuous. A linear map T between Banach lattices E and F is said to be positive if $T(x) \geq 0$ in F for every $x \in E, x \geq 0$. T is called order bounded if $T(A)$ is order bounded in F for every order bounded set A in E.

Let ω be the vector space of all scalar sequences and φ its subspace of the sequences with finitely many non zero coordinates. A sequence space λ is a linear subspace of ω containing φ provided with a topology finer than the topology of coordinatewise convergence. *A Banach sequence space* will be a sequence space λ provided with a norm which makes it a Banach lattice and an ideal in ω, i.e. such that if $|x| \leq |y|$ with $x \in \omega$ and $y \in \lambda$, then $x \in \lambda$ and $\|x\|_\lambda \leq \|y\|_\lambda$. A sectional subspace $S_k(\lambda), k \in \mathbf{N}$, is the topological subspace of λ of those sequences (α_i) such that $\alpha_i = 0$ for every $i \geq k$. Clearly $S_k(\lambda)$ is 1-complemented in λ. A Banach sequence space

λ will be called regular whenever the sequence $\{\mathbf{e_i}\}_{\mathbf{i=1}}^{\infty}$ where $\mathbf{e}_i := (\delta_{ij})_j$ (Kronecker's delta) is a Schauder base in λ.

We now discuss Orlicz spaces. A non degenerated Orlicz function M is a continuous, non decreasing and convex function defined in $\mathbf{R}^+$ such that $M(t) = 0$ if and only if $t = 0$ and $\lim_{t\to\infty} M(t) = \infty$. The Orlicz sequence space ℓ_M is the space of all sequences $a = (a_i)_{i=1}^{\infty}$ such that $\sum_{i=1}^{\infty} M(|a_i|/c) < \infty$, for some $c > 0$. The functional

$$\Pi_M(a) = \inf\{c > 0 : \sum_{i=1}^{\infty} M(|a_i|/c) \leq 1\}$$

is a norm in ℓ_M and $(\ell_M, \Pi_M(.))$ is a Banach space. We say that an Orlicz function M has the Δ_2 property at zero if $M(2t)/M(t)$ is bounded in a neighbourhood of zero. In general ℓ_M is not regular, but this is the case if and only if M satisfies the Δ_2 property at zero.

We say that the function M^* is the complementary of M if $M^*(u) := \max\{ut - M(t) : 0 < t < \infty\}$ then H^* is also an Orlicz function. Associated to M^* we can introduce a new norm on ℓ_M, defined by

$$\|a\|_M = \sup\{\sum_{n=1}^{\infty} a_n b_n : \Pi_M((b_n)) \leq 1\}$$

if $a = (a_i)_{i=1}^{\infty}$ which is equivalent to $\Pi_M(.)$. Then if M has the Δ_2-property at zero, $(\ell_M, \Pi_H(.))' = (\ell_{H^*}, \|.\|_{M^*})$ as isometric spaces. Moreover ℓ_M is reflexive if and only if M and M^* have the Δ_2-property at zero. For more information on Orlicz functions and Orlicz spaces we refer to [13].

All Orlicz spaces in this paper are considered regular and reflexive. Moreover we will always suppose that $M(1) = 1$.

Let (Ω, Σ, μ) be a measure space, we denote by $L_0(\mu)$ the space of equivalence classes, modulo equality μ-almost everywhere, of μ-measurable real-valued functions, endowed with the topology of local convergence in measure. And the space of all equivalence classes of μ-measurable X-valued functions is denoted by $L_0(\mu, X)$. By a Köthe function space $\mathcal{K}(\mu)$ on (Ω, Σ, μ), we shall mean an order dense ideal of $L_0(\mu)$, which is equipped with a norm $\|.\|_{\mathcal{K}(\mu)}$ that makes it a Banach lattice(if $f \in L_0(\mu)$ and $g \in \mathcal{K}(\mu)$ $|f| \leq |g|$, then $f \in \mathcal{K}(\mu)$ with $\|f\|_{\mathcal{K}(\mu)} \leq \|g\|_{\mathcal{K}(\mu)}$). Similarity, $\mathcal{K}(\mu, X) = \{f \in L_0(\mu, X) : \|f(.)\|_X \in \mathcal{K}(\mu)\}$, endowed with the norm $\|f\|_{\mathcal{K}(\mu,X)} = \|\|f(.)\|_X\|_{\mathcal{K}(\mu)}$.

On theory of operator ideals and tensor norms we refer to the books [16] and [4] of Pietsch and Defant and Floret respectively.

If E and F are Banach spaces and α is a tensor norm, then $E \otimes_\alpha F$ represents the space $E \otimes F$ endowed with the α-normed topology. The completion of $E \otimes_\alpha F$ is denoted by $E\hat{\otimes}_\alpha F$, and the norm of z in $E\hat{\otimes}_\alpha F$ by $\alpha(z; E \otimes F)$. If there is no risk of confusion we write $\alpha(z)$ instead of $\alpha(z; E \otimes F)$.

2. The tensor norm g_M^c and M-nuclear operators associated to an Orlicz function M

First we establish some notation. Given a Banach space E and an Orlicz function M with $M(1)$=1 such that ℓ_M is reflexive, we say that a sequence $(x_n)_{n=1}^\infty \in E^{\mathbf{N}}$ is strongly M-summing if $(\|x_n\|) \in \ell_M$ and we write $\pi_M((x_i)) := \Pi_H((\|x_n\|))$ and it is said to be weakly M-summing if $\varepsilon_M((x_i)) : = \sup_{\|x'\|\leq 1} \|(|\langle x_n, x'\rangle|)\|_M$. We denote by $\ell_M[E]$ (resp. $\ell_M(E)$) the space of all strongly (resp. weakly) M-summing in E with the norm $\pi_M(.)$ (resp. $\varepsilon_M(.)$).

The more natural approach to define a tensornorm in analogy to Saphar's tensornorm is as follows. Let E and F be Banach spaces and $z \in E \otimes F$, we define

$$g_M(z; E \otimes F) := \inf \pi_M((x_n))\, \varepsilon_{M^*}((y_n))$$

taking the infimum over all representations of z as $\sum_{n=1}^m x_n \otimes y_n$. We will write $g_M(z)$ instead of $g_M(z; E \otimes F)$ if there is not possibility of confusion.

It is possible that for some M the functional g_M does not satisfy the triangle inequality, but it is always a reasonable *quasi norm* on $E \otimes F$, see [3] and [6]. We denote $E\hat{\otimes}_{g_M} F$ the corresponding quasi Banach space.

To have a tensornorm g_M^c in [12] we took the Minkowski functional, denoted $g_M^c(z; E \otimes F)$, of the absolutely convex hull of the unit closed ball $B_{g_M} := \{z \in E \otimes F \ / \ g_M(z) \leq 1\}$ of the quasi norm g_M in $E \otimes F$, such that

$$g_M^c(z; E \otimes F) := \inf \sum_{i=1}^n \pi_M((x_{ij}))\, \varepsilon_{M^*}((y_{ij}))$$

taking the infimum over all representations of z as $\sum_{i=1}^n \sum_{j=1}^m x_{ij} \otimes y_{ij}$. Again, we will write $g_M(z)$ instead of $g_M(z; E \otimes F)$ if there is not possibility of confusion.

