Proyecciones
Vol. 27, N^o 1, pp. 97–102, May 2008.
Universidad Católica del Norte
Antofagasta - Chile
DOI: 10.4067/S0716-09172008000100006

SHARP INEQUALITIES FOR FACTORIAL n

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Received : Septiembre 2006. Accepted : March 2007

Abstract

Let n be a positive integer. We prove

$$\frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-\alpha}} \leq n! < \frac{n^{n+1}e^{-n2}\sqrt{2\pi}}{\sqrt{n-\beta}}$$

with the best possible constants

$$\alpha = 1 - 2\pi e^{-2} = 0.149663... \text{ and } \beta = 1/6 = 0.1666666...$$

This refines and extends a result of Sandor and Debnath, who proved that the double inequality holds with $\alpha = 0$ and $\beta = 1$.

2000 Mathematics Subject Classification : *Primary : 30E15; Secondary : 26D07*

Key words and phrases : *Factorial n, gamma function, Stirling's formula, Burnside's formula.*

1. Introduction

Stirling's approximation to n!,

$$n! \sim n^n e^{-n}\sqrt{2\pi n} = \alpha_n \tag{1.1}$$

plays a central role in statistical physics and probability theory. Inspired by this formula, many authors have made attempts to find a formula, which has an improvement over (1.1) and as simple as (1.1), to approximate $n!$. Such a typical result is due to Burnside [1] :

$$n! \sim \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}} = \beta_n. \tag{1.2}$$

It is known that (1.2) has great superiority over (1.1). Formula (1.2) was rediscovered by Y. Weissman [9] and caused a lively debate in the American Journal of Physics in 1983, see [6]. Schuster found some other formulas to approximate n! but they are complicated and not easy to use [8]. In a recently paper Sandor and Debnath [7] found the following inequalities for $n \geq 2$:

$$\frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n}} \leq n! < \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-1}}$$

This formula was rediscovered by Guo in very newly paper [2]. In this short note we determine the largest number α and the smallest number β such that the inequalities

$$\frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-\alpha}} \leq n! < \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-\beta}}$$

are valid for all positive integers n. Numerical computations indicate that the approximation

$$n! \sim \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-1/6}} = \gamma_n \tag{1.3}$$

gives much more accurate values for n! than α_n and β_n (see the table at the end of the paper). Throughout, we denote the gamma function Γ and its logarithmic derivative, known as psi or digamma function as

$$\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}du,\ \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

for positive real numbers x, respectively.

In order to prove our main result we need to present two lemmas.
Lemma 1.1 : For $x \geq 1$ we have

$$\sqrt{\pi}\left(\tfrac{x}{e}\right)^x\left(8x^3+4x^2+x+\tfrac{1}{100}\right)^{\frac{1}{6}} < \Gamma(x+1) < \sqrt{\pi}\left(\tfrac{x}{e}\right)^x\left(8x^3+4x^2+x+\tfrac{1}{30}\right).$$

This result is due to Karatsuba, see [4].
Lemma 1.2 : We have

$$\lim_{x\to\infty}\left(\frac{2\pi x^{2x}e^{-2x}}{[\Gamma(x)]^2}-x\right)=-\frac{1}{6}.$$

Proof : Applying Stirling's formula, we get after a little simplification

$$\lim_{x\to\infty}\left(\tfrac{2\pi x^{2x}e^{-2x}}{[\Gamma(x)]^2}-x\right)=\lim_{x\to\infty}\frac{2\pi x^{2x}e^{-2x}}{[\Gamma(x+1)]^2}\frac{2\pi x^{2x+2}e^{-2x}-x^3(\Gamma(x))^2}{2\pi x^{2x}e^{-2x}} = \lim_{x\to\infty}\frac{2\pi-x^{1-2x}e^{-2x}(\Gamma(x))^2}{2\pi\left(\frac{1}{x}\right)}$$

By L'Hospital's rule this becomes

$$\lim_{x\to\infty}\left(\tfrac{2\pi x^{2x}e^{-2x}}{[\Gamma(x)]^2}-x\right)= = \tfrac{1}{\pi}\lim_{x\to\infty}x^{1-2x}e^{2x}(\Gamma(x+1))^2\left(\psi(x)-\log x+\tfrac{1}{2x}\right).$$

Using Stirling's formula again we get

$$\lim_{x\to\infty}\left(\frac{2\pi x^{2x}e^{-2x}}{[\Gamma(x)]^2}-x\right)=\lim_{x\to\infty}\frac{1-2x(\log x-\psi(x))}{\left(\frac{1}{x}\right)}.$$

From [5] we have

$$\log x-\psi(x)=\frac{1}{2x}+\frac{1}{12x^2}+\frac{\theta}{60x^4},$$

where $0<\theta<1$. Using this relation we find that

$$\lim_{x\to\infty}\left(\frac{2\pi x^{2x}e^{-2x}}{[\Gamma(x)]^2}-x\right)=\lim_{x\to\infty}\frac{1-2\left(\frac{1}{2}+\frac{1}{12x}+\frac{\theta}{60x^4}\right)}{\left(\frac{1}{x}\right)}=-\frac{1}{6}. \square$$

2. Main Result

Our main result is the following theorem.

