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L-FUZZY CLOSURE OPERATOR *

YUELI YUE
INSTITUTE OF TECHNOLOGY, CHINA
FU–GUI SHI
UNIVERSITY OF CHINA, CHINA

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Abstract

*The aim of this paper is to study L-fuzzy closure operator in L-fuzzy topological spaces. We introduce two kinds of L-fuzzy closure operators from different point view and prove that both L-***TFCS***—the category of topological L-fuzzy closure spaces—and L-***PTFCS***—the category of topological pointwise L-fuzzy closure spaces—are isomorphic to L-***FCTOP***.*

Key words: *L-fuzzy co-topology; fuzzy remote neighborhood system; L-fuzzy closure operator*

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1. Introduction

Since Chang [2] introduced fuzzy set theory to topology, many researchers have tried successfully to generalize the theory of general topology to the fuzzy setting with crisp methods. However, in a completely different direction, Höhle [5] created the notion of a topology being viewed as an L-subset of a powerset. Then Kubiak [7] and Šostak [12] independently extended Höhle's notion to L-subsets of L^X. In [15], we established fuzzy remote neighborhood systems in L-fuzzy co-topology and prove that **TFRNS** is isomorphic to L-**FCTOP**.

It is well-known that clsoure operator (or clsoure system) plays an important role in topology and it is a very good way to characterize topology. Many authors [3, 7, 13] have studied closure operators in L-fuzzy topologyical spaces. But it is an pity that their closure operators are actually defined by the level L-topology of the L-fuzzy topology, not by L-fuzzy topology itself. In other words, their closure operators are still the closure operators in L-topologies. The aim of this paper is to study L-fuzzy closure operators in L-fuzzy topological spaces in different ways from [3, 7, 13]. We give two kinds of L-fuzzy closure operators and prove that both L-**TFCS**—the category of topological L-fuzzy closure spaces—and L-**PTFCS**—the category of topological pointwise L-fuzzy closure spaces—are isomorphic to L-**FCTOP**.

2. Preliminaries

An element a in a complete lattice L is said to be coprime if $a \leq b \vee c$ implies that $a \leq b$ or $a \leq c$. The set of nonzero $\vee$-irreducible elements(or coprimes) of L is denoted by $c(L)$. We say a is wedge below b, in symbols, $a \lhd b$ or $b \rhd a$, if for every arbitrary subset $D \subseteq L$, $\bigvee D \geq b$ implies $a \leq d$ for some $d \in D$. A complete lattice is said to be completely distributive if every element in L is the supremum of all the elements wedge below it. For more details about completely distributive lattice, please refer to [4].

By the definition of complete distributivity it is easy to see that a complete lattice L is completely distributive if and only if the operator $\bigvee : Low(L) \to L$ taking every lower set to its supremum has a left adjoint β, and in the case, $\beta(a) = \{b |\ b \lhd a\}$ for all $a \in L$. Hence the wedge below relation has the interpolation property in a completely distributive lattice, this is to say, $a \lhd b$ implies there is some $c \in L$ such that $a \lhd c \lhd b$. $\{a \in L |\ a \lhd b\}$ is called the greatest minimal family of b, in symbol $\beta(b)$.

Moreover, for $b \in L$, define $\alpha(b) = \{a \in L|\ a' \lhd b'\}$ which is called the greatest maximal family of b.

Throughout this paper L is a completely distributive lattice and M is a completely distributive lattice with an order reversing involution $'$. L^X is the set of all L-fuzzy sets on X. The set of nonzero $\vee$-irreducible elements (or coprime) in L^X is denoted by $c(L^X)$. Let $x_\lambda|A$ denote the set $\{B \in L^X | x_\lambda \not\leq B \geq A\}$ for $x_\lambda \in c(L^X)$ and $A \in L^X$. For undefined notions about categories, please refer to [1], [6] and [11].

