Proyecciones
Vol. 27, N^o 1, pp. 103–111, May 2008.
Universidad Católica del Norte
Antofagasta - Chile
DOI: 10.4067/S0716-09172008000100007

ON CONMUTATIVE LEFT-NILALGEBRAS OF INDEX 4 *

JUAN C. GUTIERREZ FERNANDEZ
UNIVERSIDADE DE SAO PAULO, BRASIL

Received : Diciembre 2007. Accepted : March 2008

Abstract

We first present a solution to a conjecture of (Correa, Hentzel, Labra, 2002) in the positive. We show that if A *is a commutative nonassociative algebra over a field of characteristic* $\neq 2, 3$, *satisfying the identity* $x(x(xx)) = 0$, *then* $L_{a^{t_1}} L_{a^{t_2}} \cdots L_{a^{t_s}} \equiv 0$ *if* $t_1 + t_2 + \cdots + t_s \geq 10$, *where* $a \in A$.

Keywords : solvable; commutative; nilalgebra;

Mathematics Subject Classification : 17A05, 17A30

*Partially Supported by FAPESP 05/01790-9, Brasil

1. Introduction

Throughout this paper the term *algebra* is understood to be a commutative not necessarily associative algebra. We will use the notations and terminology of (Fernandez, 2004). Let A be an (commutative nonassociative) algebra over a field F. We define inductively the following powers, $A^1 = A$ and $A^s = \sum_{i+j=s} A^i A^j$ for all positive integers $s \geq 2$. We shall say that A is *nilpotent* if there is a positive integer s such that $A^s = (0)$. The least such number is called the index of nilpotency of the algebra A. The algebra A is called *nilalgebra* if given $a \in A$ we have that $\mathrm{alg}(a)$, the subalgebra of A generated by a, is nilpotent. The *(principal) powers* of an element a in A are defined recursively by $a^1 = a$ and $a^{i+1} = aa^i$ for all integers $i \geq 1$. The algebra A is called *left-nilalgebra* if for every a in A there exists an integer $k = k(a)$ such that $a^k = 0$. The smallest positive integer k which this property is the *index*. Obviously, every nilalgebra is left-nilalgebra. For any element a in A, the linear mapping L_a of A defined by $x \rightarrow ax$ is called *multiplication operator* of A. An *Engel algebra* is an algebra in which every multiplication operator is nilpotent in the sense that for every $a \in A$ there exists a positive integer j such that $L_a^j = 0$.

An important question is that of the existence of simple nilalgebras in the class of finite-dimensional algebras. In (Fernandez, 2004) we proved that every nilagebra A of dimension ≤ 6 over a field of characteristic $\neq 2, 3, 5$ is solvable and hence $A^2 A$. For power-associative nilalgebras of dimension ≤ 8 over a field of characteristic $\neq 2, 3, 5$, we have shown in (Fernandez, Suazo, 2005) that they are solvable, and hence there is no simple algebra in this subclass. See also (Elgueta, Suazo, 2004; Fernandez, 2004) for power-associative nilalgebras of dimension ≤ 7.

We show now the process of linearization of identities, which is an important tool in the theory of varieties of algebras. See (Gerstenhaber, 1960; Osborn, 1972; Zhevlakov, 1982) for more information. Let P be the free commutative nonassociative polynomial ring in two generators x and y over a field F. For every $\alpha_1, \ldots, \alpha_r \in P$, the *operator linearization* $\delta[\alpha_1, \ldots, \alpha_r]$ can be defined as follows: if $p(x, y)$ is a monomial in P, then $\delta[\alpha_1, \ldots, \alpha_r]p(x, y)$ is obtained by making all the possible replacements of r of the k identical arguments x by $\alpha_1, \ldots, \alpha_r$ and summing the resulting terms if $x-$degree of $p(x, y)$ is $\geq r$, and is equal to zero in other cases. Some examples of this operator are

$$[ll]\delta[y](x^2(xy)) \qquad = 2(xy)^2 + x^2y^2$$
$$\delta[x^2, y](x^2) \quad = 2x^2y, \quad \delta[y, xy^2, x](x^2) = 0.$$

For simplicity, $\delta[\alpha : r]$ will denote $\delta[\alpha_1, \ldots, \alpha_r]$, where $\alpha_1 = \cdots = \alpha_r = \alpha$. We observe that if $p(x)$ is a polynomial in P, then $p(x + y) = p(x) + \sum_{j=1}^{\infty} \delta[y : j]p(x)$, where $\delta[y : j]p(x)$ is the sum of all the terms of $p(x + y)$ which have degree j with respect to y.

