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O_R-CONVERGENCE AND WEAK O_R-CONVERGENCE OF NETS AND THEIR APPLICATIONS*

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Abstract

In this paper, the theory of O_R-convergence and weak O_R-convergence of nets is introduced in L-topological spaces by means of neighborhoods and strong neighborhoods of fuzzy points based on Shi's O-convergence. It can be used to characterize preclosed sets, preopen sets, δ-closed sets, δ-open sets, near compactness and near S^*-compactness.

Keywords: *L-space; neighborhood; strong neighborhood; O_R-convergence; weak O_R-convergence*

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1. Introduction

As is known now, the Moore-smith convergence theory plays an important role in general topology, it not only is an significantly basic theory of fuzzy topology and fuzzy analysis but also has wide applications in fuzzy inference and some other aspects. In [18], Pu and Liu introduced the concept of Q-neighborhoods and established a systematic Moore-Smith convergence theory of fuzzy nets in [0,1]-topology. It paved a new way for the study of the fuzzy topology. Wang extended this theory to L-fuzzy set theory in [22]. Later on, all kinds of convergence theory were presented [2, 3, 4, 7, 8, 9, 12, 14]. In [19], Shi introduced the O-convergence theory of nets in terms of neighborhoods of fuzzy points in L-space. It overcomes the difficulty which the neighborhood method meets.

In this paper, our aim is to introduce the theory of O_R-convergence and weak O_R-convergence of nets in L-spaces based on Shi's O-convergence. We shall discuss its properties and use them to characterize preclosed sets, preopen sets, δ-closed sets, δ-open sets, near compactness and near S^*-compactness.

2. Preliminaries

Throughout this paper $(L, \bigvee, \bigwedge, ')$ is a completely distributive de Morgan algebra. X a nonempty set. L^X is the set of all L-fuzzy sets (or L-sets for short) on X. The smallest element and the largest element in L^X are denoted by $\underline{0}$ and $\underline{1}$.

An element a in L is called prime if $a \geq b \wedge c$ implies that $a \geq b$ or $a \geq c$. An element a in L is called co-prime if a' is a prime element [13]. The set of nonunit prime elements in L is denoted by $P(L)$. The set of nonzero co-prime elements in L is denoted by $M(L)$. The set of nonzero co-prime elements in L^X is denoted by $M(L^X)$. Members in $M(L^X)$ are also called points.

The binary relation $\prec$ in L is defined as follows: for $a, b \in L$, $a \prec b$ if and only if for every subset $D \leq L$, the relation $b \sup D$ always implies that the existence of $d \in D$ with ad [10]. In a completely distributive de Morgan algebra L, each member b is a sup of $\{a \in L \mid a \prec b\}$. In the sense of [15, 23], $\{a \in L \mid a \prec b\}$ is the greatest minimal family of b, in symbol $\beta(b)$. Moreover for $b \in L$, define $\alpha(b) = \{a \in L \mid a' \prec b'\}$ and $\alpha^*(b) = \alpha(b) \cap P(L)$.

For an L-set $G \in L^X$, $\beta(G)$ denotes the greatest minimal family of G

and $\beta^*(G) = \beta(G) \bigcap M(L^X)$.

An L-topological space (or L-space for short) is a pair $(X, \mathcal{T})$, where $\mathcal{T}$ is a subfamily of L^X which contains $\underline{0}$, $\underline{1}$ and is closed for any suprema and finite infima. $\mathcal{T}$ is called an L-topology on X. Each member of $\mathcal{T}$ is called an open L-set and its quasi-complement is called a closed L-set.

Definition 2.1. *Let $(X, \mathcal{T})$ be an L-space. $A \in L^X$ is called*

(1) regularly open [1] if $A^{-\circ} = A$, the complement of a regularly open set is called regularly closed;

(2) β-open [16] if $AA^{-\circ-}$, the complement of a β-open set is called β-closed;

(3) preopen [16] if $AA^{-\circ}$, the complement of a preopen set is called preclosed. If A is not only preopen, but also preclosed, then we call it preclopen.

Definition 2.2 ([19]). *$x_\lambda \in M(L^X)$ is said to be quasi-coincident with $B \in L^X$ if $x_\lambda \not\leq B'$.*

Definition 2.3 ([19]). *An (a regularly open, preopen, δ-open, etc.) open L-set U is called an (a regularly open, preopen, δ-open, etc.)open neighborhood of $x_\lambda \in M(L^X)$ if $X_\lambda U$. All (regularly open, preopen, δ-open, etc.)open neighborhoods of x_λ are denoted by ($\mathcal{N}_R^\circ(x_\lambda)$, $\mathcal{N}_P^\circ(x_\lambda)$, $\mathcal{N}_\delta^\circ(x_\lambda)$)$N^\circ(x_\lambda)$.*

Definition 2.4 ([20]). *Let (X, T) be an L-space. An (a regularly open, preopen, δ-open, etc.)open L-set U is called a strongly (regularly open, preopen, δ-open, etc.)open neighborhood of a fuzzy point x_λ, if $\lambda \in \beta\,(U(x))$.*