Definition 2.1[7,12]**.** An L-fuzzy co-topology is a mapping $\eta : L^X \to M$ such that

(FCT1) $\eta(1_X) = \eta(0_X) = 1$;

(FCT2) $\eta(A \vee B) \geq \eta(A) \wedge \eta(B)$ for all $A, B \in L^X$;

(FCT3) $\eta(\bigwedge_{j\in J} A_j) \geq \bigwedge_{j\in J} \eta(A_j)$ for every family $\{A_j | j \in J\} \subseteq L^X$.

The pair (L^X, η) is called an L-fuzzy co-topological space (L-FCTop, for short). A mapping $F : (L^X, \eta) \to (L^Y, \eta_1)$ is said to be continuous with respect to η and η_1 if $\eta(F_L^{\leftarrow}(B)) \geq \eta_1(B)$ for all $B \in L^Y$, where $F_L^{\leftarrow}(B)(x) = B(F(x))$ (following the notation in [14]). The category of L-FCTops is denoted by L-**FCTOP**.

Definition 2.2[15]**.** A fuzzy remote neighborhood system is a set $\mathcal{R} = \{R_{x_\lambda} | x_\lambda \in c(L^X)\}$ of mappings $R_{x_\lambda} : L^X \to M$ such that

(FRN1) $R_{x_\lambda}(1_X) = 0, R_{x_\lambda}(0_X) = 1$;

(FRN2) $R_{x_\lambda}(A) > 0 \Rightarrow x_\lambda \not\leq A$;

(FRN3) $R_{x_\lambda}(A \vee B) = R_{x_\lambda}(A) \wedge R_{x_\lambda}(B)$.

$\mathcal{R}$ will be called a topological fuzzy remote neighborhood system if it also satisfies the following equation:

(FRN4) $R_{x_\lambda}(A) = \bigvee_{B\in x_\lambda|A} \bigwedge_{y_\mu \not\leq B} R_{y_\mu}(B)$,

and $(L^X, \mathcal{R})$ is called a topological fuzzy remote neighborhood space (TFRNS, for short). A fuzzy continuous mapping between topological fuzzy remote neighborhood spaces $(L^X, \mathcal{R})$ and $(L^Y, \mathcal{S})$ is a mapping $F : L^X \to L^Y$ such that $S_{F_L^{\rightarrow}(x_\lambda)}(B) \leq R_{x_\lambda}(F_L^{\leftarrow}(B))$ for all $x_\lambda \in c(L^X)$ and $B \in L^Y$. The category of TFRNSs and fuzzy continuous mappings is denoted by **TFRNS**.

Lemma 2.3[15]**.** Let $\eta : L^X \to M$ be an L-fuzzy co-topology. Then we have

(1) $\mathcal{R}^\eta = \{R^\eta_{x_\lambda} | x_\lambda \in c(L^X)\}$ is a topological fuzzy remote neighborhood system, where $R^\eta_{x_\lambda}$ is defined by

$$R^\eta_{x_\lambda}(A) = \begin{cases} \bigvee_{B \in x_\lambda | A} \eta(B), & x_\lambda \not\leq A, \\ 0, & x_\lambda \leq A. \end{cases}$$

for $x_\lambda \in c(L^X)$ and $A \in L^X$.

(2) If η and ζ are two L-fuzzy co-topologies which determine the same topological fuzzy remote neighborhood system, then $\eta = \zeta$.

Lemma 2.4[15]**.** Let $\mathcal{R} = \{R_{x_\lambda} | x_\lambda \in c(L^X)\}$ be a fuzzy remote neighborhood system and $\eta : L^X \to M$ be defined by $\eta(u) = \bigwedge_{x_\lambda \not\leq A} R_{x_\lambda}(A)$ for all $A \in L^X$. Then η is an L-fuzzy co-topology. Furthermore, if $\mathcal{R}$ and $\mathcal{S}$ are two topological fuzzy remote neighborhood systems which determine the same L-fuzzy co-topology, then $\mathcal{R} = \mathcal{S}$.

Lemma 2.5[15]**.** **TFRNS** is isomorphic to L-**FCTOP**.