It is easy to see that g_M^c *is a tensor norm* on the class of all Banach spaces (using criterion 12.2 in [4] and bearing in mind that $\pi_M(\mathbf{e}_i) = \|\mathbf{e}_i\|_{M^*} = 1$ for every $i \in \mathbf{N}$), and that $\forall\ z \in E \otimes F \quad g_M^c(z; E \otimes F) \leq g_M(z; E \otimes F)$. We denote $E\hat{\otimes}_{g_M^c} F$ the corresponding Banach space.

Proceeding as in [3] and [18], it is easy to see that if $z \in E\hat{\otimes}_{g_M}F$, there are $(x_i)_{i=1}^{\infty} \in \ell_M[E]$ and $(y_i)_{i=1}^{\infty} \in \ell_{H^*}(F)$ such that $\pi_M((x_i))\,\varepsilon_{M^*}((y_i)) < \infty$ and $z = \sum_{i=1}^{\infty} x_i \otimes y_i$. Moreover the quasi norm of z in $E\hat{\otimes}_{g_M}F$ (again denoted by $g_M(z)$) is given by $g_M(z) = \inf \pi_M((x_i))\,\varepsilon_{M^*}((y_i))$ taking the infimum over all such representations of z as $\sum_{n=1}^{m} x_n \otimes y_n$. Similarly, if $z \in E\hat{\otimes}_{g_M^c}F$ then z can be represented as $z = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x_{ij} \otimes y_{ij}$ where $(x_{ij})_{i=1}^{\infty} \in \ell_M[E]$ for each $j \in \mathbf{N}$, $(y_{ij})_{i=1}^{\infty} \in \ell_{M^*}(F)$ for each $j \in \mathbf{N}$ and $\sum_{j=1}^{\infty} \pi_M((x_{ij}))\;\varepsilon_{M^*}((y_{ij})) < \infty$. Moreover, the norm of z in $E\hat{\otimes}_{g_M^c}F$ is $g_M^c(z) = \inf \sum_{j=1}^{\infty} \pi_M((x_{ij}))\;\varepsilon_{M^*}((y_{ij}))$ taking the infimum over all representations of z as $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x_{ij} \otimes y_{ij}$.

The topology defined by the quasi norm g_M on $E \otimes F$ is normable with norm equivalent to g_M^c. In fact, being ℓ_M a reflexive Orlicz space and following the arguments of proposition 16 of [3], we consider the bilinear onto map $R : \ell_M[E] \times \ell_{M^*}(F) \to E\hat{\otimes}_{g_M}F$ such that $R((x_i),(y_i)) = \sum_{i=1}^{\infty} x_i \otimes y_i$. R is continuous with quasi norm less or equal one. Then there exists a unique linear and continuous map $\ell_M[E] \otimes_\pi \ell_{M^*}(F) \to E\hat{\otimes}_{g_M}F$ (see [20]). This map can be extended to a continuous linear and onto map $\ell_M[E]\hat{\otimes}_\pi \ell_{M^*}(F) \to E\hat{\otimes}_{g_M}F$ which is open by the open mapping theorem. Then $E\hat{\otimes}_{g_M}F$ is isomorphic to a quotient of a Banach space and so it is a Banach space itself. In this way there is *a norm* $w_M(.;E \otimes F)$ equivalent to the quasi norm $g_M(.;E \otimes F)$ furthermore it is easy to see that $w_M(.;E \otimes F)$, $g_M(.;E \otimes F)$ and $g_M^c(.;E \otimes F)$ are equivalent with $g_M^c(.;E \otimes F) \le w_M(.;E \otimes F)$. Given the last equivalence, g_M seems appropriate for our purposes, but we need g_M^c for our main results.

To introduce M-nuclear operators, bearing in mind that every representation of $z \in E'\hat{\otimes}_{g_M^c}F$ as $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x_{ij} \otimes y_{ij}$ defines a map $T_z \in \mathcal{L}(E,F)$ such that $\forall x \in E$, $T_z(x) := \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\langle x'_{ij}, x\rangle\; y_{ij}$. Furthermore, T_z is well defined and independent of the chosen representation for z. Let $\Phi_{EF} : E'\hat{\otimes}_{g_M^c}F \to \mathcal{L}(E,F)$ be defined by $\Phi_{EF}(z) := T_z$.

Definition 1. *An operator between Banach spaces $T : E \to F$ is said to be M- nuclear if $T = \Phi_{EF}(z)$, for some $z \in E'\hat{\otimes}_{g_M^c}F$.*

Given any pair of Banach spaces E and F, the space of the M-nuclear operators $T : E \to F$ endowed with the topology of the norm $\mathbf{N}_M^c(T) := \inf\{g_M^c(z) \;/\; \Phi_{EF}(z) = T\}$ or with the equivalent quasi-norm $\mathbf{N}_M(T) := \inf\{g_M(z) \;/\; \Phi_{EF}(z) = T\}$ is denoted by $\mathcal{N}_H(E,F)$. Also $(\mathcal{N}_M(E,F), \mathbf{N}_M^c)$ denotes a component of the minimal Banach operator ideal $(\mathcal{N}_M, \mathbf{N}_M^c)$ associated to the tensor norm g_M^c. Analogously as in [12] we obtain the following result.

Theorem 2. *Let E, F be any pair of Banach spaces and an operator $T \in \mathcal{L}(E, F)$. Then the following are equivalent:*

1) T is M-nuclear.

2) T factors continuously in the following way:

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[d, "A"'] & F \\ \ell_\infty \arrow[r, "B"'] & \ell_M \arrow[u, "C"'] \end{tikzcd}$$

where B is a diagonal multiplication operator defined by a positive sequence $(b_i) \in \ell_M$.

Furthermore $\mathbf{N}_M(T) = \inf\{\|C\|\|B\|\|A\|\}$, infimum taken over all such factors.

3) T factors continuously in the following way:

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[d, "A"'] & F \\ \ell_\infty[\ell_\infty] \arrow[r, "B"'] & \ell_1[\ell_M] \arrow[u, "C"'] \end{tikzcd}$$

where B is a diagonal multiplication operator defined by a positive sequence $(b_i) \in \ell_1[\ell_M]$.

Furthermore $\mathbf{N}^c_M(T) = \inf\{\|C\|\|B\|\|A\|\}$, infimum taken over all such factors.

Associated to g_M and g^c_M, there are other important operator ideal.

Definition 3. *Let $T \in \mathcal{L}(E, F)$, we say that T is M-absolutely summing if exist a real number $C > 0$, such that for all sequences (x_i) in E, with $\varepsilon_M((x_i)) < \infty$, it satisfies that*

$$\|(T(x_i))\|_M \leq C\varepsilon_M((x_i)) \tag{2.1}$$

For $\mathcal{P}_M(E,F)$ we denote the Banach ideal of the M-absolutely summing operators $T : E \to F$ endowed with the topology of the norm $\mathbf{P}_M(T) := \inf C$, taking the infimum over all C that satisfies (2.1)

Theorem 4. *Let E and F be Banach spaces.* $\left(E \otimes_{g^c_M} F\right)' = \mathcal{P}_{M^*}(F, E')$ *isometrically.*

3. M-integral operators

The characterization of maximal operator ideal obtained in [12] was given in terms of theory of finite representability of Banach spaces and/or Banach Lattices.