Definition 2.5. *Let (X, T_1) and (Y, T_2) be two L-spaces. A map $f : (X, T_1) \to (Y, T_2)$ is called (1) almost continuous [1] if $f_L^{\leftarrow}(G) \in T_1$ for all regularly open L-set G in (Y, T_2); (2) completely continuous [5, 17] if $f_L^{\leftarrow}(G)$ is regularly open L-set in (X, T_1) for each $G \in T_2$; (3) R-irresolute [21] if $f_L^{\leftarrow}(G)$ is regularly closed in (X, T_1) for each regularly closed L-set G in (Y, T_2); (4) δ-continuous [11] if $f_L^{\leftarrow}(G)$ is δ-open in (X, T_1) for each regularly open L-set G in (Y, T_2).*

Definition 2.6 ([19]). *A net S with index set D is also denoted by $\{S(n) \mid n \in D\}$ or $\{S(n)\}_{n \in D}$. For $G \in L^X$, a net S is said to quasi-coincide with G if $\forall n \in D, S(n) \not\leq G'$.*

Definition 2.7 ([19, 22]). *Let $\alpha \in M(L)$. A net $\{s(n) \mid n \in D\}$ in L^X is called an α^--net if there exists $n_0 \in D$ such that $\forall nn_0$, $V(S(n))\alpha$, where*

$V(S(n))$ denotes the height of $S(n)$. A net $\{S(n)\}_{n\in D}$ in L^X is said to be a constant α-net if the height of each $S(n)$ is a constant value α.

Definition 2.8 ([19, 22]). *Let $\{S(n) \mid n \in D\}$ be a net in (X,T), $x_\lambda \in M(L^X)$. S eventually possesses the property P, if there exists $n_0 \in D$ such that $\forall n \geq n_0$, $S(n)$ always possesses the property P. S frequently possesses the property P, if for every $n \in D$, there always exists $n_0 \in D$ such that $n_0 \geq n$ and $S(n_0)$ possesses the property P.*

Definition 2.9 ([19]). *x_λ is an O-cluster point of S, if $\forall U \in N^\circ(x_\lambda)$, S is frequently in U. x_λ is an O-limit point of S, if $\forall U \in N^\circ(x_\lambda)$, S is eventually in U, in this case we also say that S O-converges to x_λ, denoted by $S \xrightarrow{O} x_\lambda$.*

Definition 2.10 ([20]). *Let (X,T) be an L-space, $a \in M(L)$ and $G \in L^X$. A subfamily U of L^X is called a β_a-cover of G if for any $x \in X$ with $a \notin \beta(G'(x))$, there exists an $A \in U$ such that $a \in \beta(A(x))$. A β_a-cover U of G is called open(regularly open, etc.) β_a-cover of G if each member of U is open (regularly open, etc.).*

It is obvious that U is a β_a-cover of G if and only if for any $x \in X$ it follows that $a \in \beta\left(G'(x) \vee \bigvee_{A\in U} A(x)\right)$.

Definition 2.11 ([20]). *Let (X,T) be an L-space, $a \in M(L)$ and $G \in L^X$. A subfamily U of L^X is called a Q_a-cover of G if for any $x \in X$ with $G(x) \not\leq a'$, it follows that $\bigvee_{A\in U} A(x) \geq a$. A Q_a-cover U of G is called open (regularly open, etc.) Q_a-cover of G if each member of U is open (regularly open, etc.).*

Definition 2.12 ([21]). *Let (X,T) be an L-space. $G \in L^X$ is called nearly compact if for every family $U \leq T$, it follows that*

$$\bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{A\in U} A(x)\right) \bigvee_{V\in 2^{(U)}} \bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{A\in V} A^{-\circ}(x)\right).$$

Lemma 2.13 ([21]). *Let (X,T) be an L-space and $G \in L^X$. Then G is nearly compact if and only if for any $a \in M(L)$ and any $b \in \beta^*(a)$, each open Q_a-cover of G has a finite subfamily V such that $V^{-\circ}$ is a Q_b-cover of G.*

Definition 2.14. *Let (X,T) be an L-space and $G \in L^X$. Then G is called nearly S^*-compact if for any $a \in M(L)$, each open β_a-cover of G has a finite subfamily V such that $V^{-\circ} = \{A^{-\circ} \mid A \in V\}$ is a Q_a-cover of G. (X,T) is said to be nearly S^*-compact if $\underline{1}$ is nearly S^*-compact.*

For the sake of convenience, we introduced the following concept.

Definition 2.15. *Let $A \in L^X$. $cl_\delta(A) = \bigwedge\{V \mid AV^{\circ -}, V \in T'\}$ is called δ-closure of A. The δ-interior of A, written as $int_\delta(A)$, is defined to be $cl_\delta(A')'$.*

It can be proved that Definition 2.15 is equivalent to the notion of δ-closure in [11] when $L = [0,1]$.