3. L-fuzzy closure operator

In this section, we study L-fuzzy closure operator in a different direction from [3, 7, 13].

Lemma 3.1. Let (L^X, η) be an L-fuzzy co-topological space and define $C^\eta_A : L^X \to M$ by

$$C^\eta_A(B) = \begin{cases} 0, & A \not\leq B, \\ \eta(B) \wedge \bigwedge_{A \leq D \not\geq B} \eta(D)', & A \leq B. \end{cases}$$

Then $\{C^\eta_A\}_{A \in L^X}$ satisfying the following properties.

(1) $C^\eta_{0_X}(0_X) = 1$;
(2) $C^\eta_A(B) > 0 \Rightarrow A \leq B$;
(3) $C^\eta_A(B) \wedge C^\eta_D(E) \leq C^\eta_{A \vee D}(B \vee E)$;
(4) $\bigwedge_{t \in T} C^\eta_{A_t}(B_t) \leq C^\eta_{\bigwedge_{t \in T} B_t}(\bigwedge_{t \in T} B_t)$;
(5) $C^\eta_A(B) \wedge C^\eta_B(E) \leq C^\eta_A(E)$;
(6) $C^\eta_A(B) = \bigwedge_{A \leq D \leq B} C^\eta_D(B)$;
(7) $C^\eta_A(A) = \eta(A)$;

(8) $C^\eta_A(B) = \begin{cases} 0, & A \not\leq B, \\ C^\eta_B(B) \wedge \bigwedge_{A\leq D \not\geq B}(C^\eta_D(D))', & A \leq B. \end{cases}$

Proof. We prove (3).

$$\begin{aligned} C^\eta_A(B) \wedge C^\eta_D(E) &= \eta(B) \wedge \bigwedge_{A\leq F\not\geq B} \eta(F)' \wedge \eta(E) \wedge \bigwedge_{D\leq G\not\geq E} \eta(G)' \\ &\leq \eta(B\vee E) \wedge \bigwedge_{A\leq F\not\geq B} \eta(F)' \wedge \bigwedge_{D\leq G\not\geq E} \eta(G)' \\ &\leq \eta(B\vee E) \wedge \bigwedge_{A\vee D\leq H\not\geq B\vee E} \eta(H)' \\ &= C^\eta_{A\vee D}(B\vee E). \end{aligned}$$

Remark 3.2. (1) The value $C^\eta_A(B)$ can be interpreted as the degree to which B is the closure of A. When $M = \{0,1\}$, that is to say η is an L-co-topology, and define $\bar{A} = B$ when $C^\eta_A(B) = 1$, then the operator $^- : L^X \to L^X$ is just the closure operator induced by η.

(2) The readers can easily show that C^η_A can be written as follows:

$$C^\eta_A(B) = \begin{cases} 0, & A \not\leq B, \\ \bigwedge_{x_\lambda \not\leq B} R^\eta_{x_\lambda}(B) \wedge \bigwedge_{x_\lambda\leq B}(R^\eta_{x_\lambda}(A))', & A \leq B. \end{cases}$$

Definition 3.3. An L-fuzzy closure operator is a set $\mathcal{C} = \{C_A | A \in L^X\}$ of mappings $C_A : L^X \to M$ such that:

(FC1) $C_{0_X}(0_X) = 1$;

(FC2) $C_A(B) \wedge C_D(E) \leq C_{A\vee D}(B \vee E)$;

(FC3) $\bigwedge_{t\in T} C_{A_t}(B_t) \leq C_{\bigwedge_{t\in T} B_t}(\bigwedge_{t\in T} B_t)$.