Lemma 1. *(Zhevlakov, 1982) Let* $p(x, y)$ *be a commutative nonassociative polynomial* *of x-degree $\leq n$. If F is a field of characteristic either zero or $\geq n$, and the F-algebra A satisfies the identity $p(x,y)$, then A satisfies all linearizations of $p(x,y)$.*

2. Left-nilalgebras of index 4

Throughout this section F is a field of characteristic different from 2 or 3 and all the algebras are over F. We will study left-nilalgebras of index ≤ 4, that is the variety V of algebras over the field F satisfying the identity

$$x^4 = 0. \tag{2.1}$$

Let A be an algebra in V. For simplicity, we will denote by L and U the multiplication operators, L_x and L_{x^2} respectively, where x is an element in A. The following known result is a basic tool in our investigation. See (Correa, Hentzel, Labra, 2002; Elduque, Labra, 2007).

Lemma 2. *Let A be a commutative left-nilalgebra of index 4. Then A satisfies the identities*

$$xx = -x(xx), \quad xx = (x)^3 = x(x(xx)), \tag{2.2}$$

and $p(x) = 0$, for every monomial $p(x)$ with x-degree ≥ 7. Furthermore, we have

$$[lcl]L_x = -LU - 2L^3, \tag{2.3}$$

$$L_{xx^2} = -U - 2UL^2 - 2LUL + 4L^4, \tag{2.4}$$

$$L_{x(xx)} = -LU - 2LUL - 2LUL - 4L^3U - 12L^5, \tag{2.5}$$

$$L_{x(x(xx))} = 2LU + 4LUL + 4L^4U + 8L^6, \tag{2.6}$$

and also

Table i. Multiplication identities of degree 5.

	ULU	LU^2	UL^3	LUL^2	L^2UL	L^3U	L^5
U^2L	0	-1	2	0	0	-2	-8

and two identities of x-degree 6 which may be written as

Table ii. Multiplication identities of degree 6.

	UL^2U	$(LU)^2$	L^2U^2	UL^4	LUL^3	L^2UL^2	L^3UL	L^4U	L^6
U^3	-2	-2	2	-8	-8	0	-4	8	40
$(UL)^2$	-1	-1	1	-4	-2	2	0	4	24

We note that, for example, Table i means that $U^2L = -LU^2 + 2UL^3 - 2L^3U - 8L^5$. From the identities (2.3-2.6) we get that for any $a \in A$ the associative algebra A_a generated by all L_c with $c \in \mathrm{alg}(a)$ is in fact generated by L_a and L_{a^2}. Furthermore, every algebra in V is a nilalgebra of index ≤ 7.

We now pass to study homogeneous identities in A with x-degree ≥ 7 and y-degree 1. From the relation $0 = \delta[y,x,x,x](x^4) = 2y(x(x^3x^3)) + 4y(x^3(xx^3)) + 2x(y(x^3x^3)) + 4x^3(y(xx^3)) + 4x(x^3(yx^3)) + 4x^3(x(yx^3)) + 4x^3(x^3(xy)) = 2x((x^3)^2y) + 4x(x^3(x^3y)) + 4x^3(x(x^3y)) + 4x^3(x^3(xy)) = 2[LL_{x^3x^3} + 2LL_{x^3}L_{x^3} + 2L_{x^3}LL_{x^3} + 2L_{x^3}L_{x^3}L](y)$ we have

$$L^3U^2 = -2L^3UL - L^4UL - 5L^5U - 20L^7, \tag{2.7}$$

since we can use the reductions (2.3-2.6) and replace the occurrences of $(UL)^2$. Multiplying the identity of Table i by U from the left, replacing first the occurrences of U^3 and next using reductions from Table i, Table ii and above identity we get a new identity as follows:

$$\begin{aligned} 0 &= U^3L + ULU^2 - 2U^2L^3 + 2UL^3U + 8UL^5 = [-2UL^2UL - 2(LU)^2L + 2L^2U^2L - 8UL^5 \\ &\quad -8LUL^4 - 4L^3UL^2 + 8L^4UL + 40L^7] + ULU^2 - 2U^2L^3 + 2UL^3U + 8UL^5 \\ &= -2UL^2UL + [2LUL^2U + 2L(LU)^2 + [4L^3UL^2 + 2L^4UL + 10L^5U + 40L^7] \\ &\quad +8LUL^4 + 4L^2UL^3 - 4L^3UL^2 - 8L^5U + 48L^7] + 2L^2U^2L - 8UL^5 - 8LUL^4 \\ &\quad -4L^3UL^2 + 8L^4UL + 40L^7 + ULU^2 + [[-2L^2U^2L + 4LUL^4 - 4L^4UL - 16L^7] \\ &\quad -4UL^5 + 4L^3UL^2 + 16L^7] + 2UL^3U + 8UL^5, \end{aligned}$$

that is,

$$\begin{aligned} ULU^2 = 2\Big(&UL^2UL - UL^3U - LUL^2U - L^2ULU + \\ &2UL^5 - 2LUL^4 - 2L^2UL^3 - 3L^4UL - L^5U - 16L^7\Big). \end{aligned} \tag{2.8}$$

Next, we can reduce the relation $0 = \delta[y,x,x,xx]x^4$ using the above identities. This yields

$$UL^5 = -LUL^4 + \frac{1}{2}L^2UL^3 + \frac{3}{4}L^4UL + \frac{3}{4}L^5U + 8L^7. \tag{2.9}$$

Now combining (2.8) and (2.9) we obtain $ULU^2 = 2ULUL - 2UL^3U - 2LUL^2U - 2L^2ULU - 8LUL^4 - 2L^2UL^3 - 3L^4UL + L^5U$. Thus, we have three identities of x-degree 7 and y-degree 1 which may be written as multiplication identities:

Table iii. Multiplication identities of degree 7.

	UL^2UL	UL^3U	$LULU$	$L(LU)^2$	LUL^4	LUL^3	L^3UL	L^4UL	L^5U	L^7
L^3U	0	0	0	0	0	0	−2	−1	−5	−20
UL^5	0	0	0	0	−1	1/2	0	3/4	3/4	8
ULU^2	2	−2	−2	−2	−8	−2	0	−3	1	0

In an analogous way, using successively the identities

$$0 = \delta[y,x,x,x(x(x^2x))]x^4, \quad 0 = \delta[y,x,x,x^2x]x^4, \quad 0 = \delta[y,x,x^2,x(x^2x)]x^4,$$

multiplying the second identity of Table ii with the operator U from the left and replacing the occurrences of UUL, and finally using $0 = \delta[y,x,x^3,x^2x]x^4$, we obtain the following 5 multiplication identities:

Table iv. Multiplication identities of degree 8.

	UL^4U	LUL^2UL	LUL^3U	L^2ULU	L^2UL^4	L^4UL^2	L^5UL	L^6U	L^8
L^3ULU	0	0	0	0	0	$-1/2$	-2	$-11/2$	-20
ULU^2	-4	-2	-2	0	2	$-5/2$	13	$31/2$	32
$(UL)^2$	0	1	0	-1	-12	$-11/4$	$-7/2$	$25/4$	36
UL^3UL	-1	-1	-1	0	-4	$-11/2$	-3	$9/2$	0
L^3UL^3	0	0	0	0	0	$-3/4$	$-3/2$	$-3/4$	-8

Now, relations $0 = \delta[y,x,x^3,x(x^2x^2)]x^4$, $0 = \delta[y,x,x^2,x(x(x^2x^2))]x^4$, $0 = \delta[y,x^2,x^2,(x^2x^2)]x^4$, $0 = \delta[y,x,x^2x^2,x^2x^2]x^4$, $0 = \delta[y,x^2,x^3,x^2x^2]x^4$, and multiplying the relation determined by the last row of Table iii with the operator U from the left and first replacing the occurrences of UUL, imply the following 6 multiplication identities:

Table v. Multiplication identities of degree 9.