In present paper, we give another characterization of such ideals by considering the structure of finite dimensional subspaces of Orlicz spaces involved. The behavior of the Orlicz sequences spaces under ultraproducts is also crucial.

On ultraproducts of Banach spaces we refer to [8]. We only set the notation we will use. Let D be a non empty index set and $\mathcal{U}$ a non-trivial ultrafilter in D. Given a family $\{X_d,\ d \in D\}$ of Banach spaces, $(X_d)_{\mathcal{U}}$ denotes the corresponding ultraproduct Banach space. If every $X_d,\ d \in D$, coincides with a fixed Banach space X the corresponding ultraproduct is named an ultrapower of X and is denoted by $(X)_{\mathcal{U}}$. Recall that if every $X_d,\ d \in D$ is a Banach lattice, $(X_d)_{\mathcal{U}}$ has a canonical order which makes it a Banach lattice. If we have another family of Banach spaces $\{Y_d,\ d \in D\}$ and a family of operators $\{T_d \in \mathcal{L}(X_d, Y_d),\ d \in D\}$ such that $\sup_{d\in D} \|T_d\| < \infty$, then $(T_d)_{\mathcal{U}} \in \mathcal{L}((X_d)_{\mathcal{U}}, (Y_d)_{\mathcal{U}})$ denotes the canonical ultraproduct operator.

We now give a local definition which has been inspired in Gordon and Lewis definition of local unconditional structure.

Definition 5. Given a sequence space λ, we say that a Banach space X has an $S_k(\lambda)$-local unconditional structure if there exists a real constant $c > 0$ such that for every finite dimensional subspace F of X, there is a section $S_n(\lambda)$ of λ and linear operators $u : F \to S_n(\lambda)$ and $v : S_n(\lambda) \to X$ such that $\|u\|\|v\| \le c$ and $v\,u = I_{F,X}$.

The constant c which appears in above definition is called a $S_k(\lambda)$-local unconditional structure constant of X and in this case we say that X has c-$S_k(\lambda)$-local unconditional structure. If a Banach space X has c-$S_k(\lambda)$-local unconditional structure for every $c > C$ we say that it has C^+-$S_k(\lambda)$-local unconditional structure.

The following definition was introduced by Pelczyǹnski and Rosenthal [15] in 1975.

Definition 6. A Banach space X has the *uniform projection property* if there is a $b > 0$ such that for each natural number n there is a natural number $m(n)$ such that for every n-dimensional subspace $M \subset X$ there exists a k-dimensional and b-complemented subspace Z of X containing M with $k \leq m(n)$.

The constant b of the above definition is called a uniform projection property constant of X, and in this case we say that X has the b-uniform projection property. If X has the b-uniform projection property for every $b > B$ we say that X has the B^+-uniform projection property.

We need to remark the following aspects involving Orlicz spaces.

The class of Banach spaces with the uniform projection property is quite large and includes, for example the reflexive Orlicz spaces, see [14]. In particular they have the $1+\varepsilon$-uniform projection property for every $\varepsilon > 0$. Furthermore If $1 \leq p \leq \infty$ then, the Bochner space $L_p(\mu, E)$ and $\ell_p(E)$ has the b-uniform projection property if E does, see [8]. We highlight that the uniform projection property is stable under ultrapowers, see [8]. Moreover from [17].

Proposition 7. If ℓ_M is reflexive, then every ultrapower of $\ell_1[\ell_M]$ (of ℓ_M) has 1^+-$S_r(\ell_1)[S_k(\ell_M)]$-local unconditional structure (resp. 1^+-$S_k(\ell_M)$-local unconditional structure).

According to the general theory of tensor norms and operator ideals, the normed ideal of M-integral operators $(\mathcal{I}_M, \mathbf{I}_M)$ is the maximal operator ideal associated to the tensor norm g_M^c in the sense of Defant and Floret [4], or in an equivalent way, the maximal normed operator ideal associated to the normed ideal of M-nuclear operators in the sense of Pietsch [16]. From [4], for every pair of Banach spaces E and F, an operator $T : E \to F$ is M-integral if and only if $J_F T \in (E \otimes_{(g_\lambda^c)'} F')'$.

For every pair of Banach spaces E, F we define the finitely generated tensor norm g'_M such that if $M \in FIN(E)$ and $N \in FIN(F)$, for every $z \in M \otimes N$, $g'_M(z; M \otimes N) := \sup\{|\langle z, w\rangle| \ / \ g_M(w; M' \otimes N') \leq 1\}$. Clearly $g'_M = (g_M^c)'$ since the unit ball in $M' \otimes_{g_M^c} N'$ is the convex hull of the unit ball of $M' \otimes_{g_M} N'$. But we remark that $E' \otimes_{g_M^c} F'$ (and no $E' \otimes_{g_M} F'$) is an isometric subspace of $(E \otimes_{g'_M} F)'$ because g_M^c is finitely generated, see [4], 15.3.

In this case we define $\mathbf{I}_M(T)$ to be the norm of $J_F\, T$ considered as an element of the topological dual of the Banach space $E \otimes_{g'_\lambda} F'$. Remark that

$\mathbf{I}_M(T) = \mathbf{I}_M(J_F\, T)$ as a consequence of F' be canonically complemented in F'''.

First we give a non trivial example of M-integral operators.

Theorem 8. Let(Ω, Σ, μ) a measure space and let ℓ_M be a reflexive Orlicz sequence space. Then every order bounded operator $S : L_\infty(\mu) \to \ell_M$ and every order bounded operator $S : L_\infty(\mu) \to \ell_1[\ell_M]$ are M-integral with $\mathbf{I}_M(S) = \|S\|$.

Proof. We will only give the proof if $S : L_\infty(\mu) \to \ell_M$ is an order bounded operator since the proof in the other case is similar.

The predual space of ℓ_M is ℓ_M*, which is regular space because M^* has the Δ_2 property at zero. Then, the linear span $\mathcal{T}$ of the set $\{\mathbf{e}_i,\, i \in \mathbf{N}\}$ is dense in ℓ_M* and by the representation theorem of maximal operator ideals (see 17.5 in [4]) and the density lemma (theorem 13.4 in [4]) we only have to see that $S \in (L_\infty(\mu) \otimes_{g'_M} \mathcal{T})'$.