$\mathcal{C}$ is called a topological L-fuzzy closure operator if it also satisfies the following condition

(FC4) $C_A(B) = \begin{cases} 0, & A \not\leq B, \\ C_B(B) \wedge \bigwedge_{A\leq D\not\geq B}(C_D(D))', & A \leq B. \end{cases}$

The pair $(L^X, \mathcal{C})$ is called topological L-fuzzy closure space. A fuzzy continuous mapping between L-fuzzy closure spaces $(L^X, \mathcal{C}^1)$ and $(L^Y, \mathcal{C}^2)$ is a mapping $F : X \to Y$ such that $C^1_{F^\leftarrow_L(B)}(F^\leftarrow_L(B)) \geq C^2_B(B)$ for all $B \in L^Y$. The category of topological L-fuzzy closure spaces and continuous mappings is denoted by L-**TFCS**.

Lemma 3.4. Let $\mathcal{C} = \{C_A | A \in L^X\}$ be an L-fuzzy closure operator. Then $\eta^{\mathcal{C}} : L^X \to M$ defined by $\eta^{\mathcal{C}}(A) = C_A(A)$ is an L-fuzzy co-topology.

Lemma 3.5. (1) Let η be an L-fuzzy co-topology. Then $\eta = \eta^{\mathcal{C}^{\eta}}$.

(2) If $\mathcal{C}$ is topologocal L-fuzzy closure operator, then $\mathcal{C} = \mathcal{C}^{\eta^{\mathcal{C}}}$.

Lemma 3.6. (1) $F : (L^X, \mathcal{C}^1) \to (L^Y, \mathcal{C}^2)$ is continuous, then $F : (L^X, \eta^{\mathcal{C}^1}) \to (L^Y, \eta^{\mathcal{C}^2})$ is continuous.

(2) $F : (L^X, \eta^1) \to (L^Y, \eta^2)$ is continuous, then $F : (L^X, \mathcal{C}^{\eta^1}) \to (L^Y, \mathcal{C}^{\eta^2})$ is continuous.

Theorem 3.7. L-**TFCS** is isomorphic to L-**FCTOP**.

Question 3.8. We know that B is the closure of A if and only if B is the smallest closed set containing A. In fact, the L-fuzzy closure operator studied in Lemma 3.1 is just defined according to this kind of definition. We can also define the closure of A from other ways, such as from the direction of adherent point of A,

$$C_A^{\eta *}(B) = \bigwedge_{x_\lambda \not\leq B} R^{\eta}_{x_\lambda}(A) \wedge \bigwedge_{x_\lambda \leq B} (R^{\eta}_{x_\lambda}(A))'.$$

From Remark 3.2 (2), we know that there is some difference in this form and that in Lemma 3.1. We wonder whether $C_A^{\eta *}(B)$ is equal to $C_A^{\eta}(B)$ or not.

4. Pointwise L-fuzzy closure operator

In this section, we give one definition of Pointwise L-fuzzy closure operator and study the relationship between this kind of closure operator and fuzzy remote neighborhood system.

Lemma 4.1. Let $\eta : L^X \to M$ be an L-fuzzy co-topology and define $C^{\eta}_{x_\lambda} : L^X \to M$ by $C^{\eta}_{x_\lambda}(A) = \bigwedge_{B \in x_\lambda | A} \eta(B)'$. Then $\mathcal{C}^{\eta} = \{C^{\eta}_{x_\lambda} | x_\lambda \in c(L^X)\}$ satifies:

(1) $C^{\eta}_{x_\lambda}(1_X) = 1, C^{\eta}_{x_\lambda}(0_X) = 0;$

(2) $C^{\eta}_{x_\lambda}(A) < 1 \Rightarrow x_\lambda \not\leq A$;
(3) $C^{\eta}_{x_\lambda}(A \vee B) = C^{\eta}_{x_\lambda}(A) \vee C^{\eta}_{x_\lambda}(B)$;
(4) $C^{\eta}_{x_\lambda}(A) = \bigwedge_{B \in x_\lambda | A} \bigvee_{y_\mu \not\leq B} C^{\eta}_{y_\mu}(B)$.

Proof. From the definition of $C^{\eta}_{x_\lambda}$, we know that $C^{\eta}_{x_\lambda}(A) = R^{\eta}_{x_\lambda}(A)'$. By (FRN1)–(FRN4), we have (1)–(4).