	LUL^4U	$(L^2U)^2L$	L^7U	L^9
L^6UL	0	0	-7	-48
$L(LU)^2$	0	0	-217	$-4510/3$
UL^4UL	1	0	$-587/2$	$-6155/3$
L^2UL^3U	0	0	$29/3$	$422/9$
UL^2ULU	0	0	$1318/3$	$27988/9$
L^5UL^2	0	0	-23	$-496/3$

The author used a simples MAPLE language program to check these identities. We now present a solution of a Conjecture of (Correa, Hentzel, Labra, 2002) in the positive. We see that for every $a \in A$, the associative algebra A_a, generated by the multiplication operators L_a and L_{a^2}, is nilpotent of index ≤ 10.

Theorem 1. *Let A be an algebra over a field F of characteristic $\neq 2,3$, satisfying $x^4 = 0$. Then every monomial in P of x-degree ≥ 10 and y-degree 1 is an identity in A. In particular, $L_a^{10} = 0$ for all $a \in A$.*

Proof. First we shall prove that every monomial of x-degree 10 and y-degree 1 is an identity in A. Multiplying the operators in the first line of Table v with L from the left and from the right, and the operators in the first line of Table iv with U from the left and from the right and next using reductions from Tables i-v we see that we only need to prove that $L^2UL^4U = 0, L^8U = 0$ and $L^{10} = 0$ are multiplication identities in A. Now, for any x in A we have

$$\begin{aligned}[ll]L^7UL &= L(L^6UL) = -7L^8U - 48L^{10},\\ L^6UL^2 &= (L^6UL)L = -7L^7UL - 48L^{10} = 49L^8U + 288L^{10},\\ L^6UL^2 &= L(L^5UL^2) = -23L^8U - 496/3L^{10}.\end{aligned}$$

Therefore

$$27L^8U + 170L^{10} = 0. \tag{2.10}$$

Now,

$$\begin{aligned}
[ll]L^5UL^3 &= (L^5UL^2)L = -23L^7UL - 496/3L^{10} = 161L^8U + 2816/3L^{10},\\
L^5UL^3 &= L^2(L^3UL^3) = -3/4L^6UL^2 - 3/2L^7UL - 3/4L^8U - 8L^{10}\\
&= -27L^8U - 152L^{10},
\end{aligned}$$

and hence

$$141L^8U + 818L^{10} = 0. \tag{2.11}$$

Next

$$\begin{aligned}
[ll]L^3UL^3U &= L(L^2UL^3U) = 29/3L^8U + 422/9L^{10},\\
L^3UL^3U &= (L^3UL^3)U = -3/4L^4UL^2U - 3/2L^5ULU - 3/4L^6UU - 8L^8U\\
&= -3/4L(L^3UL^2U) - 3/2L^2(L^3ULU) - 3/4L^3(L^3U^2) - 8L^{10}\\
&= 9/4L^6UL^2 + 15/4L^7UL + 667/4L^8U + 2345/2L^{10}\\
&= 1003L^8U + 3281/2L^{10},
\end{aligned}$$

so that

$$17880L^8U + 28685L^{10} = 0. \tag{2.12}$$

Combining (2.10-2.12) we obtain that $L^8U = 0$ and $L^{10} = 0$. Now, we have by Table v that $0 = (L^2UL^3U)L = L^2(UL^3UL) = -L^2UL^4U - L^3UL^2UL - L^3UL^3U - 4L^4UL^4 - 11/2L^6UL^2 - 3L^7UL + 9/2L^8U = -L^2UL^4U - (L^3UL^2U)L - 4L(L^3UL^3)L = -L^2UL^4U$. Therefore, we have $L^2UL^4U = 0$.

In an analogous way, we can see that every monomial of x-degree 11 and y-degree 1 is an identity in A. This proves the theorem. □

Now we shall investigate two subvarieties of V. We start in Subsection 2.1 with the class of all nilalgebras in V of index ≤ 5 and next in Subsection 2.2 we study the multiplication identities of the variety of all the nilalgebras in V of index ≤ 6.

2.1. The identity x((xx)(xx))=0

We will now consider the class of all algebras in V satisfying the identity $x(xx) = 0$. First, linearization $\delta[y]\{x(x^2)^2\}$ implies

$$L_{x^2x^2} = -4LUL, \tag{2.13}$$

and identity $\delta[y]\{x^2x^3\} = 0$ forces

$$UU = -2ULL + 2LUL + 4L^4. \tag{2.14}$$

Next, using above identity and $\delta[y, x^2]\{x(x^2)^2\} = 0$ we get that $0 = 4UUL + 4LUU + 8LL_{x^3}L = 4(UUL + LUU - 2LLUL - 4L^5) = 8(-UL^3 + LULL + 2L^5 - LULL + LLUL +$

$2L^5 - LLUL - 2L^5) = 8(-UL^3 + 2L^5)$. Hence $UL^3 = 2L^5$. Now idnetity $L_{x(x^2x^2)} = 0$ and relations (2.5) and (2.14) imply $L^2UL = -L^3U - 4L^5$. Thus, we have the following multiplication identities.