Theorem 2.1 : For any positive integer n the following double inequality holds

$$\frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-\alpha}} \leq n! < \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-\beta}},$$

where the constants $\alpha = 1 - 2\pi e^{-2} = 0.149663...$ and $\beta = \dfrac{1}{6} = 0.166666...$ are best possible.

Proof. : Set

$$h(x) = \frac{2\pi x^{2x}e^{-2x}}{[\Gamma(x)]^2} - x, x > 0.$$

We show that h is strictly decreasing on $(0, \infty)$. Differentiating h, we get

$$h' = \frac{4\pi \left(\frac{x}{e}\right)^{2x}(\log x - \psi(x) - \Gamma(x))^2}{(\Gamma(x))^2}.$$

Hence in order to show that $h'(x) < 0$, it sufficies to show that

$$\left(\frac{\Gamma(x+1)}{x^x e^{-x}\sqrt{2\pi x}}\right)^2 - 2x(\log x - \psi(x)) > 0.$$

From the left inequality of Lemma1.1 we obtain for $x \geq 1$

$$\left(\frac{\Gamma(x+1)}{x^x e^{-x}\sqrt{2\pi x}}\right)^2 > \left(8 + \frac{4}{x} + \frac{1}{x^2} + \frac{1}{100x^3}\right)^{\frac{1}{3}}$$

In [3] it was proved that

$$x(\log x - \psi(x)) < \frac{1}{2} + \frac{1}{12x}.$$

Employing these two inequalities, we find that for $x \geq 1$

$$\left(\frac{\Gamma(x+1)}{x^x e^{-x}\sqrt{2\pi x}}\right)^2 - 2x(\log x - \psi(x)) > \left(8 + \frac{4}{x} + \frac{1}{x^2} + \frac{1}{100x^3}\right)^{\frac{1}{3}} - 1 - \frac{1}{6x} > 0.$$

Hence, h is strictly decreasing on $(1, \infty)$. Using $h(1) = 2\pi e^{-2} - 1$ and $h(\infty) = -\dfrac{1}{6}$ by Lemma 1.2, we get for any positive integer n

$$-\frac{1}{6} = h(\infty) < h(n) = \frac{2\pi n^{2n+2}e^{-2n}}{(n!)^2} - n < h(1) = 2\pi e^{-2} - 1,$$

for which the proof follows. □

The following table shows that γ_n has great superiority over α_n and β_n, where α_n, β_n and γ_n are as defined by (1.1), $(1,2)$ and $(1,3)$.

n	$n!$	α_n	β_n	γ_n
1	1	0.92213	1.02750	1.01015
2	2	1.91900	2.03331	2.00433
3	6	5.83620	6.07151	6.00541
4	24	23.50617	24.22261	24.01174
5	120	118.01916	120.91079	120.03673
6	720	710.07818	724.62384	720.15071
7	5040	4980.39583	5068.04888	5040.76647
8	40320	39902.39545	40517.97261	40324.65478
9	362880	359536.87284	364474.04470	362912.87998

References

[1] W. Burnside, A rapidly convergent series for logN!, Messenger Math., 46, pp. 157-159, (1917).

[2] S. Guo, Monotonicty and concavity properties of some functions involving the gamma function with applications, J. Inequal. Pure Appl. Math. 7, No. 2, article 45, (2006)

[3] M. Fichtenholz, Differential und integralrechnung II, Verlag Wiss, Berlin, (1978).

[4] E. A. Karatsuba, On the asymptotic representation of the Euler gamma function by Ramanujan, J. Comp. Appl. Math. 135.2, pp. 225-240, (2001).

[5] E. A. Karatsuba, On the computation of the Euler constant γ, Numerical Algorithms, 24, pp. 83-97, (2000).

[6] N. D. Mermin, Improving and improved analytical approximation to n!, Amer. J. Phys., 51, pp. 776, (1983).

[7] J. Sandor and L. Debnath, On certain inequalities involving the constant e and their applications, J. Math. Anal. Appl. 249, pp. 569-582, (2000).

[8] W. Schuster, Improving Stirling's formula, Arch. Math. 77, pp. 170-176, (2001).

[9] Y. Weissman, An improved analytical approximation to n!, Amer. J. Phys., 51, No. 9, (1983).

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