Remark 4.2. The value $C^{\eta}_{x_\lambda}(A)$ can be interpreted as the degree to which x_λ is an adherent point of A.

Definition 4.3. A pointwise L-fuzzy closure operator is a set $\mathcal{C} = \{C_{x_\lambda} | x_\lambda \in c(L^X)\}$ of mappings $C_{x_\lambda} : L^X \to M$ such that
(FPC1) $C_{x_\lambda}(1_X) = 1, C_{x_\lambda}(0_X) = 0$;
(FPC2) $C_{x_\lambda}(A) < 1 \Rightarrow e \not\leq A$;
(FPC3) $C_{x_\lambda}(A \vee B) = C_{x_\lambda}(A) \vee C_{x_\lambda}(B)$.
and $(L^X, \mathcal{C})$ is called an pointwise L-fuzzy closure space (L-PFCS, for short). $\mathcal{C}$ will be called a topological L-fuzzy closure operator if it also satisfies the following equation:
(FPC4) $C_{x_\lambda}(A) = \bigwedge_{B \in x_\lambda | A} \bigvee_{y_\mu \not\leq B} C_{y_\mu}(B)$,
and $(L^X, \mathcal{C})$ is called a topological pointwise L-fuzzy closure space (L-PTFCS, for short). A fuzzy continuous mapping between topological L-fuzzy closure spaces $(L^X, \mathcal{C}^1)$ and $(L^Y, \mathcal{C}^2)$ is a mapping $F : X \to Y$ such that $C^1_{x_\lambda}(A) \leq C^2_{F^{\to}_L(x_\lambda)}(F^{\to}_L(A))$ for all $x_\lambda \in c(L^X)$ and $A \in L^X$. The category of L-PTFCS and continuous mappings is denoted by L-**PTFCS**.

Remark 4.4. When $L = \{0, 1\}$, the definition of pointwise L-fuzzy closure operator is just the definition of fuzzifying closure operator in [11].

It is easy to verify the next two theorems:

Theorem 4.5. (1) Let $(L^X, \mathcal{C})$ be a topological pointwise L-fuzzy closure space and define $R^{\mathcal{C}}_{x_\lambda} : L^X \to M$ by $R^{\mathcal{C}}_{x_\lambda}(A) = \mathcal{C}_{x_\lambda}(A)'$. Then $\mathcal{R}^{\mathcal{C}} = \{R^{\mathcal{C}}_{x_\lambda} | x_\lambda \in c(L^X)\}$ is a topological fuzzy remote neighborhood system.

(2) Let $(L^X, \mathcal{R})$ be a topological fuzzy remote neighborhood space and define $C^{\mathcal{R}}_{x_\lambda} : L^X \to M$ by $C^{\mathcal{R}}_{x_\lambda}(A) = R_{x_\lambda}(A)'$. Then $\mathcal{C}^{\mathcal{R}} = \{C^{\mathcal{R}}_{x_\lambda} | x_\lambda \in c(L^X)\}$

is a topological pointwise L-fuzzy closure space.

Theorem 4.6. (1) If $F:(L^X,\mathcal{C}^1)\to(L^Y,\mathcal{C}^2)$ is continuous, then $F:(L^X,\mathcal{R}^{\mathcal{C}^1})\to(L^Y,\mathcal{R}^{\mathcal{C}^2})$ is continuous.

(2) If $F:(L^X,\mathcal{R}^1)\to(L^Y,\mathcal{R}^2)$ is continuous, then $F:(L^X,\mathcal{C}^{\mathcal{R}^1})\to(L^Y,\mathcal{C}^{\mathcal{R}^2})$ is continuous.

From Theorem 4.5 and Theorem 4.6, we have one main theorem in this paper.

Theorem 4.7. L-**PTFCS** is isomorphic to **TFRNS**. Hence, L-**PTFCS** is isomorphic to L-**FCTOP**.