Given $z \in L_\infty(\mu) \otimes_{g'_M} \mathcal{T}$ and $\varepsilon > 0$, let X and Y be finite dimensional subspaces of $L_\infty(\mu)$ and $\mathcal{T}$ respectively such that $z \in X \otimes Y$ and

$$g'_M(z; X \otimes Y) \leq g'_M(z; L_\infty(\mu) \otimes \mathcal{T}) + \varepsilon. \tag{3.1}$$

Let $\{\mathbf{g}_s\}_{s=1}^m$ be a basis for Y and let $k \in \mathbf{N}$ be such that $\forall\ 1 \leq s \leq m \quad \mathbf{g}_s = \sum_{i=1}^k c_{si}\mathbf{e}_i$. Then $\forall\ f \in X, \ \ \forall\ 1 \leq s \leq m$

$$\langle S, f \otimes \mathbf{g}_s\rangle = \langle f, S'(\mathbf{g}_s)\rangle = \left\langle f, \left(\sum_{i=1}^k c_{si}\right) S'(\mathbf{e}_i)\right\rangle =$$

$$\left\langle f \otimes \sum_{j=1}^k c_{sj}\ \mathbf{e}_j, \sum_{i=1}^k S'(\mathbf{e}_i) \otimes \mathbf{e}_i \right\rangle.$$

Then if U denotes the tensor $U := \sum_{i=1}^k S'(\mathbf{e}_i) \otimes \mathbf{e}_i \in L_\infty(\mu)' \otimes \lambda$, by bilinearity we get $\forall\ z \in X \otimes Y \quad \langle z, S\rangle = \langle U, z\rangle$.

Given $\nu > 0$, for every $1 \leq i \leq k$ there is $f_i \in L_\infty(\mu)$ such that $\|f_i\| \leq 1$ and $\|S'(\mathbf{e}_i)\| \leq |\langle S'(\mathbf{e}_i), f_i\rangle| + \nu$. Then $f := \sup_{1\leq i\leq k} f_i$ lies in the closed unit ball of $L_\infty(\mu)$. On the other hand, ℓ_H is a dual lattice and hence it is order complete. By the Riesz-Kantorovich theorem (see theorem 1.13 in [1] for instance), the modulus $|S|$ of the operator S exists in $\mathcal{L}(L_\infty(\mu), \ell_M)$. By the lattice properties of ℓ_M we have

$$\pi_M((S'(e_i)) = \pi_M\left(\sum_{i=1}^k \|S'(\mathbf{e}_i)\|\ \mathbf{e}_i\right) \leq \pi_M\left(\sum_{i=1}^k |\langle S'(\mathbf{e}_i), f_i\rangle|\ \mathbf{e}_i\right)$$

$$+\nu\pi_M\left(\sum_{i=1}^{k}\mathbf{e}_i\right)\leq\leq\pi_M\left(\sum_{i=1}^{k}|\langle S(f_i),\mathbf{e}_i\rangle|\,\mathbf{e}_i\right)+\nu\pi_M\left(\sum_{i=1}^{k}\mathbf{e}_i\right)$$

$$\leq\pi_M\left(\sum_{i=1}^{k}\langle|S(f_i)|,\mathbf{e}_i\rangle\right)+\nu\pi_M\left(\sum_{i=1}^{k}\mathbf{e}_i\right)\leq\pi_M\left(\sum_{i=1}^{k}\langle|S|(|f_i|),\mathbf{e}_i\rangle\;\mathbf{e}_i\right)$$

$$+\nu\pi_M\left(\sum_{i=1}^{k}\mathbf{e}_i\right)\leq\pi_M\left(\sum_{i=1}^{k}\langle|S|(|f|),\mathbf{e}_i\rangle\;\mathbf{e}_i\right)+\nu\pi_M\left(\sum_{i=1}^{k}\mathbf{e}_i\right)=$$

$$=\pi_M\left(\,|S|(|f|)\right)+\nu\pi_M\left(\sum_{i=1}^{k}\mathbf{e}_i\right)\leq\|\;|S|\;\|+\nu\pi_M\left(\sum_{i=1}^{k}\mathbf{e}_i\right).$$

Moreover $\varepsilon_{M^*}((\mathbf{e}_i)_{i=1}^k)\leq 1$. Hence, denoting by I_X and I_Y the corresponding inclusion maps into $L_\infty(\mu)$ and ℓ_H respectively, we have

$$\begin{aligned}
&|\langle S,z\rangle|=|\langle U,z\rangle|=|\langle U,((I_X)'\otimes(I_Y)')(z)\rangle|\leq\\
&\leq g_M^c(U;X\otimes Y)\;g'_M(((I_X)'\otimes(I_Y)')(z);X'\otimes Y')\leq\\
&\leq g_M(U;X\otimes Y)\;g'_M(((I_X)'\otimes(I_Y)')(z);X'\otimes Y')\leq\\
&\leq(g_M(U;L_\infty\otimes(\ell_M))+\varepsilon)g'_M(z;L_\infty(\mu)\otimes\ell_{M^*})\leq\\
&\leq g'_M(z;L_\infty(\mu)\otimes\ell_{M^*}\;\;(\pi_M((S'(\mathbf{e}_i))\;\varepsilon_{M^*}((\mathbf{e}_i))+\varepsilon)\leq\\
&\leq g'_M(z;L_\infty(\mu)\otimes\ell_{M^*})\left(\|\;|S|\;\|+\nu\pi_M\left(\sum_{i=1}^{k}\mathbf{e}_i\right)+\varepsilon\right)
\end{aligned}$$

and ν being arbitrary $|\langle S,z\rangle|\leq g'_M(z;L_\infty(\mu)\otimes\ell_{M^*})(\|\;|S|\;\|+\varepsilon)$. Finally, by since ε is arbitrary we get $|\langle S,z\rangle|\leq g'_M(z;L_\infty(\mu)\otimes\ell_{M^*}\;\|\;|S|\;\|$. But from [1] theorem 1.10, $|S|(\chi_\Omega)=\sup\{|S(f)|,|f|\leq\chi_\Omega\}$ and as ℓ_M is order continuous

$$\|\;|S|\;\|=\|\;|S|(\chi_\Omega)\|=\sup\{\|\;|S(f)|\;\|,\|f\|\leq 1\}=\|S\|.$$

Then S is $M-$integral with $\mathbf{I}_M(S)\leq\|S\|$. But as $(\mathcal{I}_M,\mathbf{I}_M)$ is a Banach operators ideal, $\|S\|\leq\mathbf{I}_M(S)$, hence $\mathbf{I}_M(S)=\|S\|$.

Corollary 9. Let (Ω, Σ, μ) be a measure space and $n, k \in \mathbf{N}$. Then every operator $T : L_\infty(\mu) \to S_k(\ell_M)$ and every operator $T : L_\infty(\mu) \to S_n(\ell_1)[S_k(M)]$ satisfy that $\mathbf{I}_M(T) = \|T\|$.

Proof. The result follows easily from theorem 3, since every operator $T : L_\infty(\mu) \to S_k(\ell_M)$ ($T : L_\infty(\mu) \to S_n(\ell_1)[S_k(\ell_M)]$ in the other case) is order bounded and $S_k(\ell_M)$ (resp. $S_n(\ell_1)[S_k(\ell_M)]$) is reflexive hence order continuous.

For our next theorem we need a very deep technical result of Lindenstrauss and Tzafriri [14] which gives us a kind of "uniform approximation" of finite dimensional subspaces by finite dimensional sublattices in Banach lattices.