Obviously we have the following theorem.

Lemma 2.16. *For each $A \in L^X$, $cl_\delta(A) \in T'$ and $int_\delta(A) \in T$.*

Lemma 2.17. *Let $A \in L^X$, then $cl_\delta(A) = \bigwedge\{V \mid AV, V$ is regularly closed$\}$.*

Lemma 2.18. *Let $A \in L^X$, then $A^- cl_\delta(A)$ and $int_\delta(A) A^\circ$.*

Lemma 2.19. *If A is β-open, then $A^- = cl_\delta(A)$; If A is β-closed, then $A^\circ = int_\delta(A)$.*

Definition 2.20. *An L-set G is called δ-closed if $A = cl_\delta(A)$; The complement of a δ-closed set is called δ-open.*

Lemma 2.21. *Each regular open L-set is δ-open and each regular closed L-set is δ-closed.*

3. O_R-convergence and weak O_R-convergence of nets

Definition 3.1. *$x_\lambda \in M(L^X)$ is said to be weak quasi-coincident with $B \in L^X$ if $\lambda \notin \beta(B'(x))$.*

Definition 3.2. *Let (X,T) be an L-space, $x_\lambda \in M(L^X)$ and $S = \{S(n) \mid n \in D\}$ a net in L^X. Then*

(1) x_λ is an O_R-cluster point of S, if $\forall U \in N^\circ(x_\lambda)$, S is frequently in $U^{-\circ}$.

(2) x_λ is an O_R-limit point of S, if $\forall U \in N^\circ(x_\lambda)$, S is eventually in $U^{-\circ}$, in this case we also say that S O_R-converges to x_λ, denoted by $S \xrightarrow{O_R} x_\lambda$.

Definition 3.3. *Let $\{S(n) \mid n \in D\}$ be a net in (X,T), $x_\lambda \in M(L^X)$. x_λ is called a weak O_R-cluster point of S, if for each strongly open neighborhood U of x_λ, S is frequently in $U^{-\circ}$. x_λ is called a weak O_R-limit point of S, if for each strongly open neighborhood U of x_λ, S is eventually in $U^{-\circ}$, in this case, we also say that S weakly O_R-converges to x_λ, denoted by $S \stackrel{WO_R}{\longrightarrow} x_\lambda$.*

Theorem 3.4. *Let S be a net in (X,T) and $x_\lambda \in M(L^X)$. Then the following conditions are equivalent.*

(1) *x_λ is an O_R-cluster point of S.*

(2) *$\forall U \in N_P^\circ(x_\lambda)$, S is frequently in $U^{-\circ}$.*

(3) *$\forall U \in N_R^\circ(x_\lambda)$, S is frequently in U.*

Proof. (1) $\Rightarrow$ (2) Suppose that x_λ is an O_R-cluster point of S. If $U \in N_P^\circ(x_\lambda)$, then $U^{-\circ} \in N^\circ(x_\lambda)$. By the hypothesis of (1) S is frequently in $U^{-\circ-\circ}$. S is frequently in $U^{-\circ}$ since $U^{-\circ-\circ}U^{-\circ}$.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1) Suppose that the given condition hold for a net S and let $U \in N^\circ(x_\lambda)$, then $U^{-\circ} \in N_R^\circ(x_\lambda)$. By the hypothesis of (3) S is frequently in $U^{-\circ}$. Therefore x_λ is an O_R-cluster point of S. $\Box$

Analogous to the proof of Theorem 3.4 we can easily obtain the following result.

Theorem 3.5. *Let S be a net in (X,T) and $x_\lambda \in M(L^X)$. Then the following conditions are equivalent.*

(1) *x_λ is an O_R-limit point of S.*

(2) *$\forall U \in N_P^\circ(x_\lambda)$, S is eventually in $U^{-\circ}$.*

(3) *$\forall U \in N_R^\circ(x_\lambda)$, S is eventually in U.*

For weak O_R-convergence, we have same conclusions as Theorem 3.4 and Theorem 3.5. We omit them.

Theorem 3.6. *Let S be a net in (X,T) and $x_\lambda \in M(L^X)$. Then*

(1) *x_λ is a weak O_R-cluster point of S if and only if for each strongly δ-open neighborhood U of x_λ, S is frequently in U.*

(2) *x_λ is a weak O_R-limit point of S if and only if for each strongly δ-open neighborhood U of x_λ, S is eventually in U.*

Proof. (1) Sufficiency. Suppose that U is a strongly open neighborhood of x_λ, then $U^{-\circ}$ is a strongly δ-open neighborhood of x_λ. By the hypothesis, S is frequently in $U^{-\circ}$. Therefore x_λ is a weak O_R-cluster point of S.