Theorem 4.8. Let $\mathcal{C}=\{C_{x_\lambda}|x_\lambda\in M(L^X)\}$ be a pointwise L-fuzzy closure operator. Then the following statements are equivalent:

(1) $C_{x_\lambda}(A)=\bigwedge_{B\in x_\lambda|A}\bigvee_{y_\mu\not\leq B}C_{y_\mu}(B)$;
(2) $C_{x_\lambda}(A)=\bigwedge_{B\in x_\lambda|A}(C_{x_\lambda}(B)\vee\bigvee_{y_\mu\not\leq B}C_{y_\mu}(B))$.
(3) $C_{x_\lambda}(A)=\bigwedge_{B\in x_\lambda|A}(C_{x_\lambda}(A)\vee\bigvee_{y_\mu\not\leq B}C_{y_\mu}(B))$.
(4) $C_{x_\lambda}(A)=\bigwedge_{B\in x_\lambda|A}(C_{x_\lambda}(B)\vee\bigvee_{y_\mu\not\leq B}C_{y_\mu}(A))$.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) is trivial. We only prove (1) $\Leftrightarrow$ (4). (1)$\Rightarrow$(4) is trivial. Now suppose (4) holds, i.e,

$$C_{x_\lambda}(A)=\bigwedge_{B\in x_\lambda|A}(C_{x_\lambda}(B)\vee\bigvee_{y_\mu\not\leq B}C_a(A)).$$

Let

$$\begin{aligned}t\in\alpha(C_{x_\lambda}(A))&=\alpha(\bigwedge_{B\in x_\lambda|A}(C_{x_\lambda}(B)\vee\bigvee_{y_\mu\not\leq B}C_a(A)))\\&=\bigcup_{B\in x_\lambda|A}\alpha(C_{x_\lambda}(B)\vee\bigvee_{y_\mu\not\leq B}C_{y_\mu}(A)).\end{aligned}$$

Then there exists some $B\in x_\lambda|A$ such that

$$(1)\ t\in\alpha(C_{x_\lambda}(B));\quad(2)\ \forall y_\mu\not\leq B,\ t\in\alpha(C_{y_\mu}(A)).$$

It is clear that the meet of fuzzy sets containing A and fulfilling (1) and (2) is still of such kind. So we can define B_t to be the minimal fuzzy set containing A and fulfilling (1) and (2) , i.e., $t \in \alpha(C_{x_\lambda}(B_t))$ and $t \in \alpha(C_{y_\mu}(A))$ for all $y_\mu \not\leq B_t$. Thus, $\forall y_\mu \not\leq B_t$, it follows from $t \in \alpha(C_{y_\mu}(A))$ that there exists $W_{y_\mu} \in y_\mu|A$ such that

$$(3)\ t \in \alpha(C_{y_\mu}(W_{y_\mu})),\quad (4)\ \forall z_\gamma \not\leq W_{y_\mu},\ t \in \alpha(C_{z_\gamma}(A)).$$

It is easy to check that $B_t \wedge W_{y_\mu}$ satisfies (1) and (2). Hence, by the minimality of B_t, it follows that $B_t \leq B_t \wedge W_{y_\mu}$. Therefore $B_t \leq W_{y_\mu}$. Then we can get that $\forall y_\mu \not\leq B_t,\ C_{y_\mu}(B_t) \leq C_{y_\mu}(W_{y_\mu})$.
Then $t \in \alpha(C_{y_\mu}(B_t))$ Thus, $t \geq \bigvee_{y_\mu \not\leq B_t} C_{y_\mu}(B_t))$. Therefore, $\bigwedge_{B \in x_\lambda|A} \bigvee_{y_\mu \not\leq B} C_{y_\mu}(B) \leq t$. From the arbitrariness of t, we have $C_{x_\lambda}(A) \geq \bigwedge_{B \in x_\lambda|A} \bigvee_{y_\mu \not\leq B} C_{y_\mu}(B)$. Since $C_{x_\lambda}(A) \leq \bigwedge_{B \in x_\lambda|A} \bigvee_{y_\mu \not\leq B} C_{y_\mu}(B)$ is obvious, we have $C_{x_\lambda}(A) = \bigwedge_{B \in x_\lambda|A} \bigvee_{y_\mu \not\leq B} C_{y_\mu}(B)$, as desired.