Lemma 10. Let $\varepsilon > 0$ and $n \in \mathbf{N}$ be fixed. There is a natural number $h(n, \varepsilon)$ such that for every Banach lattice X and every subspace $F \subset X$ of dimension $\dim(F) = n$ there are $h(n, \varepsilon)$ disjoints elements $\{z_i,\ 1 \le i \le h(n, \varepsilon)\}$ and an operator A from F into the linear span G of $\{z_i,\ 1 \le i \le h(n, \varepsilon)\}$ such that

$$\forall\, x \in F \quad \|A(x) - x\| \le \varepsilon\, \|x\|.$$

Theorem 11. Let ℓ_M be a regular Orlicz sequence space, G an abstract M-space, and X a Banach space with c-$S_k(\ell_M)$ or c-$S_k(\ell_1)[S_n(\ell_M)]$-local uniform structure . Then every operator $T : G \longrightarrow X$ is M-integral and $\mathbf{I}_M(T) \le c\, \|T\|$.

Proof. We will prove the case where X has c-$S_k(\ell_M)$-local unconditional structure since the other case is similar. By the representation theorem of maximal operator ideals (see 17.5 in [4]), we only need to show that $J_X\, T \in (G \otimes_{g'_M} X')'$.

Given $z \in G \otimes X'$ and $\varepsilon > 0$, let $P \subset G$ and $Q \subset X'$ be finite dimensional subspaces and let $z = \sum_{i=1}^n f_i \otimes x'_i$ be a *fixed* representation of z with $f_i \in P$ and $x'_i \in Q$, $i = 1, 2, .., n$ such that

$$g'_M(z; G \otimes X') \le g'_M(z; P \otimes Q) \le g'_M(z; G \otimes X') + \varepsilon.$$

From lemma 10 we have a finite dimensional sublattice P_1 of G and an operator $A : P \to P_1$ so that $\forall\, f \in P, \quad \|A(f) - f\| \le \varepsilon \|f\|$. Then, if id_G denotes the identity map on G we have

$$|\langle J_X T, z\rangle| = \left|\sum_{i=1}^n \langle T(f_i), x'_i\rangle\right| \le \left|\sum_{i=1}^n \langle\, T(id_G - A)(f_i), x'_i\rangle\right| + \left|\sum_{i=1}^n \langle T\ A(f_i), x'_i\rangle\right|$$

$$\leq \varepsilon \|T\| \sum_{i=1}^{n} \|f_i\| \, \|x_i'\| + \left| \sum_{i=1}^{n} \langle T\, A(f_i), x_i' \rangle \right| .$$

Let $X_1 := T(P_1)$. As X has $S_k(\ell_M)$-local unconditional structure, hence there are $k \in \mathbf{N}$, $u : X_1 \to S_k(\ell_M)$ and $v : S_k(\ell_M) \to X$ such that $I_{X_1,X} = v\, u$ and $\|u\| \, \|v\| \leq c$. Let $X_2 := v\, u(X_1)$ which is a finite dimensional subspace of X containing X_1 and $I_{X_1,X_2} = v\, u$. Put $K_2 : X''' \longrightarrow X_2' = X'''/X_2^{\circ}$ be the canonical quotient map. Then

$$\sum_{i=1}^{n} \langle T\,(A(f_i)), x_i' \rangle = \sum_{i=1}^{n} \langle I_{X_1,X_2}\, T\,(A(f_i)), K_2(x_i') \rangle =$$

$$\sum_{i=1}^{n} \langle v\, u\, T\,(A(f_i)), K_2(x_i') \rangle =$$

$$= \sum_{i=1}^{n} \langle u\, T\,(A(f_i)), v'\, K_2(x_i') \rangle = \langle u\, T, \sum_{i=1}^{n} A(f_i) \otimes v'\, K_2(x_i') \rangle$$

with $\sum_{i=1}^{n} A(f_i) \otimes v'\, K_2(x_i') \in P_1 \otimes (S_k(\ell_M))'$ and $u\, T : P_1 \to S_k(\ell_M)$. Since P_1 is a reflexive abstract M-space it is lattice isometric to some $L_\infty(\mu)$ space, hence by corollary 9 this map is M-integral with $\mathbf{I}_M(u\, T) \leq \|u\| \|T\|$. Then

$$\left| \sum_{i=1}^{n} \langle T\,(A(f_i)), x_i' \rangle \right| = \left| \left\langle u\, T, \sum_{i=1}^{n} A(f_i) \otimes v'\, K_2(x_i') \right\rangle \right| \leq$$

$$\leq \mathbf{I}_M(u\, T)\; g_M'(\sum_{i=1}^{n} A(f_i) \otimes v'\, K_2(x_i'); P_1 \otimes S_k(\ell_M)) \leq$$

$$\leq \|u\| \, \|T\| \; g_M' \left((A \otimes v'\, K_2)(z); P_1 \otimes S_k(\ell_M) \right) \leq$$

$$\leq \|u\| \, \|T\| \, \|A\| \, \|v'\| \, \|K_2\| \; g_M'(z; P \otimes Q) \leq$$

$$\leq (1+\varepsilon)\; c\, \|T\| \; g_M'(z; P \otimes Q) \leq (1+\varepsilon)\; c\, \|T\| \; (g_M'(z; G \otimes X') + \varepsilon)$$

and since ε is arbitrary we obtain $|\langle J_X T, z \rangle| \leq c\, \|T\| \;\; g_M'\,(z; G \otimes X')$

Concerning to characterization theorem of M-integral operators we have:

Theorem 12. Let ℓ_M be a regular Orlicz sequence space and let E and F be Banach spaces. The following statements are equivalent:

1) $T \in \mathcal{I}_M(E, F)$.

2) $J_F T$ factors continuously in the following way:

$$\begin{tikzcd} E \arrow[r, "J_F\,T"] \arrow[d, "A"'] & F'' \\ L_\infty(\mu) \arrow[r, "B"'] & X \arrow[u, "C"'] \end{tikzcd}$$

where X is an ultrapower of $\ell_1[\ell_M]$ and B is a lattice homomorphism. Furthermore $\mathbf{I}_M(T)$ is equivalent to $\inf\{\|D\|\|B\|\|A\|\}$, taking it over all such factors.

Proof. 1) $\Longrightarrow$ 2). Let $D := \{(P,Q) : P \in FIN(E), Q \in FIN(F')\}$ where $FIN(Y)$ is the set of finite dimensional subspace of a Banach space Y, endowed with the natural inclusion order

$$(P_1,Q_1) \leq (P_2,Q_2) \Longleftrightarrow P_1 \subset P_2,\ Q_1 \subset Q_2.$$