Necessity. Suppose that x_λ is a weak O_R-cluster point of S and U is a strongly δ-open neighborhood of x_λ. Then there exists a regularly open L-set C such that CU and $x_\lambda \in \beta(C)$ since

$$\begin{aligned} x_\lambda \in \beta(U) &= \beta\left(\bigvee\{C \mid CU, C \text{ is regularly open}\}\right) \\ &= \bigcup\{\beta(C) \mid CU, C \text{ is regularly open}\}. \end{aligned}$$

By the hypothesis, S is frequently in CU.

(2) This is analogous to the proof of (1). □

It is easy to prove the following theorem.

Theorem 3.7. *Let S be a net in (X, T), T a subnet of S and $x_\lambda, x_\mu \in M(L^X)$. Then*

(1) $S \xrightarrow{O} x_\lambda$ implies that $S \xrightarrow{O_R} x_\lambda$;

(2) $S \xrightarrow{O_R} x_\lambda$ implies that $S \xrightarrow{WO_R} x_\lambda$;

(3) $S \xrightarrow{O_R} x_\lambda$ implies that x_λ is an O_R-cluster point of S;

(4) $S \xrightarrow{WO_R} x_\lambda$ implies that x_λ is a weak O_R-cluster point of S;

(5) x_λ is an O-cluster point of S implies that x_λ is an O_R-cluster point of S;

(6) x_λ is an O_R-cluster point of S implies that x_λ is a weak O_R-cluster point of S;

(7) If $x_\lambda x_\mu$ and x_λ is an O_R-cluster point of S, then x_μ is also an O_R-cluster point of S;

(8) $S \xrightarrow{O_R} x_\lambda x_\mu \Rightarrow S \xrightarrow{O_R} x_\mu$;

(9) If $x_\lambda x_\mu$ and x_λ is a weak O_R-cluster point of S, then x_μ is also a weak O_R-cluster point of S;

(10) $S \xrightarrow{WO_R} x_\lambda x_\mu \Rightarrow S \xrightarrow{WO_R} x_\mu$;

(11) $S \xrightarrow{O_R} x_\lambda \Rightarrow T \xrightarrow{O_R} x_\lambda$;

(12) $S \xrightarrow{WO_R} x_\lambda \Rightarrow T \xrightarrow{WO_R} x_\lambda$;

(13) x_λ is an O_R-cluster point of T implies that x_λ is an O_R-cluster point of S;

(14) x_λ is a weak O_R-cluster point of T implies that x_λ is a weak O_R-cluster point of S;

(15) x_λ is an O_R-cluster point of S if and only if S has a subnet R such that $R \xrightarrow{O_R} x_\lambda$.

(16) x_λ is a weak O_R-cluster point of S if and only if S has a subnet R such that $R \xrightarrow{WO_R} x_\lambda$.

Theorem 3.8. *Let $x_\lambda \in M(L^X)$, B be β-open. Then the following conditions are equivalent.*

(1) *x_λ quasi-coincides with B^-.*

(2) *There exists a net S quasi-coinciding with B such that $S \xrightarrow{O_R} x_\lambda$.*

(3) *There exists a net S quasi-coinciding with B such that x_λ is an O_R-cluster point of S.*

Proof. (1) $\Rightarrow$ (2) Suppose that x_λ quasi-coincides with B^-. Then $\forall U \in N_R^\circ(x_\lambda), U \not\leq B^{-'}$, i.e., $B^- U'$. Hence $B \not\leq U'$. This implies that $U \not\leq B'$. Take $S(U) \in M(L^X)$ such that $S(U)U$, $S(U) \not\leq B'$. We obtain a net $\{S(U) \mid U \in N_R^\circ(x_\lambda)\}$ O_R-converging to x_λ and it quasi-coincides with B.

(2) $\Rightarrow$ (3) is obvious by Theorem 3.7(3).

(3) $\Rightarrow$ (1) Let $\{S(n)\}_{n\in D}$ be a net quasi-coinciding with B and x_λ is an O_R-cluster point of S. If $x_\lambda(B^-)'$, then $\forall n \in D$, there exists $n_0 \in D$ such that $n_0 \geq n$ and $S(n_0)(B^-)'^{-\circ} = B^{-\circ-'}B'$ since B is β-open, which contradicts that S quasi-coincides with B. $\square$

Corollary 3.9. *Let (X, T) be an L-space and $A \in L^X$. Then the following conditions are equivalent:*

(1) *A is preclosed.*

(2) *For any net S quasi-coinciding with A°, if $S \xrightarrow{O_R} x_\lambda$, then $x_\lambda \not\leq A'$.*

(3) *For any net S quasi-coinciding with A°, if x_λ is an O_R-cluster point of S, then $x_\lambda \not\leq A'$.*

Proof. (1) $\Rightarrow$ (2) Suppose that $x_\lambda A'$. Then $A' \in N_P^\circ(x_\lambda)$. By Theorem 3.5 there exists $n_0 \in D$ such that $\forall n \geq n_0$, $S(n)A'^{-\circ} = A^{\circ-'}A^{\circ'}$, which contradicts that S quasi-coincides with A°. Therefore $x_\lambda \not\leq A'$.