Table vi. Multiplication identities of degree 5.

	ULU	LUL^2	L^3U	L^5
UUL	0	2	0	0
LUU	0	−2	−2	−4
L^2UL	0	0	−1	−4
UL^3	0	0	0	2

From Table ii, we can prove that

$$(UL)^2 = -UL^2U - (LU)^2 + 2L^3UL + 4L^4U + 16L^6, \tag{2.15}$$

and $\delta[x^2]\{x^2(x(x(xy))) - 2x(x(x(x(xy))))\} = 0$ forces

$$(UL)^2 + UL^2U + 2L^3UL + 4L^6 = 0. \tag{2.16}$$

Combining (2.15) and (2.16), we have $(LU)^2 = 4L^6$ and $(UL)^2 = -UL^2U + 2L^4U + 4L^6$. Now, we can check easily the following multiplication identities.

Table vii. Multiplication identities of degree 6.

	$ULLU$	L^4U	L^6
UUU	−2	4	8
$UULL$	0	0	4
$ULUL$	−1	2	4
$LUUL$	0	2	0
$LULU$	0	0	4

	$ULLU$	L^4U	L^6
$LLUU$	0	−4	−4
UL^4	0	0	2
LUL^3	0	0	2
L^2UL^2	0	1	0
L^3UL	0	−1	−4

Theorem 2. *Let A be an algebra over a field F of characteristic $\neq 2$ or 3, satisfying the identities $x^4 = 0$ and $x(x^2x^2) = 0$. Then every monomial in P of x-degree ≥ 7 and y-degree 1 is an identity in A. In particular, $L_a^7 = 0$ for all $a \in A$. Furthermore, the algebra generated by L_x and L_{x^2} is spanned, as vector space, by* $L, U, L^2, UL, LU, L^3, UL^2, LUL, L^2U, L^4, ULU, LUL^2, L^3U, L^5, UL^2U, L^4U, L^6$.

Proof. We shall prove that every monomial of x-degree ≥ 7 and y-degree 1 is an identity in A. Multiplying the operators in the first line of Table vii with L and U from the left and from the right, and the operators in the first line of Table vi with U from the left and from the right, and next using reductions from Tables i-vii we see that we only need to prove that $LUL^2U = 0, L^5U = 0$ and $L^7 = 0$ are multiplication identities in A. Now, we have $0 = \delta[y, x^2x^2]\{x(x^2)^2\} = 4L_{x^2x^2}UL + 4LUL_{x^2x^2} = -16LULUL - 16LULUL = -32LULUL = -32(LU)^2L = -2^7L^7$, so that $L^7 = 0$. Also $0 = LULUL = L(UL)^2 = -LUL^2U + 2L^5U$. Therefore, $LUL^2U = 2L^5U$. Finally, from Table vi we have that $0 = (L^2UL + L^3U + 4L^5)L^2 = L^2UL^3 + L^3UL^2 = L^3UL^2 = L(L^2UL^2) = L^5U$. This proves the theorem. □

2.2. The identity x(x((xx)(xx)))=0

In this subsection we consider the class of all algebras in V satisfying the identity $x(x(xx)) = 0$. Because we use linearization process of identities and $x(x(x^2x^2))$ has degree 6, we need consider the field F of characteristic not 5 (2 or 3.)