For every $(P_0,Q_0) \in D$, $R(P_0,Q_0) := \{(P,Q) \in D : (P_0,Q_0) \subset (P,Q)\}$ and $\mathcal{R} = \{R(P,Q),(P,Q) \in D\}$. $\mathcal{R}$ is filter basis in D, and according to Zorn's lemma, let $\mathcal{D}$ be an ultrafilter on D containing $\mathcal{R}$. If $d \in D$, P_d and Q_d denote the finite dimensional subspaces of E and F' respectively so that $d = (P_d,Q_d)$. For every $d \in D$, if $z \in P_d \otimes Q_d$, $J_F T_{|P_d\otimes Q_d} \in (P_d \otimes_{g'_M} Q_d)' = M'_d \otimes_{g_M} Q'_d = \mathcal{N}_M(P_d,Q'_d)$. Then from theorem 2 of characterization of M-nuclear operators, $J_F T_{|P_d\otimes Q_d}$ factors as

$$\begin{tikzcd} P_d \arrow[r, "J_F T_{|P_d\otimes Q_d}"] \arrow[d, "A_d"'] & Q'_d \\ \ell_\infty[\ell_\infty] \arrow[r, "B_d"'] & \ell_1[\ell_M] \arrow[u, "C_d"'] \end{tikzcd}$$

where B_d is a positive diagonal operator and $\|A_d\|\ \|B_d\|\ \|C_d\| \leq \mathbf{N}^c_M(T_{|P_d\otimes Q_d})+ \varepsilon = \mathbf{I}_M(T_{|P_d\otimes Q_d}) + \varepsilon$. Then

$$\|A_d\|\ \|B_d\|\ \|C_d\| \leq\ \mathbf{I}_M(T_{|P_d\otimes Q_d}) + \varepsilon \leq\ \mathbf{I}_M(T) + \varepsilon$$

Without loss of generality we can suppose that $\|A_d\| = \|C_d\| = 1$. We define $W_E : E \to (M_d)_{\mathcal{D}}$ such that $W_E(x) = (x_d)_{\mathcal{D}}$ so that $x_d = x$ if $x \in M_d$

and $x_d = 0$ if $x \notin M_d$. In the same way we define $W_{F'} : F' \to (Q_d)_{\mathcal{D}}$ such that $W_{F'}(a) = (a_d)_{\mathcal{D}}$ so that $a_d = a$ if $a \in Q_d$ and $a_d = 0$ if $a \notin Q_d$. Then we have the following commutative diagram:

$$\begin{tikzcd}
E \arrow[rr, "J_F T"] \arrow[d, "W_E"'] & & F'' \\
(P_d)_{\mathcal{D}} \arrow[r, "(J_F T_{|P_d \otimes Q_d})_{\mathcal{D}}"'] \arrow[d, "(A_d)_{\mathcal{D}}"'] & (Q'_d)_{\mathcal{D}} \arrow[r, "I"'] & ((Q_d)_{\mathcal{D}})' \arrow[u, "W'_{F'}"'] \\
(\ell_\infty[\ell_\infty])_{\mathcal{D}} \arrow[r, "(B_d)_{\mathcal{D}}"'] & (\ell_1[\ell_M])_{\mathcal{D}} \arrow[u, "(C_d)_{\mathcal{D}}"'] &
\end{tikzcd}$$

where I is the canonical inclusion map. As in [14] $((\ell_1[\ell_M])_{\mathcal{D}})''$ is a 1-complemented subspace of some ultrapower $((\ell_1[\ell_M])_{\mathcal{D}})_{\mathcal{U}}$ which from [19] is another ultrapower $(\ell_1[\ell_M])_{\mathcal{U}_1}$ with projection Q, the result follows with $A = (A_d)_{\mathcal{D}}$, $B = ((B_d)_{\mathcal{D}})''$ which is a lattice homomorphism,
$C = P_{F''''}\ (W'_{F'}\ I\ (C_d)_{\mathcal{D}})''\ Q$, where $P_{F''''}$ is the projection of F'''' in F'', and $X = (\ell_1[\lambda])_{\mathcal{U}_1}$, having in mind that as $(\ell_\infty[\ell_\infty])_{\mathcal{D}}$ is an abstract M-space, there is a measure space such that $L_\infty(\mu) = ((\ell_\infty[\ell_\infty])_{\mathcal{D}})''$, where equality means that the spaces are lattice isometric.

2) $\Longrightarrow$ 1) As $(\mathcal{I}_M, \mathbf{I}_M)$ is a operator ideal, it follows easily from theorem 3 and proposition 3

The following new formulation of the preceding characterization theorem is needed in our context:

Theorem 13. Let ℓ_M be an Orlicz space. For every pair of Banach spaces E and F, the following statements are equivalent:

1) $T \in \mathcal{I}_M(E, F)$.

2) There exists a σ-finite measure space $(\mathcal{O}, \mathcal{S}, \nu)$ and a Köthe function space $\mathcal{K}(\nu)$ which is complemented in a space with $S_k(\ell_1)[S_n(\ell_M)]$-local unconditional structure, such that $J_F T$ factors continuously in the following way:

$$\begin{tikzcd} E \arrow[r, "J_F\,T"] \arrow[d, "A"'] & F'' \\ L_\infty(\nu) \arrow[r, "B"'] & \mathcal{K}(\nu) \arrow[u, "C"'] \end{tikzcd}$$

where B is a multiplication operator for a positive function of $\mathcal{K}(\nu)$. Furthermore $\mathbf{I}_M(T) = \inf\{\|C\|\|B\|\|A\|\}$, taking the infimum over all such factors.

Proof. Starting from the theorem 12, as $\ell_1[\ell_M]$ has finite cotype, it is order continuous ([7], 4.6), and for [13], theorem 1.a.9 $\ell_1[\ell_M]$ can be decomposed into an unconditional direct sum of a family of mutually disjoint ideals $\{X_h, h \in H\}$ having a positive weak unit, and then from 1.b.14 in [13], as every X_h is order isometric to a Köthe space of functions defined on a probability space $(\mathcal{O}_h, \mathcal{S}_h, \nu_h)$, then $(\ell_1[\ell_M])_{\mathcal{U}}$ is order isometric to a Köthe function space $\mathcal{K}(\nu^1)$ over a measure space $(\mathcal{O}^1, \mathcal{S}^1, \nu^1)$, hence we can substitute $(\ell_1[\ell_M])_{\mathcal{U}}$ for $\mathcal{K}(\nu^1)$ in 12. If we denote $z := B(\chi_\Omega)$ with $z = \sum_{i=1}^\infty y_{h_i}$ with $y_{h_i} \in X_{h_i}$ for every $i \in \mathbf{N}$, then $B(L_\infty(\mu))$ is contained in the unconditional direct sum of $\{X_{h_i}, i \in \mathbf{N}\}$ which is is order isometric to a space of Köthe function space $\mathcal{K}(\nu)$ over a σ-finite measure space $(\mathcal{O}, \mathcal{S}, \nu)$ which is 1-complemented in $\mathcal{K}(\nu^1)$.

Now, since $\mathcal{K}(\nu)$ is order complete, there exists $g := \sup_{\|f\|_{L_\infty(\mu)}} B(f)$ in $\mathcal{K}(\nu)$. Then the operators $B_1 : L_\infty(\mu) \to L_\infty(\nu)$ and $B_2 : L_\infty(\nu) \to \mathcal{K}(\nu)$, such that $B_1(f)(\omega) := B(f)(\omega)/g(\omega)$, for all $f \in L_\infty(\mu)$, $\omega \in \mathcal{O}$ with $g(\omega) \neq 0$ and $B_1(f)(\omega) = 0$ otherwise, and $B_2(h)(\omega) := g(\omega)h(\omega)$ for all $h \in L_\infty(\nu)$, $\omega \in \mathcal{O}$, satisfy that $B = B_2B_1$ and B_2 is a multiplication operator for a positive element $g \in \mathcal{K}(\nu)$.