(2) $\Rightarrow$ (1) $\forall x_\lambda \not\leq A^{\circ-'}$, by Theorem 3.8 there exists a net quasi-coinciding with A° such that $S \xrightarrow{O_R} x_\lambda$. By the hypothesis of (2) $x_\lambda \not\leq A'$. It implies that $A'A^{\circ-'}$, i.e., $A^{\circ-}A$. Therefore A is preclosed.

(1) $\Leftrightarrow$ (3) is analogous to the proof of (1) $\Leftrightarrow$ (2). $\square$

Corollary 3.10. *Let (X, T) be an L-space and $A \in L^X$. Then the following conditions are equivalent:*

(1) *A is preopen.*

(2) *$\forall x_\lambda A, S \xrightarrow{O_R} x_\lambda$ implies that S is eventually in $A^{-\circ}$.*

(3) *$\forall x_\lambda A$, if x_λ is O_R-cluster point of S, then S is frequently in $A^{-\circ}$.*

Proof. (1) $\Rightarrow$ (2) is obvious by Theorem 3.5.

(2) $\Rightarrow$ (1) $\forall x_\lambda A^{-\circ} = A^{-'-'}$, by Theorem 3.8 there exists a net quasi-coinciding with $A^{-'}$ such that $S \xrightarrow{O_R} x_\lambda$. If $x_\lambda A$, by the hypothesis of (2) S is eventually in $A^{-\circ}A^{-}$, which contradicts that S quasi-coincides with $A^{-'}$. Thus $x_\lambda \not\leq A$. It implies that $AA^{-\circ}$. Therefore A is preopen.

(1) $\Leftrightarrow$ (3) is analogous to the proof of (1) $\Leftrightarrow$ (2). $\Box$

Corollary 3.11. *Let (X, T) be an L-space and $A \in L^X$. Then A is preclopen if one of the following conditions is true.*

(1) *For any net S quasi-coinciding with A°, if $S \xrightarrow{O_R} x_\lambda$, then $x_\lambda \not\leq A'$*

(2) *$\forall x_\lambda A, S \xrightarrow{O_R} x_\lambda$ implies that S is eventually in $A^{-\circ}$*

(3) *For any net S quasi-coinciding with A°, if x_λ is an O_R-cluster point of S, then $x_\lambda \not\leq A'$*

(4) *$\forall x_\lambda A$, if x_λ is O_R-cluster point of S, then S is frequently in $A^{-\circ}$.*

Proof. Suppose that the condition (1) is satisfied. By Corollary 3.9 A is preclosed. Now we prove that A is preopen, i.e., $AA^{-\circ}$. $\forall x_\lambda A^{-\circ} = A^{-'-'}$, there exists a net S quasi-coinciding with $A^{-'}$ such that $S \xrightarrow{O_R} x_\lambda$. By the hypothesis of (1) and $A^{-'} = A^{'\circ}$, it follows that $x_\lambda \not\leq A$. This implies that $AA^{-\circ}$. Therefore A is preclopen.

Suppose that the condition (2) is satisfied. By Corollary 3.10, A is preopen. Now we prove that A is preclosed, i.e., $A^{\circ-}A$. $\forall x_\lambda A^{\circ-'}$, there exists a net S quasi-coinciding with A° such that $S \xrightarrow{O_R} x_\lambda$. If $x_\lambda A'$, by the hypothesis of (2) S is eventually in $A'^{-\circ} = A^{\circ-'}A^{\circ'}$, which contradicts that S quasi-coincides with A°. Thus $x_\lambda \not\leq A'$. It implies that $A'A^{\circ-'}$, i.e., $A^{\circ-}A$. Therefore A is preclopen.

The other cases can achieved from the similar progress. $\Box$

Theorem 3.12. *Let $x_\lambda \in M(L^X)$, $B \in L^X$. Then the following conditions are equivalent.*

(1) *x_λ weak quasi-coincides with $cl_\delta(B)$.*

(2) *There exists a net S quasi-coinciding with B such that $S \xrightarrow{WO_R} x_\lambda$.*

(3) *There exists a net S quasi-coinciding with B such that x_λ is a weak O_R-cluster point of S.*

Proof. (1) $\Rightarrow$ (2) Suppose that x_λ weak quasi-coincides with $cl_\delta(B)$. Then for each strongly open neighborhood U of x_λ, $U \not\leq cl_\delta(B)'$, i.e., $U \not\leq \bigvee\{C \mid CB', C \text{ is regularly open}\}$. Thus $U^{-\circ} \not\leq B'$. Take $S(U) \in M(L^X)$ such that $S(U)U^{-\circ}, S(U) \not\leq B'$. We obtain a net

$$\{S(U) \mid U \text{ is a strongly open neighborhood of } x_\lambda\}.$$

It weak O_R-converges to x_λ and quasi-coincides with B.

(2) $\Rightarrow$ (3) is obvious by Theorem 3.7(4).