From linearization $\delta[y]\{x(x(xx))\}$, we get the multiplication identity $L_{x(x^2x^2)}+LL_{x^2x^2}+4L^2UL = 0$ and now Lemma 2 forces

$$LUU = -2LUL^2 - 2L^3U - 4L^5. \tag{2.17}$$

The relation $0 = \delta[y, x^2]\{x(x(x^2x^2))\} = UL_{x^2x^2} + 4LL^{x^2x^3}L + 4ULUL + 4LUUL + 8L^2L_{x^3}L + 4L^2UU$ implies

$$LUL^3 = -\frac{1}{2}\Big(L^2UL^2 + L^3UL\Big), \tag{2.18}$$

since we can use identities from Tables i-v. Next, by $0 = \delta[y, x^3]\{x(x(x^2x^2))\}$ and $0 = \delta[y, x^2, x^2]\{x(x(x^2x^2))\}$ we get

$$[lcl]L^4UL = -3L^5U - 16L^7, \tag{2.19}$$

$$L^2ULU = -L^3UL^2 + 5L^5U + 28L^7, \tag{2.20}$$

and identities $0 = \delta[y, x^2, x, x]\{x(x(x^2x^2))\}$ and $0 = \delta[y, x^2, x^3]\{x(x(x^2x^2))\}$ imply

$$[lcl]UL^4U = -\frac{1}{2}L^2UL^2U + 24L^6U + 62L^8, \tag{2.21}$$

$$L^2UL^2U = 48L^6U + 156L^8. \tag{2.22}$$

Now, identity $0 = \delta[y, xx^2]\{xx^3\}$ forces

$$L^6U = -2L^8. \tag{2.23}$$

Theorem 3. *Let A be a commutative algebra over a field F of characteristic not $2, 3$ or 5 , satisfying the identities $x^4 = 0$ and $x(x(x^2x^2)) = 0$. Then every monomial in P of x-degree ≥ 9 and y-degree 1 is an identity in A. In particular, $L_a^9 = 0$ for all $a \in A$.*

Proof. By Tables i-v, we only need to prove that $LUL^4U = 0$, $L^2UL^2UL = 0$, $L^7U = 0$ and $L^9 = 0$ are multiplication identities in A. From (2.19-2.23) may be deduced immediately $L^7U = -2L^9$ and $2L^9 = 2L^8L = -L^6UL = -L^2(L^4UL) = 3L^7U + 16L^9 = -6L^9 + 16L^9 = 10L^9$. Therefore $L^9 = 0$ and $L^7U = 0$ are identities in A. Now $L^2UL^2UL = (L^2UL^2U)L = 48L^6UL + 156L^9 = 0$ and $LUL^4U = L(UL^4U) = -(1/2)L^3UL^2U + 24L^7U + 62L^9 = -(1/2)L(L^2UL^2U) = -24L^7U - 78L^9 = 0$. This proves the theorem. □

References

[1] Albert, A. A.. Power-associative rings. *Trans. Amer. Math. Soc.* 64 : pp. 552-593, (1948).

[2] Correa, I., Hentzel I.R., Labra, A. (2002). On the nilpotence of the multiplication operator in commutative right nil algebras. *Comm. in Algebra* 30, pp. 3473-3488, (2002).

[3] Elduque, A., Labra, A.. On the Classification of commutative right-nilalgebras of dimension at most four. *Comm. in Algebra* 35 : pp. 577-588, (2007).

[4] Elgueta, L., Suazo, A.. Solvability of commutative power-associative nilalgebras of nilindex 4 and dimension ≤ 8. *Proyecciones* 23 : pp. 123-129, (2004).

[5] Elgueta, L., Fernandez, J. C. G., Suazo, A.. Nilpotence of a class of commutative power-associative nilalgebras. *Journal of Algebra* 291 : pp. 492-504, (2005).

[6] Fernandez, J. C. G.. On commutative power-associative nilalgebras. *Comm. in Algebra* 32 : pp. 2243-2250, (2004).

[7] Fernandez, J. C. G., Suazo, A.. Commutative power-associative nilalgebras of nilindex 5. *Result. Math.* 47 : pp. 296-304, (2005).

[8] Gerstenhaber, M.. On nilalgebras and linear varieties of nilpotent matrices II. *Duke Math. J.* 27 : pp. 21-31, (1960).

[9] Osborn, J. M.. Varieties of algebras. *Adv. in Math.* 8 : pp. 163-396, (1972).

[10] Zhevlakov, K. A., Slinko, A. M., Shestakov, I. P., Shirshov, A. I.. Rings that are nearly associative. *Academic Press, New York, (1982).*

Juan C. Gutierrez Fernandez
Departamento de Matemtica-IME,
Universidade de So Paulo,
Caixa Postal 66281,
CEP 05315-970,
So Paulo, SP,
Brazil
e-mail : jcgf@ime.usp.br