4. On equality between M-nuclear and M-integral operators

Finally, using the preceding characterization theorems we give some properties of M-nuclear and M-integral operators. Let us establish now a necessary condition for equality between components of M-nuclear and M-integral operator ideals. First, we introduce a new operator ideal, which is contained in the ideal of the M-integral operators.

Definition 14. Given E and F Banach spaces, let ℓ_M be a Orlicz sequence. We say that $T \in \mathcal{L}(E,F)$ is **strictly M-integral** if exist a σ-finite measure space $(\mathcal{O}, \mathcal{S}, \nu)$ and a Köthe function space $\mathcal{K}(\nu)$ which is complemented in a space with $S_k(\ell_1)[S_n(\ell_M)]$-local unconditional structure, such that T factors continuously in the following way:

$$\begin{tikzcd} E \arrow[rr, "T"] \arrow[d, "A"'] & & F \\ L_\infty(\nu) \arrow[rr, "B"'] & & \mathcal{K}(\nu) \arrow[u, "C"'] \end{tikzcd}$$

where B is a multiplication operator for a positive function of $\mathcal{K}(\nu)$. endowed with the topology of the norm $\mathbf{SI}_M(T) = \mathbf{I}_M(T)$.

Obviously, if F is a dual space, or it is complemented in its bidual space, then $\mathcal{SI}_M(E,F) = \mathcal{I}_M(E,F)$.

Theorem 15. Let ℓ_H be a Orlicz sequence space, and let E and F be Banach spaces, such that E' satisfies the Radon-Nikodým property then, $\mathcal{N}_M(E,F) = \mathcal{SI}_M(E,F)$.

Proof. Let $T \in \mathcal{SI}_{\lambda^c}(E,F)$ Were E' has the Radon-Nikodým property and.

a) First, we suppose that B is an multiplication operator for a function $g \in \mathcal{K}(\nu)$ with finite measure support D. We denote ν_D the restriction of ν to D.

As $(\chi_D A) : E \to L_\infty(\nu_D)$, then $(\chi_D A)' : (L_\infty(\nu_D))' \to E'$ and the restriction of $(\chi_D A)'_{L_1(\nu_D)} : L_1(\nu_D) \to E'$, thus, for every $x \in E$ and $f \in L_1(\nu_D)$

$$\langle x, (\chi_D A)'(f)\rangle = \langle \chi_D A(x), f\rangle = \int_D \chi_D A(x)\, f d(\nu_D).$$

As E' has the Radon-Nikodým property, by III(5) of [2], we have that $(\chi_D A)'$ has a Riesz representation, therefore exist a function $\phi \in L_\infty(\nu_D, E')$ such that for every $f \in L_1(\nu_D)$

$$(\chi_D A)'(f) = \int_D f\phi d(\nu_D).$$

Then, for every $x \in E$, we have that $\chi_D A(x)(t) = \langle \phi(t), x \rangle$, ν_D-almost everywhere in D, and then $B(\chi_D A)(x) =< g\phi(.), x >$, ν_D-almost everywhere in D. Let $g\phi$ this last operator, and we can consider it as $\mathcal{K}(\nu_D, E')$ element. As the simple functions are dense in $\mathcal{K}(\nu_D, E')$, $g\phi$ can be approximated by a sequence of simple functions $((S_k)_{k=1}^{\infty})$.

We suppose $S_k = \sum_{j=1}^{m_k} x'_{kj} \chi_{A_{kj}}$, where $\{A_{ki} : i = 1, .., m\}$ is a family of ν-measure set of Ω pairwise disjoint. For each $k \in \mathbf{N}$, we can interpret S_k as a map $S_k : E \to \mathcal{K}(\nu)$ such that $S_k(x) = \sum_{j=1}^{m_k} < x'_{kj}, x > \chi_{A_{kj}}$ with norm less or equal than the norm of S_k in $\mathcal{K}(\nu, E')$.

Clearly for all $k \in \mathbf{N}$, S_k is M-nuclear since it has finite rang, but we need to evaluate its M-nuclear norm coinciding with it M-integral norm. Let $S_k^1 : E \to L_\infty(\nu)$ be such that

$$S_k^1(x) = \sum_{j=1}^{m_k} \frac{\langle x'_{kj}, x \rangle}{\|x'_{kj}\|} \chi_{A_{kj}}$$

and let $S_k^2 : L_\infty(\nu) \to \mathcal{K}(\nu)$ be such that $S_k^2(f) = \sum_{j=1}^{m_k} \|x'_{kj}\| f \chi_{A_{kj}}$.

Then $\|S_k^1\| \leq 1$ and $\|S_k^2\| \leq \|S_k\|_{\mathcal{K}(\nu,E')}$ and $S_k = S_k^2 \; S_k^1$. But as $\mathcal{K}(\nu)$ is a complemented subspace of space with $S_k(\ell_1)(S_n(\ell_M))$-local unconditional structure, from 11, there is $K > 0$ such that $\mathbf{I}_M(S_k^2) \leq K \; \|S_k^2\| \leq K \; \|S_k\|_{\mathcal{K}(\nu,E')}$, hence $\mathbf{N}_M^c(S_k^2) \leq K \; \|S_k^2\| \leq \|S_k\|_{\mathcal{K}(\nu,E')}$, hence $\mathbf{N}_M^c(S_k) \leq K \; \|S_k\|_{\mathcal{K}(\nu,E')}$.

Then, as $(S_k)_{k=1}^{\infty}$ converges in the $\mathcal{K}(\nu_D, E')$ space, it is a Cauchy sequence in $\mathcal{N}_M(E, \mathcal{K}(\nu_D))$, and since this is complete, $(S_k)_{k=1}^{\infty}$ converges to $g\phi$, that is to say, $g\phi \in \mathcal{N}_M(E, \mathcal{K}(\nu_D))$. Therefore, $g\phi = B\chi_D A$ is M-nuclear and so T is also M-nuclear.

b) Now, if g is any element of $\mathcal{K}(\nu)$, g it can be approximated in norm by means of a sequence $(t_n)_{n=1}^{\infty}$ of simple functions with finite measure support, and therefore by a), the sequence $T_n = CB_{t_n}A$ is a Cauchy sequence in $\mathcal{N}_M(E, F)$ converging to T in $\mathcal{L}(E, F)$, and then $T \in \mathcal{N}_M(E, F)$.

As consequence of the former result and of the factorization theorems 13 and 2, we obtain the following metric properties of g_M^c and $(g_M^c)'$.

Theorem 16. $(g_M^c)'$ is a totally accessible tensor norm.

Proof. Since $(g_M^c)'$ is finitely generated, it is sufficient to prove that the map $F \otimes_{(g_M^c)'} E \hookrightarrow \mathcal{P}_{M^*}(E', F'')$, is a isometric.