(3) $\Rightarrow$ (1) Let $\{S(n)\}_{n\in D}$ be a net quasi-coinciding with B and x_λ is a weak O_R-cluster point of S. If x_λ does not weak quasi-coincides with $cl_\delta(B)$, then $x_\lambda \in \beta(cl_\delta(B)')$. Hence there exists a regularly open L-set C such that CB' and $x_\lambda \in \beta(C)$ since

$$\begin{aligned} x_\lambda \in \beta(cl_\delta(B)') &= \beta\left((\bigwedge\{A \mid BA, A \text{ is regularly closed}\})'\right) \\ &= \beta(\bigvee\{A' \mid BA, A \text{ is regularly closed}\}) \\ &= \bigcup\{\beta(C) \mid CB', C \text{ is regularly open}\}. \end{aligned}$$

Then $\forall n \in D$, there exists $n_0 \in D$ such that $n_0 \geq n$ and $S(n_0)CB'$, which contradicts that S quasi-coincides with B. □

Corollary 3.13. *Let (X, T) be an L-space and $A \in L^X$. Then the following conditions are equivalent.*

(1) *A is δ-closed.*

(2) *For any net S quasi-coinciding with A, if $S \xrightarrow{WO_R} x_\lambda$, then $x_\lambda \notin \beta(A')$.*

(3) *For any net S quasi-coinciding with A, if x_λ is a weak O_R-cluster point of S, then $x_\lambda \notin \beta(A')$.*

Proof. (1) $\Rightarrow$ (2) Suppose that $x_\lambda \in \beta(A')$. By Theorem 3.6, S is eventually in A' since A is δ-closed, which contradicts that S quasi-coincides with A.

(2) $\Rightarrow$ (1) Suppose that $x_\lambda \notin \beta(cl_\delta(A)')$. Then there exists a net S quasi-coinciding with A such that $S \xrightarrow{WO_R} x_\lambda$. By the hypothesis of (2), it follows that $x_\lambda \notin \beta(A')$. Therefore $A' cl_\delta(A)'$, i.e., $cl_\delta(A)A$. By Lemma 2.18 we know that $Acl_\delta(A)$. Therefore A is δ-closed.

(1) $\Leftrightarrow$ (3) This proof is analogous to the proof of (1) $\Leftrightarrow$ (2). □

Corollary 3.14. *Let (X, T) be an L-space and $A \in L^X$. Then the following conditions are equivalent.*

(1) *A is δ-open.*

(2) *$\forall x_\lambda \in \beta(A)$, $S \xrightarrow{WO_R} x_\lambda$ implies that S is eventually in A.*

(3) *$\forall x_\lambda \in \beta(A)$, if x_λ is weak O_R-cluster point of S, then S is frequently in A.*

Proof. (1) $\Rightarrow$ (2) is obvious by Theorem 3.6.

(2) $\Rightarrow$ (1) Suppose that $x_\lambda \in \beta(A)$. If $x_\lambda \notin \beta(int_\delta(A)) = \beta(cl_\delta(A')')$, then there exists a net S quasi-coinciding with A' such that $S \xrightarrow{WO_R} x_\lambda$.

By the hypothesis of (2), S is eventually in A, which contradicts that S quasi-coincides with A'. Thus $x_\lambda \in \beta(int_\delta(A))$. It implies that $Aint_\delta(A)$. By Lemma 2.18 we know that $int_\delta(A)A$. Therefore A is δ-open.

(1) $\Leftrightarrow$ (3) This proof is analogous to the proof of (1) $\Leftrightarrow$ (2). $\square$

Theorem 3.15. *Let $f : (X, T_1) \to (Y, T_2)$ be a R-irresolute L-value Zadeh's type mapping. Then*

(1) For any net S in L^X, if $S \xrightarrow{O_R} x_\lambda$, then $f_L^{\to}(S) \xrightarrow{O_R} f_L^{\to}(x_\lambda)$.

(2) For any net S in L^X, if x_λ is an O_R-cluster point of S, then $f_L^{\to}(x_\lambda)$ is an O_R-cluster point of $f_L^{\to}(S)$.

Proof. (1) Suppose that $U \in N_R^\circ(f_L^{\to}(x_\lambda))$. Then $f_L^{\leftarrow}(U) \in N_R^\circ(x_\lambda)$. Since $S \xrightarrow{O_R} x_\lambda$, there exists $n_0 \in D$ such that $\forall n \geq n_0$ $S(n)f_L^{\leftarrow}(U)$. This implies that $f_L^{\to}(S) \xrightarrow{O_R} f_L^{\to}(x_\lambda)$ by

$$f_L^{\to}(S(n))f_L^{\to}(f_L^{\leftarrow}(U))U.$$

(2) This is analogous to the proof of (1). $\square$

Theorem 3.16. *Let $f : (X, T_1) \to (Y, T_2)$ be an almost continuous L-value Zadeh's type mapping. Then*

(1) For any net S in L^X, if $S \xrightarrow{O} x_\lambda$, then $f_L^{\to}(S) \xrightarrow{O_R} f_L^{\to}(x_\lambda)$.