It is well-known that derived operator and closure operator have close relationship in L-topology. In the following discussion, we study the relationship between pointwise L-fuzzy derived operator and pointwise L-fuzzy closure operator.

Lemma 4.9[7]**.** Let $A \in L^X$, $x_\lambda \in c(L^X)$ and define $A - x_\lambda \in L^X$ by

$$A - x_\lambda(y) = \begin{cases} A(y), & x \neq y, \\ \bigvee_{\lambda \not\leq \gamma, x_\gamma \leq A} \gamma, & x = y. \end{cases}$$

Then
(1) $\bigvee_{t \in T}(A_t - x_\lambda) = \bigvee_{t \in T} A_t - x_\lambda$;
(2) $x_\lambda \not\leq A \ \Rightarrow\ A - x_\lambda = A$.

Definition 4.10. A topological pointwise L-fuzzy derived operator is a set $\mathcal{D} = \{D_{x_\lambda} | x_\lambda \in c(L^X)\}$ of mappings $D_{x_\lambda} : L^X \to M$ such that
(FD1) $D_{x_\lambda}(0_X) = 0$;
(FD2) $D_{x_\lambda}(A) < 1 \Rightarrow x_\lambda \not\leq A - x_\lambda$;
(FD3) $D_{x_\lambda}(A \vee B) = D_{x_\lambda}(A) \vee D_{x_\lambda}(B)$;
(FD4) $D_{x_\lambda}(A) = \bigwedge_{B \in x_\lambda|A - x_\lambda} \bigvee_{y_\mu \not\leq B} D_{y_\mu}(B)$.
$(L^X, \mathcal{D})$ is called a topological pointwise L-fuzzy derived space (L-TFDS, for short).

It is easy to verify the following two theorems.

Theorem 4.11. (1) Let $(L^X, \mathcal{C})$ be a topological pointwise L-fuzzy closure space and define $D^{\mathcal{C}}_{x_\lambda} : L^X \to M$ by $D^{\mathcal{C}}_{x_\lambda}(A) = C_{x_\lambda}(A - x_\lambda)$. Then $\mathcal{D}^{\mathcal{C}} = \{R^{\mathcal{C}}_{x_\lambda} | x_\lambda \in c(L^X)\}$ is a topological pointwise L-fuzzy derived operator.

(2) Let $(L^X, \mathcal{D})$ be a topological pointwise L-fuzzy derived operator and define $C^{\mathcal{D}}_{x_\lambda} : L^X \to M$ by

$$C^{\mathcal{D}}_{x_\lambda}(A) = \begin{cases} 1, & x_\lambda \leq A, \\ D_{x_\lambda}(A), & x_\lambda \not\leq A. \end{cases}$$

Then $\mathcal{C}^{\mathcal{D}} = \{C^{\mathcal{D}}_{\lambda} | x_\lambda \in c(L^X)\}$ is a topological pointwise L-fuzzy closure operator.

Theorem 4.12. (1) Let $\mathcal{C}$ be a topological pointwise L-fuzzy closure operator. Then $\mathcal{C} = \mathcal{C}^{\mathcal{D}^{\mathcal{C}}}$.

(2) Let $\mathcal{D}$ be a topological pointwise L-fuzzy closure operator. Then $\mathcal{D} \geq \mathcal{D}^{\mathcal{C}^{\mathcal{D}}}$.

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Yueli Yue
Department of Mathematics
Beijing Institute of Technology
Beijing, 100081,
P. R. China
e-mail : yueliyue@163.com

Fu-Gui Shi
Department of Mathematics
Ocean University of China,
Qingdao, 266071,
P. R. China
e-mail : fuguishi@bit.edu.cn