Let $z = \sum_{i=1}^{n} \sum_{j=1}^{l_i} y_{ij} \otimes x_{ij} \in F \otimes_{(g_M^c)'} E$, and let $H_z \in \mathcal{P}_{M^*}(E', F'')$ be the canonical map associated to z, that is to say,

$H_z\left(x'\right) = \sum_{i=1}^{n}\sum_{j=1}^{l_i}\left\langle x_{ij}, x'\right\rangle y_{ij}$ for all $x' \in E'$, con $H_z \in \mathcal{L}\left(E', F\right) \subset \mathcal{L}\left(E', F''\right)$.

Applying the theorem 15.5 of [4] for $\alpha = (g_\lambda^c)'$, the theorem 4, and the equality $(g_M^c)'' = g_M^c$ since g_M^c finitely generated, we have that inclusion

$$F \otimes_{\overleftarrow{(g_M^c)'}} E \hookrightarrow \left(F' \otimes_{g_M^c} E'\right)' \to \mathcal{P}_{M^*}\left(E', F''\right)$$

is an isometry, and therefore by proposition 12.4 in [4] we obtain

$$\mathbf{P}_{M^*}\left(H_z\right) = \overleftarrow{(g_M^c)'}\left(z; F \otimes E\right) \leq (g_M^c)'\left(z; F \otimes E\right).$$

Now, given N, a finite dimensional subspace of F such that $z \in N \otimes_{(g_M^c)'} E$, there exists $V \in (N \otimes_{(g_M^c)'} E)' = \mathcal{I}(N, E')$ such that $\mathbf{I}_M(V) \leq 1$ and $(g_M^c)'(z; N \otimes E) = \langle z, V\rangle$. Clearly enough $V \in \mathcal{SI}_M(N, E') = \mathcal{I}_M(N, E')$ because E' is a dual space, and N', being finite dimensional, has the Radon-Nikodým property. Therefore by theorem 15, $V \in \mathcal{N}_M\left(N, E'\right)$ and by theorem 2, given $\epsilon > 0$, there is a factorization V in the way

$$\begin{tikzcd} N \arrow[r, "T"] \arrow[d, "A"'] & E' \\ \ell_\infty[\ell_\infty] \arrow[r, "B"'] & \ell_1[\ell_M] \arrow[u, "C"'] \end{tikzcd}$$

such that $\|C\|\|B\|\|A\| \leq \mathbf{N}_M^c(V) + \epsilon = \mathbf{I}_M^c(V) + \epsilon \leq 1 + \epsilon$.

As $\ell_\infty[\ell_\infty]$ has the extension metric property, (to see proposition 1, C.3.2. in [16]), A can be extended to a continuous map $\overline{A} \in \mathcal{L}\left(F, \ell_\infty[\ell_\infty]\right)$ such that $\left\|\overline{A}\right\| = \|A\|$. By theorem 2 again, $W := CB\overline{A}$ is in $\mathcal{N}_M\left(F, E'\right)$, so there is a representation $w =: \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} y'_{ij} \otimes x'_{ij} \in F'\widehat{\otimes}_{g_M^c} E'$ of W verifying

$$\sum_{i=1}^{\infty}\pi_M\left(\left(y'_{ij}\right)\right)\varepsilon_{M^*}\left(\left(x'_{ij}\right)\right) \leq \mathbf{N}_M^c\left(W\right) + \epsilon \leq \|C\|\,\|B\|\left\|\overline{A}\right\| + \epsilon \leq 1 + 2\epsilon.$$

Then, $(g_M^c)'(z; F \otimes E) \leq (g_M^c)'(z; N \otimes E) = \langle z, V\rangle = \langle z, W\rangle$ it follows that

$$(g_M^c)'(z; F \otimes E) \leq g_M^c(w)\mathbf{P}_{M^*}(H_z) \leq (1 + 2\epsilon)\mathbf{P}_{M^*}(H_z)$$

whence $(g_M^c)'(z; F \otimes E) \leq \mathbf{P}_{M^*}(H_z)$, and the equality is obvious.

Finally, as consequence of the former theorem and of proposition 15.6 of [4], we have:

Corollary 17. g_M^c is an accessible tensor norm.

References

[1] Aliprantis, C. D., Burkinshaw, O.: *Positive operators.* Pure and Applied Mathematics 119. Academic Press, Newe York, (1985).

[2] Diestel, J. and Uhl, J. J. Jr.: *Vector measures.* Mathematical Surveys and Monographs. Number 15. American Mathematical Society. U. S. A. (1977).

[3] De Grande-De Kimpe, N.: *Λ-mappings between locally convex spaces,* Indag. Math. 33, pp. 261-274, (1971).

[4] Defant, A. and Floret, K.: *Tensor norms and operator ideals.* North Holland Math. Studies. Amsterdam. (1993).

[5] Dubinsky, E. and Ramanujan, M. S.: *On M-nuclearity.* Mem. Amer. Math. Soc. 128, (1972).

[6] Harksen, J.: *Tensornormtopologien.* Dissertation, Kiel, (1979).

[7] Haydon, R., Levy, M., Raynaud, Y.: *Randomly normed spaces.* Hermann, (1991).

[8] Heinrich, S.: *Ultraproducts in Banach spaces theory.* J. reine angew. Math. 313, pp. 72-104, (1980).

[9] Johnson, W. B.: *On finite dimensional subspaces of Banach spaces with local unconditional structure.* Studia Math. 51, pp. 225-240, (1974).

[10] Komura, T., Komura, Y.: *Sur les espaces parfaits de suites et leurs généralisations,* J. Math. Soc. Japan, 15,3, pp. 319-338, (1963).

[11] Lacey, H. E.: *The isometric theory of Classical Banach spaces,* Springer Verlag. Berlin, Heidelberg, New York, (1974).

[12] Loaiza, G., López Molina, J.A., Rivera, M.J.: *Characterization of the Maximal Ideal of Operators Associated to the Tensor Norm Defined by an Orlicz Function.* Zeitschrift für Analysis und ihre Anwendungen (Journal for Analysis and its Applications) 20, no.2, pp. 281–293, (2001).

[13] Lindenstrauss, J., Tzafriri, L.: *Classical Banach spaces I and II*, Springer Verlang. Belin, Heidelberg, New York, (1977).

[14] Lindenstrauss, J., Tzafriri, L.: *The uniform approximation property in Orlicz spaces*, Israel J. Math. 23, 2, pp. 142-155,(1976).

[15] Pelczyński, A., Rosenthal, H. P.: *Localization techniques in L^p spaces,* Studia Math. 52, pp. 263-289, (1975).

[16] Pietsch, a.: *Operator Ideals.* North Holland Math. Library. Amsterdam, New York. (1980).

[17] Rivera, M.J.: *On the classes of $\mathcal{L}^\lambda$, $\mathcal{L}^{\lambda,g}$ and quasi-$\mathcal{L}^E$ spaces.* Preprint

[18] Saphar, P.: *Produits tensoriels topologiques et classes d'applications lineáires.* Studia Math. 38, pp. 71-100, (1972).

[19] Sims, B.: *"Ultra"-techniques in Banach space theory,* Queen's Papers in Pure and Applied Mathematics, 60. Ontario, (1982).

[20] Tomášek, S: *Projectively generated topologies on tensor products,* Comentations Math. Univ. Carolinae, 11, 4 (1970).

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