(2) For any net S in L^X, if x_λ is an O-cluster point of S, then $f_L^{\to}(x_\lambda)$ is an O_R-cluster point of $f_L^{\to}(S)$.

Proof. (1) Suppose that $U \in N_R^\circ(f_L^{\to}(x_\lambda))$. Then $f_L^{\leftarrow}(U) \in N^\circ(x_\lambda)$. Since $S \xrightarrow{O} x_\lambda$, there exists $n_0 \in D$ such that $\forall n \geq n_0$ $S(n)f_L^{\leftarrow}(U)$. This implies that $f_L^{\to}(S) \xrightarrow{O_R} f_L^{\to}(x_\lambda)$ by

$$f_L^{\to}(S(n))f_L^{\to}(f_L^{\leftarrow}(U))U.$$

(2) This is analogous to the proof of (1). $\square$

Theorem 3.17. *Let $f : (X, T_1) \to (Y, T_2)$ be a completely continuous L-value Zadeh's type mapping. Then*

(1) For any net S in L^X, if $S \xrightarrow{O_R} x_\lambda$, then $f_L^{\to}(S) \xrightarrow{O} f_L^{\to}(x_\lambda)$.

(2) For any net S in L^X, if x_λ is an O_R-cluster point of S, then $f_L^{\to}(x_\lambda)$ is an O-cluster point of $f_L^{\to}(S)$.

Proof. (1) Suppose that $U \in N^{\circ}(f_L^{\rightarrow}(x_\lambda))$. Then $f_L^{\leftarrow}(U) \in N_R^{\circ}(x_\lambda)$. Since $S \xrightarrow{O_R} x_\lambda$, there exists $n_0 \in D$ such that $\forall n \geq n_0$ $S(n) f_L^{\leftarrow}(U)$. This implies that $f_L^{\rightarrow}(S) \xrightarrow{O} f_L^{\rightarrow}(x_\lambda)$ by

$$f_L^{\rightarrow}(S(n)) f_L^{\rightarrow}(f_L^{\leftarrow}(U)) U.$$

(2) This is analogous to the proof of (1). $\square$

For weak O_R-convergence, we have the similar three conclusions as above since $f_L^{\rightarrow}(x_\lambda) \in \beta(U)$ implies $x_\lambda \in \beta(f_L^{\leftarrow}(U))$. They are also omitted here.

Theorem 3.18. *Let $f : (X, T_1) \rightarrow (Y, T_2)$ be an L-value Zadeh's type mapping. Then the following conditions are equivalent.*

(1) *f is δ-continuous.*

(2) *For any net S in L^X, if $S \xrightarrow{WO_R} x_\lambda$, then $f_L^{\rightarrow}(S) \xrightarrow{WO_R} f_L^{\rightarrow}(x_\lambda)$.*

(3) *For any net S in L^X, if x_λ is a weak O_R-cluster point of S, then $f_L^{\rightarrow}(x_\lambda)$ is a weak O_R-cluster point of $f_L^{\rightarrow}(S)$.*

Proof. (1) $\Rightarrow$ (2) Suppose that U is a strongly regularly open neighborhood of fuzzy point $f_L^{\rightarrow}(x_\lambda)$ and net $S \xrightarrow{WO_R} x_\lambda$. Then $x_\lambda \in \beta(f_L^{\leftarrow}(U))$. Thus $f_L^{\leftarrow}(U)$ is a strongly δ-open neighborhood of fuzzy point x_λ since f is δ-continuous. There exists $n_0 \in D$ such that $\forall n \geq n_0, S(n) f_L^{\leftarrow}(U)$. Thus $f_L^{\rightarrow}(S(n)) f_L^{\rightarrow}(f_L^{\leftarrow}(U)) U$ for any $n \geq n_0$. Therefore $f_L^{\rightarrow}(S) \xrightarrow{WO_R} f_L^{\rightarrow}(x_\lambda)$.

(2) $\Rightarrow$ (1) Suppose that A is a regularly open L-set in (Y, T_2). $\forall x_\lambda \in \beta(f_L^{\leftarrow}(A))$, let $S \xrightarrow{WO_R} x_\lambda$. By the hypothesis of (2) $f_L^{\rightarrow}(S) \xrightarrow{WO_R} f_L^{\rightarrow}(x_\lambda)$. There exists $n_0 \in D$ such that $\forall n \geq n_0, f_L^{\rightarrow}(S(n)) A$ since $f_L^{\rightarrow}(x_\lambda) \in \beta(f_L^{\rightarrow}(f_L^{\leftarrow}(A))) \leq \beta(A)$. It implies that $S(n) f_L^{\leftarrow}(A)$. Thus $f_L^{\leftarrow}(A)$ is δ-open by Corollary 3.14. Therefore f is δ-continuous.

(1) $\Leftrightarrow$ (3) This is analogous to the proof of (1) $\Leftrightarrow$ (2). $\square$

4. Characterizations of near (compactness) S^*-compactness

Theorem 4.1. *An L-set G is nearly compact in (X, T) if and only if $\forall a \in M(L)$, $\forall b \in \beta^*(a)$, each constant b-net quasi-coinciding with G has an O_R-cluster point x_a quasi-coinciding with G.*

Proof. Suppose that G is nearly compact. For $a \in M(L)$ and $b \in \beta^*(a)$, let $\{S(n) \mid n \in D\}$ be a constant b-net quasi-coinciding with G. Suppose that S has no O_R-cluster point x_a quasi-coinciding with G. Then for each

$x_a \not\leq G'$, there exist $U_x \in N^\circ(x_a)$ and $n_x \in D$ such that $\forall n \geq n_x$, $S(n)U_x^{-\circ}$. Take $\Phi = \{U_x \mid x_a \not\leq G'\}$, then Φ is an open Q_a-cover of G. Since G is nearly compact, Φ has a finite subfamily $\Psi = \{U_{x^i} \mid i = 1, 2, \cdots, k\}$ such that $\Psi^{-\circ}$ is a Q_b-cover of G. Since D is a directed set, there exists $n_0 \in D$ such that $n_0 \geq n_{x^i}$ for each ik. Thus we can obtain that $\forall n \geq n_0$, $S(n) \bigvee \{U_{x^i}^{-\circ} \mid i = 1, 2, \cdots, k\}$. This contradicts that $\Psi^{-\circ}$ is a Q_b-cover of G. Therefore S has an O_R-cluster point $x_a \not\leq G'$.

Conversely suppose that $\forall a \in M(L)$, $\forall b \in \beta^*(a)$, each constant b-net quasi-coinciding with G has an O_R-cluster point $x_a \not\leq G'$. We now prove that G is nearly compact. Let Φ be an open Q_a-cover of G. If for each finite subfamily Ψ of Φ, $\Psi^{-\circ}$ is not a Q_b-cover of G, then for each finite subfamily Ψ of Φ, there exists $S(\Psi) \in M(L^X)$ with height b such that $S(\Psi) \not\leq G'$ and $S(\Psi) \not\leq \bigvee \Psi^{-\circ}$. Take $S = \{S(\Psi) \mid \Psi \text{ is a finite subfamily of } \Phi\}$, then S is a constant b-net quasi-coinciding with G. By $b \in \beta^*(a)$ we can take $s \in \beta^*(a)$ such that $b \in \beta^*(s)$. Then S has an O_R-cluster point $x_s \not\leq G'$. Hence for each finite subfamily Ψ of Φ we have that $x_s \not\leq \bigvee \Psi$(because if $x_s \bigvee \Psi$, then there exists an $A \in \Psi$ such that $x_s A$, i.e., A is an open neighborhood of x_s, hence there exists a finite subfamily Ψ_0 of Φ such that $\Psi \leq \Psi_0$ and $S(\Psi_0)A^{-\circ} \bigvee \Psi^{-\circ} \bigvee \Psi_0^{-\circ}$, this contradicts the definition of S), in particular $x_s \not\leq B$ for each $B \in \Phi$. But since Φ is an open Q_a-cover of G, we know that there exists $B \in \Phi$ such that $x_s B$, this yields a contradiction with $x_s \not\leq B$. So G is nearly compact. □

Theorem 4.2. *An L-set G is nearly compact in $(X, \mathcal{T})$ if and only if $\forall a \in M(L)$, $\forall b \in \beta^*(a)$, each b^--net quasi-coinciding with G has an O_R-cluster point x_a quasi-coinciding with G.*

Proof. The sufficiency is obvious, we need only to prove the necessity.

Let G be nearly compact, $a \in M(L)$, $b \in \beta^*(a)$ and $\{S(n) \mid n \in D\}$ be an b^--net quasi-coinciding with G. Then there exists $n_0 \in D$ such that $\forall n \geq n_0$, $S(n)b$. Put $E = \{n \in D \mid n \geq n_0\}$ and

$$T = \{T(n) \mid n \in E, V\left(T(n)\right) = b, \text{ the support point of } T(n) \text{ is same as } S(n)\}.$$

Then T is a constant b-net quasi-coinciding with G. Let x_a be an O_R-cluster point of T. It is easy to see that x_a is also an O_R-cluster point of S. □

Analogous to the proof of Theorem 4.1 and Theorem 4.2 we can easily obtain the following two results.

Theorem 4.3. *An L-set G is near S^*-compact in $(X, \mathcal{T})$ if and only if $\forall a \in M(L)$, each constant a-net quasi-coinciding with G has a weak O_R-cluster point $x_a \notin \beta(G')$.*

Theorem 4.4. *An L-set G is near S^*-compact in $(X, \mathcal{T})$ if and only if $\forall a \in M(L)$, each a^--net quasi-coinciding with G has a weak O_R-cluster point $x_a \notin \beta(G')$.*

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